mathematics

Article

On Some Families of Codes Related to the Even Linear Codes Meeting the Grey–Rankin Bound

Iliya Bouyukliev [1], Stefka Bouyuklieva [2,*] and Maria Pashinska-Gadzheva [1]

[1] Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 5000 Veliko Tarnovo, Bulgaria
[2] Faculty of Mathematics and Informatics, St. Cyril and St. Methodius University of Veliko Tarnovo, 5000 Veliko Tarnovo, Bulgaria
* Correspondence: stefka@ts.uni-vt.bg

Abstract: Bounds for the parameters of codes are very important in coding theory. The Grey–Rankin bound refers to the cardinality of a self-complementary binary code. Codes meeting this bound are associated with families of two-weight codes and other combinatorial structures. We study the relations among six infinite families of binary linear codes with two and three nonzero weights that are closely connected to the self-complementary linear codes meeting the Grey–Rankin bound. We give a construction method and partial classification results for such codes. The properties of the codes in the studied families and their relations help us in constructing codes of a higher dimension from codes with a given dimension.

Keywords: linear code; residual code; two-weight code; Grey–Rankin bound

MSC: 94B05; 05A18

Citation: Bouyukliev, I.; Bouyuklieva, S.; Pashinska-Gadzheva, M. On Some Families of Codes Related to the Even Linear Codes Meeting the Grey–Rankin Bound. *Mathematics* **2022**, *10*, 4588. https://doi.org/10.3390/math10234588

Academic Editor: Patrick Solé

Received: 9 November 2022
Accepted: 30 November 2022
Published: 3 December 2022

1. Introduction

We have two main goals with this study. The first is to explore and describe in detail the relationships between derivative codes of codes meeting the Grey–Rankin bound. Our motivation for looking for these connections is that they help us in constructing codes of higher dimensions. The other goal is to construct new codes and partially classify codes with larger parameters on the basis of proven relations, properties, and known results. More precisely, we perform the following: First, we give a general theoretical construction for $(k+2)$-dimensional codes of the considered type on the basis of already known k-dimensional codes. Second, we find all inequivalent codes of a given dimension that have as residual a code with fixed properties with the help of previously developed algorithms.

The (Hamming) weight of a vector v in a vector space over the binary field $\mathbb{F}_2$ is given by the number of its nonzero coordinates. A binary linear $[n,k,d]$ code C is a k-dimensional subspace of the vector space $\mathbb{F}_2^n$ with minimal weight (or distance) d, where d is the smallest weight among all nonzero codewords in the code. The elements of C are called *codewords*. A *subcode* of C is a linear subspace of the code.

A code C is self-complementary whenever $x \in C$ implies $\overline{x} \in C$, where $\overline{x}$ denotes the complement of the binary vector x, obtained by replacing each 0 in x by 1, and each 1 by 0. The binary linear code C is self-complementary if and only if it contains the all-ones vector $\mathbf{1} = (1, \ldots, 1)$ (see, for example, [1]). For any length n and minimal distance d, the Grey–Rankin bound [2–4] is an upper bound for the cardinality of a binary self-complementary code C. It states that

$$|C| \leq \frac{8d(n-d)}{n-(n-2d)^2} \tag{1}$$

provided that the right-hand side is positive. The bound also holds for nonlinear codes, but we consider only linear codes here.

There is significant research on the self-complementary codes meeting the Grey–Rankin bound. Some of their subcodes are two-weight projective codes. Therefore, they are associated with strongly regular graphs [5–7] and uniformly packed codes [8]. Codes of this type can also be considered as divisible codes [9] or self-orthogonal codes [10]. Another important connection of these codes is with the bent functions [11,12] and bent vectorial functions [13]. The set of all codewords with minimal weight in a binary linear self-complementary code of even length, meeting the Grey–Rankin bound, constitutes the set of blocks of a quasisymmetric SDP design [14,15]. Hence, inequivalent codes yield nonisomorphic SDP designs and the number of inequivalent codes and its associated SDP design coincide. Jungnickel and Tonchev [15] showed that the numbers of nonisomorphic quasisymmetric SDP designs and inequivalent self-complementary codes with Parameters (2) and (3) grow exponentially with m from an argument by Kantor [6].

Linear codes of dimension $k \leq 7$ meeting the Grey–Rankin bound and some families of codes related to them have been well-studied. For codes of this type with a dimension $k > 7$, there are only partial results. There are four inequivalent self-complimentary codes with parameters [28,7,12] and [36,7,16] as shown in [16,17]. A lower bound on the number of $[120, 9, 56]$ and $[136, 9, 64]$ codes is presented in [10].

The paper is organized as follows. In Section 2, we present some important properties of the subcodes and residual codes of a linear code, and give some information about the weights in two- and three-weight codes. Section 3 is devoted to the subcodes of the self-complementary codes, projective complementary codes, and a new equivalence relation in the set of binary linear codes with given length and dimension, called SC-equivalence. Section 4 presents six classes of binary linear codes connected to the Grey–Rankin bound. We give a theoretical construction of codes in these families in Section 5. Some computational results are shown in Section 6.

2. Preliminaries

In this section, we present some basic notations, definitions, and theorems.

The dual code of C is $C^\perp = \{u \in \mathbb{F}_2^n : u \cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n = 0$ for all $v \in C\}$. If C is an $[n, k, d]$ code, then $C^\perp$ is an $[n, n-k, d^\perp]$ code, and $d^\perp$ is called *dual distance* of C. If $d^\perp(C) = 1$, then the code has zero coordinates. Removing these zero coordinates, we obtain a linear code with the same dimension and the same weights, but with a smaller length, which we call the *effective length* of C. If $d^\perp(C) > 1$, the effective length of C is n. If the effective length of C is less than n, we consider the code obtained from C by removing the zero coordinates, as the same code, and as the length of C we take its effective length. If $d^\perp(C) = 2$, then the code has equal coordinates. We call the code *projective*, if $d^\perp(C) \geq 3$.

Definition 1. *The residual code $Res(C, c)$ with respect to a codeword $c \in C$ is the restriction of C to the zero coordinates of c.*

Lower bounds on the minimal and dual distance of a residual code in the binary case are given by the following theorem and proposition, respectively.

Theorem 1 ([18] Theorem 2.7.1)**.** *Suppose C is a binary $[n, k, d]$ code, and suppose $c \in C$ has weight w where $w < 2d$. Then, $Res(C, c)$ is an $[n-w, k-1, d']$ code with $d' \geq d - w + \lceil w/2 \rceil$.*

The proposition follows directly from [18], Theorem 1.5.7.

Proposition 1. *Suppose C is a binary $[n, k, d]$ code with dual distance $d^\perp$, $c \in C$, and the dimension of $Res(C, c)$ is $k-1$. Then, the dual distance of $Res(C, c)$ is at least $d^\perp$.*

A matrix whose rows form a basis of C is called a *generator matrix* of C. *The weight enumerator* $W(y)$ of a code C is given by $W(y) = \sum_{i=0}^{n} A_i y^i$, where A_i is the number of codewords of weight i. By $[n, k, \{w_1, \ldots, w_s\}]$ we denote a linear code whose nonzero weights are in the set $\{w_1, \ldots, w_s\}$. A linear code over a finite field with the Hamming metric is called Δ-divisible if the weights of all codewords are divisible by Δ. Surveys on divisible codes are given in [9,19]. Applications include subspace codes, partial spreads, vector space partitions, and distance optimal codes [19].

Two binary codes are equivalent if one can be obtained from the other with a permutation of the coordinates. A permutation $\sigma \in S_n$ for which $C = \sigma(C)$ is an automorphism of C. The set of all automorphisms of C with composition of permutations as the group operation forms the automorphism group of C, denoted by $\mathrm{Aut}(C)$.

The parameters of a linear self-complementary code meeting the Grey–Rankin bound are

$$[2^{2m-1} - 2^{m-1}, 2m+1, 2^{2m-2} - 2^{m-1}] \tag{2}$$

or

$$[2^{2m-1} + 2^{m-1}, 2m+1, 2^{2m-2}] \tag{3}$$

for even lengths [1]. A self-complementary code with Parameter (2) or (3) has a unique weight enumerator for each m. In this paper, all self-complementary codes that are considered are even-weight binary projective linear codes.

A binary projective code has no zero coordinates and no equal columns in its generator matrix. Hence, its dual distance is $d^{\perp} \geq 3$. Pless power moments are sequences of equations involving binomial coefficients and Stirling numbers relating the weight distributions of C and $C^{\perp}$ [20]. The first three equations for a three-weight projective code with nonzero weights α, β and γ are as follows:

$$\begin{array}{rcrcrcl} A_\alpha & + & A_\beta & + & A_\gamma & = & 2^k - 1 \\ \alpha A_\alpha & + & \beta A_\beta & + & \gamma A_\gamma & = & 2^{k-1} n \\ \alpha^2 A_\alpha & + & \beta^2 A_\beta & + & \gamma^2 A_\gamma & = & 2^{k-2} n(n+1) \end{array} \tag{4}$$

Solving the corresponding linear system with variables A_α, A_β and A_γ, we obtain

$$\begin{aligned} A_\alpha &= \frac{2^{k-2}n(n+1) - (\beta+\gamma)2^{k-1}n + \beta\gamma(2^k - 1)}{(\alpha-\beta)(\alpha-\gamma)} \\ A_\beta &= \frac{2^{k-2}n(n+1) - (\gamma+\alpha)2^{k-1}n + \alpha\gamma(2^k - 1)}{(\beta-\gamma)(\beta-\alpha)} \\ A_\gamma &= \frac{2^{k-2}n(n+1) - (\beta+\alpha)2^{k-1}n + \alpha\beta(2^k - 1)}{(\gamma-\beta)(\gamma-\alpha)} \end{aligned}$$

If the code is two-weight, the system is:

$$\begin{array}{rcrcl} A_\alpha & + & A_\beta & = & 2^k - 1 \\ \alpha A_\alpha & + & \beta A_\beta & = & 2^{k-1} n \end{array} \tag{5}$$

and therefore

$$A_\alpha = \frac{(2^k - 1)\beta - 2^{k-1}n}{\beta - \alpha}, \quad A_\beta = \frac{2^{k-1}n - (2^k - 1)\alpha}{\beta - \alpha}. \tag{6}$$

For the two-weight codes that we considered, we obtained $B_2 = 0$, where B_2 is the number of the codewords of weight 2 in the dual code. This means that the studied two-weight codes in this research are projective.

3. Subcodes, Projective Complementary Codes, and SC Equivalence

Let C be a binary projective $[n, k]$ linear code with a generator matrix G. In this section, we characterise the subcodes of C with dimension $k - 1$. We present some relations between these subcodes for a self-complementary code C.

3.1. Subcodes of Dimension $k-1$

If $a \in \mathbb{F}_2^k$, then aG is a codeword in the code C. Hence, each vector of the k-dimensional vector space $\mathbb{F}_2^k$ defines a codeword in C. If we consider all nonzero vectors of the k-dimensional vector space as the columns of a $k \times (2^k - 1)$ matrix S_k, we obtain a generator matrix of the well-known simplex code $\mathcal{S}_k$.

Let $a = (a_1, a_2, \ldots, a_k) \in \mathbb{F}_2^k$ be a nonzero vector, and B_a be the set of all solutions of the equation $a_1x_1 + a_2x_2 + \cdots + a_kx_k = 0$. Then, B_a is a linear subspace of $\mathbb{F}_2^k$ with dimension $k-1$. Using this subspace, we obtain a subcode of C as

$$C_a = \{bG,\ b \in B_a\},\ \ \dim C_a = k-1.$$

Conversely, any subcode of dimension $k-1$ can be considered as a set $\{w = vG,\ v \in \mathbb{F}_2^k\}$ where the vectors v are the solutions of a linear equation with k variables. In this way, we obtain a bijection between the set of nonzero vectors in $\mathbb{F}_2^k$ and the set of all subspaces of C with dimension $k-1$. Hence, the number of these subspaces is $2^k - 1$.

Moreover, if we fix $b \in \mathbb{F}_2^k$, we can consider it as a solution of $2^{k-1} - 1$ linear equations with k variables. It follows that any codeword bG belongs to exactly $2^{k-1} - 1$ subcodes of dimension $k-1$.

Let us see what the effective length of these subcodes is. As C is a projective binary code, the effective length of its subcode of dimension $k-1$ cannot be less than $n-1$. If C_i is the set of all codewords in C that have zeros in the i-th coordinate, then C_i is a subcode of C with dimension $k-1$ and effective length $n-1$. Thus, $C_1, C_2, \ldots, C_n$ are all subcodes of C with dimension $k-1$ and effective length $n-1$. It follows that the subcodes of dimension $k-1$ and effective length n are $2^k - 1 - n$. An interesting observation is that the simplex code is the only projective code for which all subcodes of dimension $k-1$ have the effective length $n-1$.

3.2. SC-Equivalence

Let C be a self-complementary projective code. The number of subcodes containing the all-ones vector is $N_{all-one} = 2^{k-1} - 1$. Therefore, the number of subcodes not containing that vector is $2^k - 1 - N_{all-one} = 2^{k-1}$. If $n < 2^{k-1}$, the subcodes of C with effective length n that do not contain the all-ones vector are $2^{k-1} - n$. If $n \geq 2^{k-1}$, all subcodes with effective length n contain the all-ones vector. For example, for the Reed–Muller code $\mathcal{RM}(1,m)$ the length is $n = 2^m$ and the dimension $k = m+1$, so $n = 2^{k-1}$; therefore, all its $[2^m, m]$ subcodes either contain the all-ones vector or have effective length $n-1$.

Next, we introduce the following construction for a given even integer n:

- If C is an $[n, k-1]$ linear code that is not self-complementary, $\widehat{C} = C \cup (\mathbf{1} + C)$. This means that if G is a generator matrix of C, the matrix $\widehat{G} = \begin{pmatrix} 1 & \ldots & 1 \\ & G & \end{pmatrix}$ is a generator matrix of $\widehat{C}$.
- If C' is an $[n-1, k-1]$ linear code that is not self-complementary, $\widehat{C'} = (0|C') \cup (1|\mathbf{1} + C')$. This means that if G' is a generator matrix of C', the matrix $\widehat{G'} = \begin{pmatrix} 1 & \ldots & 1 \\ & G' & 0 \end{pmatrix}$ is a generator matrix of $\widehat{C'}$.

We can now introduce an equivalence relation as follows:

Definition 2. *Two binary linear codes C_1 and C_2 with effective length either n or $n-1$ that do not contain the all-ones vector are self-complementary (SC)- equivalent if $\widehat{C_1}$ and $\widehat{C_2}$ are equivalent self-complementary codes with length n.*

With this equivalence relation in mind, we have that, for code C, all subcodes not containing the all-ones vector define a self-complementary equivalence class (SCE). Thus, if we have a representative from the class, we know all codes of the class up to equivalence. For our research, we consider only codes of that class with a special weight characteristics. Let C be a binary projective linear code that does not contain the all-ones vector. Code C is self-complementary saving weight (SCSW) if all codes in SCE have the same weight enumerators.

3.3. Projective Complementary Codes

Let C be a projective $[n,k]$ code with generator matrix G. We can reorder the columns in the matrix S_k such that the obtained matrix is $S'_k = (G|\overline{G})$. The matrix S'_k generate a simplex code (more precisely, matrices S_k and S'_k generate equivalent codes, and we call them both simplex codes). Then, the matrix $\overline{G}$ generates a code of length $2^k - 1 - n$ called the projective complementary code of C and denoted by $\overline{C}$.

If C has two nonzero weights w_1 and w_2, then the weights of $\overline{C}$ are $2^{k-1} - w_1$ and $2^{k-1} - w_2$ since all words of the simplex code have constant weight 2^{k-1}. Using Pless Power Moments (4), we can calculate the weight enumerator of the projective complementary code C when we know the nonzero weights.

4. Classes of Codes Connected to Grey–Rankin Bound and the Relations between Them

Let C be a self-complementary binary projective code that meets the Grey–Rankin bound. All such codes are fully defined by projective two-weight codes. In this section, we define six families of codes connected to Grey–Rankin bound, four two-weight codes and two three-weight codes. We present the main relations among these six families. Let $k = 2s$, $s \geq 2$. We introduce the following notations:

$$\begin{array}{ll} t_k = 2^{k-2} & t_{k^\pm} = t_k \pm 2^{s-1} \\ T_k = 2t_k = 2^{k-1} & T_{k^\pm} = T_k \pm 2^{s-1} \\ T_{k+1} = 2^k & \end{array}$$

Obviously, $T_{k^+} + T_{k^-} = 2^k$ and $t_{k^+} + t_{k^-} = 2^{k-1}$.

Now, we define the following families of codes:

- Four families of two-weight linear codes with the following parameters ($k = 2s$):
  - Φ_{k^-} with parameters $[T_{k^-}, k, \{t_{k^-}, t_k\}]$
  - Φ_{k^+} with parameters $[T_{k^+}, k, \{t_k; t_{k^+}\}]$
  - Φ'_{k^-} with parameters $[T_{k^-} - 1, k, \{t_{k^-}, t_k\}]$
  - Φ'_{k^+} with parameters $[T_{k^+} - 1, k, \{t_k; t_{k^+}\}]$
- There are two families of three-weight codes with the following parameters:
  - Ψ_k with parameters $[2^k, k+1, \{T_{k^-}, 2^{k-1}, T_{k^+}\}]$
  - Ψ'_k with parameters $[2^k - 1, k+1, \{T_{k^-}, 2^{k-1}, T_{k^+}\}]$

The parameters, weight distributions, and number of inequivalent codes from the different families for $s \leq 4$ are presented in the following sections. The weight distributions of the codes in the considered families are:

$$\Phi_{k^-}:\ 1 + T_{k^-}y^{t_{k^-}} + (T_{k^+} - 1)y^{t_k}$$

$$\Phi_{k^+}:\ 1 + (T_{k^-} - 1)y^{t_k} + T_{k^+}y^{t_{k^+}}$$

$$\Phi'_{k^-}:\ 1 + T_{k^+}y^{t_{k^-}} + (T_{k^-} - 1)y^{t_k}$$

$$\Phi'_{k^+}:\ 1 + (T_{k^+} - 1)y^{t_k} + T_{k^-}y^{t_{k^+}}$$

$$\Psi_k:\ 1 + T_{k^-}y^{T_{k^-}} + (2T_k - 1)y^{T_k} + T_{k^+}y^{T_{k^+}}$$

$$\Psi'_k:\ 1 + T_{k^+}y^{T_{k^-}} + (2T_k - 1)y^{T_k} + T_{k^-}y^{T_{k^+}}$$

Remark 1. *If C is a code in one of the families Φ_{k^-}, Φ_{k^+} or Ψ_k, then the code $\widehat{C} = C \cup (\mathbf{1} + C)$ has the same nonzero weights as C with an additional weight n for the all-ones vector, where n is the length of the code. Similarly, if C' is a code in one of the families Φ'_{k^-}, Φ'_{k^+} or Ψ'_k, then the code*

$\widehat{C'} = ((0, C') \cup (1, \mathbf{1} + C))$ *has the same nonzero weights as* C' *with an additional weight for the all-ones vector.*

In Table 1, we give examples for the parameters, the weight distributions, and the number of inequivalent codes in the families $\Phi_{k^\pm}$ and $\Phi'_{k^\pm}$.

Table 1. Families $\Phi_{k^\pm}$ and $\Phi'_{k^\pm}$.

		$s = 2$	$s = 3$	$s = 4$
Φ_{k^-}	$[n, k, \{w_1, w_2\}]$	[6,4,{2,4}]	[28,6,{12,16}]	[120,8,{56,64}]
	$W(y)$	$1 + 6y^2 + 9y^4$	$1 + 28y^{12} + 35y^{16}$	$1 + 120y^{56} + 135y^{64}$
	$\sharp$	1	7	
Φ_{k^+}	$[n, k, \{w_1, w_2\}]$	[10,4,{4,6}]	[36,6,{16,20}]	[136,8,{64,72}]
	$W(y)$	$1 + 5y^4 + 10y^6$	$1 + 27y^{16} + 36y^{20}$	$1 + 119y^{64} + 136y^{72}$
	$\sharp$	1	5	≥91,337
Φ'_{k^-}	$[n, k, \{w_1, w_2\}]$	[5,4, {2,4}]	[27,6, {12,16}]	[119,8, {56,64}]
	$W(y)$	$1 + 10y^2 + 5y^4$	$1 + 36y^{12} + 27y^{16}$	$1 + 136y^{56} + 119y^{64}$
	$\sharp$	1	5	≥91,337
Φ'_{k^+}	$[n, k, \{w_1, w_2\}]$	[9,4,{4,6}]	[35,6,{16,20}]	[135,8,{64,72}]
	$W(y)$	$1 + 9y^4 + 6y^6$	$1 + 35y^{16} + 28y^{20}$	$1 + 135y^{64} + 120y^{72}$
	$\sharp$	1	7	

Proposition 2. *From a code in one of the families* Φ_{k^-}, Φ_{k^+}, Φ'_{k^-}, Φ'_{k^+}, *one can construct codes from the other three families.*

Proof. Let A be a code from the family Φ_{k^-}. Then, its projective complementary code $\overline{A}$ has length $2^k - 1 - T_{k^-} = T_{k^+} - 1$ and weights $2^{k-1} - t_k = 2^{k-1} - 2^{k-2} = t_k$ and $2^{k-1} - t_{k^-} = t_{k^+}$. Hence, $\overline{A} \in \Phi'_{k^+}$.

Now, take the code $\widehat{A} = A \cup (\mathbf{1} + A)$. It has T_{k^-} subcodes of dimension k and effective length $T_{k^-} - 1$, and none of them contains the all-ones vector. Since $t_k + t_{k^-} = T_{k^-}$, the nonzero weights in $\widehat{A}$ are t_k, t_{k^-} and T_{k^-}; therefore, t_k and t_{k^-} are the nonzero weights in the subcodes of effective length $T_{k^-} - 1$. It follows that these subcodes belong to the family Φ'_{k^-}. If $\widehat{A}'$ is one of these subcodes, its projective complementary code belongs to the family Φ_{k^+}.

Similarly, taking a code from Φ_{k^+}, we obtain codes in the other three families.

If B is a code from the family Φ'_{k^+}, then its projective complementary code belongs to the family Φ_{k^-}. Therefore, from B we can construct codes from the other three families. The same follows for the codes in Φ'_{k^-}. □

Proposition 3. *From a code from the family* Ψ_k, *one can construct a code in* Ψ'_k *and vice versa.*

Proof. Take a code $C \in \Psi_k$. Then, the subcodes of $\widehat{C} = C \cup (\mathbf{1} + C)$ with effective length $2^k - 1$ and dimension $k + 1$ belong to the family Ψ'_k.

If $C' \in \Psi'_k$, we consider the $[2^k, k + 2]$ code $\widehat{C'} = ((0, C') \cup (1, \mathbf{1} + C'))$. $\widehat{C'}$ is a self-complementary code with nonzero weights T_{k^-}, 2^{k-1}, T_{k^+} and 2^k. The number of its subcodes with effective length 2^k and dimension $k + 1$ that do not contain the all-ones vector is $2^{k+1} - 2^k = 2^k$ (see the first paragraph in Section 3.2). All these subcodes belong to the family Ψ_k. □

Next, we prove that, from any code in one of the four families with two-weight codes, one can construct a code from Ψ'_k.

Proposition 4. *Let $A \in \Phi_{k^{\pm}}$ be a code with a generator matrix G_A, and $G_{\overline{A}}$ be a generator matrix of its projective complementary code. Then, the matrix*

$$\hat{G}_A = \begin{pmatrix} 1\dots 1 & 0\dots 0 \\ G_A & G_{\overline{A}} \end{pmatrix}$$

generates a code in the family Ψ'_k.

Proof. Obviously, the code generated by $\hat{G}_A$ has the same length as the simplex code $\mathcal{S}_k$, namely, $2^k - 1$. Since $(G_A|G_{\overline{A}})$ generates the simplex code of dimension k, the code $\langle (G_A|G_{\overline{A}})\rangle$ has only one nonzero weight, namely, 2^{k-1}. If v is a codeword in this code, then $v = (v_1, v_2)$, where $v_1 \in A$, $v_2 \in \overline{A}$. If $\mathrm{wt}(v_1) = t_k$ then $\mathrm{wt}(v_2) = 2^{k-1} - t_k = t_k$ and so $\mathrm{wt}(\mathbf{1} + v_1, v_2) = T_{k^{\pm}} - t_k + t_k = T_{k^{\pm}}$. If $\mathrm{wt}(v_1) = t_{k^{\pm}}$ then $\mathrm{wt}(v_2) = 2^{k-1} - t_{k^{\pm}} = t_{k^{\mp}}$ and so $\mathrm{wt}(\mathbf{1} + v_1, v_2) = T_{k^{\pm}} - t_{k^{\pm}} + t_{k^{\mp}} = t_k + t_{k^{\mp}} = T_{k^{\mp}}$. Hence, the code $\hat{A}$ has nonzero weights $2^{k-1} = T_k$, T_{k^+} and T_{k^-}. It follows that $\hat{A} \in \Psi'_k$. □

Proposition 5. *Let $A' \in \Phi'_{k^{\pm}}$ with generator matrix $G_{A'}$ and $G_{\overline{A'}}$ is the generator matrix of the projective complementary code. Then, the matrix*

$$\hat{G}_A = \begin{pmatrix} 1 & 1\dots 1 & 0\dots 0 \\ 0 & & \\ \vdots & G_{A'} & G_{\overline{A'}} \\ 0 & & \end{pmatrix}$$

is a generator matrix of a code in Ψ_k.

Proof. The proof is similar to the proof of the previous proposition. The length of the code generated by $\hat{G}_A$ is 2^k. This code is denoted by $\hat{A}$. Since $(G_{A'}|G_{\overline{A'}})$ generates the simplex code of dimension k, it has only one nonzero weight, namely, 2^{k-1}. If $v \in \langle (G_{A'}|G_{\overline{A'}})\rangle$, then $v = (v_1, v_2)$, where $v_1 \in A'$, $v_2 \in \overline{A'}$. If $\mathrm{wt}(v_1) = t_k$ then $\mathrm{wt}(v_2) = 2^{k-1} - t_k = t_k$ and so $\mathrm{wt}(1, \mathbf{1} + v_1, v_2) = 1 + T_{k^{\pm}} - 1 - t_k + t_k = T_{k^{\pm}}$. If $\mathrm{wt}(v_1) = t_{k^{\pm}}$ then $\mathrm{wt}(v_2) = 2^{k-1} - t_{k^{\pm}} = t_{k^{\mp}}$ and so $\mathrm{wt}(1, \mathbf{1} + v_1, v_2) = 1 + T_{k^{\pm}} - 1 - t_{k^{\pm}} + t_{k^{\mp}} = t_k + t_{k^{\mp}} = T_{k^{\mp}}$. Hence, the code $\hat{A}$ has nonzero weights $2^{k-1} = T_k$, T_{k^+} and T_{k^-}. It follows that $\hat{A} \in \Psi_k$. □

Remark 2. *We denote by Ψ_{S_k} (or Ψ'_{S_k}) the set of codes from the family Ψ_k (or Ψ'_k) that contain the simplex code of dimension k as a subcode. Codes from these sets can be obtained with the constructions described in Propositions 4 and 5. For example, the matrix*

$$\left(\begin{array}{cc} 111111 & 000000000 \\ \hline 000110 & 000111111 \\ 001001 & 011001111 \\ 010010 & 111010011 \\ 100001 & 101110101 \end{array}\right)$$

generates a code in the set Ψ'_{S_4}. Table 2 presents examples for the parameters, the weight distributions, and the number of inequivalent codes in the families Ψ_{S_k} and Ψ'_{S_k}.

Table 2. Families Ψ_{S_k} and Ψ'_{S_k}.

		$\boldsymbol{\Psi_k}$	$\boldsymbol{\Psi'_k}$
$s=2$	$[n,k,\{w_1,w_2,w_3\}]$	[16,5,{6,8,10}]	[15,5,{6,8,10}]
	$W(y)$	$1+6y^6+15y^8+10y^{10}$	$1+10y^6+15y^8+6y^{10}$
	$\sharp$	1	1
$s=3$	$[n,k,\{w_1,w_2,w_3\}]$	[64,7,{28,32,36}]	[63,7,{28,32,36}]
	$W(y)$	$1+28y^{28}+63y^{32}+36y^{36}$	$1+36y^{28}+63y^{32}+28y^{36}$
	$\sharp$	4	4
$s=4$	$[n,k,\{w_1,w_2,w_3\}]$	[256,9,{120,128,136}]	[255,9,{120,128,136}]
	$W(y)$	$1+120y^{120}+255y^{128}+136y^{136}$	$1+136y^{120}+255y^{128}+120y^{136}$

In this section, we prove some properties for the residual codes of the codes from the considered families.

Lemma 1. *If C is a binary code divisible by 2^m for an integer $m \geq 2$, then its residual code $Res(C,c)$ with respect to a codeword c is divisible by 2^{m-1}.*

Proof. Without loss of generality, we can consider c in the form $c = (\underbrace{11\ldots1}_{w}\underbrace{00\ldots0}_{n-w})$ where $w = \mathrm{wt}(c)$ and n is the length of the code. Then, for any codeword $v_2 \in Res(C,c)$, there is a vector $v_1 \in \mathbb{F}_2^w$, such that $v = (v_1, v_2) \in C$. Then, $c + v = (\mathbf{1} + v_1, v_2) \in C$ is a codeword of weight $w - \mathrm{wt}(v_1) + \mathrm{wt}(v_2)$. Since 2^m divides w, $\mathrm{wt}(v_1) + \mathrm{wt}(v_2)$ and $w - \mathrm{wt}(v_1) + \mathrm{wt}(v_2)$, then 2^{m-1} divides $\mathrm{wt}(v_2)$. Hence, $Res(C,c)$ is divisible by 2^{m-1}. $\square$

Theorem 2. *If C is a code in from the family $\Phi_{k^\pm}$ (or $\Phi'_{k^\pm}$), then its residual code with respect to a codeword of weight $t_{k^\pm}$ belongs to the family Ψ_k (resp. Ψ'_k).*

Proof. If $C \in \Phi_{k^-}$ then $d = t_{k^-}$; therefore, the dimension of the considered residual code is $k-1$. In the other case, namely, $C \in \Phi_{k^+}$, $t_{k^+} = 2^{k-2} + 2^{s-1} < 2.2^{k-2} = 2d$; therefore, via Theorem 1, the dimension of the residual code with respect to a codeword of weight t_{k^+} is also $k-1$.

Let now consider the weights in these codes. Suppose that $c \in C$, $\mathrm{wt}(c) = t_{k^\pm}$ and $c = (\underbrace{11\ldots1}_{t_{k^\pm}}\underbrace{00\ldots0}_{t_k})$. Take residual code $Res(C,c)$. If $v = (v_1, v_2) \in \Phi_{k^\pm}$, $v_1 \in \mathbb{F}_2^{t_{k^\pm}}$, $v_2 \in \mathbb{F}_2^{t_k}$, then $v_2 \in Res(C,c)$ and

$$\begin{aligned} \mathrm{wt}(v) = \mathrm{wt}(v_1) + \mathrm{wt}(v_2) &= t_k \text{ or } t_{k^\pm} \\ \mathrm{wt}(c+v) = \mathrm{wt}(\mathbf{1}+v_1) + \mathrm{wt}(v_2) = t_{k^\pm} - \mathrm{wt}(v_1) + \mathrm{wt}(v_2) &= t_k \text{ or } t_{k^\pm} \end{aligned}$$

It follows that

$$\begin{aligned} \mathrm{wt}(v_1) + \mathrm{wt}(v_2) &= t_k \text{ or } t_{k^\pm} \\ -\mathrm{wt}(v_1) + \mathrm{wt}(v_2) &= 0 \text{ or } \mp 2^{s-1}; \end{aligned}$$

therefore, $\mathrm{wt}(v_2) = t_{(k-2)}$, $t_{(k-2)^-}$ or $t_{(k-2)^+}$. Hence, residual code $Res(C,c)$ has length $T_{k^\pm} - T_{k^\pm} = 2^{k-2}$, dimension $k-1$ and weights $t_{(k-2)}$, $t_{(k-2)^-}$ and $t_{(k-2)^+}$. This proves that $Res(C,c) \in \Psi_k$. $\square$

Remark 3. *We can prove the above theorem by using that the weights of the codes in the families $\Phi_{k^\pm}$ and $\Phi'_{k^\pm}$ are divisible by 2^{s-1}; therefore, through Lemma 1, their residual codes are divisible* *by 2^{s-2}. Moreover, the minimal and maximal nonzero weights in the considered residual codes are $t_{(k-2)^-}$ and $t_{(k-2)^+}$, so all possible nonzero weights are the same as in the codes from the family Ψ_k.*

Theorem 3. *If C is a code in from the family Ψ_k (resp. Ψ'_k), then its residual code with respect to a codeword c has at most five nonzero weights. If $C \in \Psi_{S_k}$ (resp. $C \in \Psi'_{S_k}$) and $v \in C$ has weight $T_{k^\pm}$, then $Res(C,v) \in \Phi_{k^\mp}$ (resp. $\Phi'_{k^\mp}$).*

Proof. Since the codes in the families Ψ_k and Ψ'_k are divisible by 2^{s-1}, through Lemma 1, their residual codes are divisible by 2^{s-2}. Moreover, the minimal weight of $Res(C,c)$ is at least $T_{k^-}/2 = 2^{k-2} - 2^{s-2}$, and the maximal weight is at most $2^{k-2} + 3.2^{s-2}$. There are exactly five integers in the interval $[2^{k-2} - 2^{s-2}, 2^{k-2} + 3.2^{s-2}]$ that are divisible by 2^{s-2}.

If $C \in \Psi'_{S_k}$, then $C = \mathcal{S}_k \cup (v + \mathcal{S}_k)$, and any codeword in the coset $v + \mathcal{S}_k$ has weight T_{k^+} or T_{k^-}. Moreover, all codewords of weight $T_{k^\pm}$ belong to the coset $v + \mathcal{S}_k$. If $y \in Res(C,v)$, then $(x,y) \in \mathcal{S}_k$ and $v + (x,y) \in v + \mathcal{S}_k$ for a suitable vector x (as in the proof of Lemma 1, we can consider v in the form $v = (\underbrace{11\ldots1}_{T_{k^\pm}}\underbrace{00\ldots0}_{T_{k^\mp}})$). Hence

$$\begin{aligned} \mathrm{wt}(x) + \mathrm{wt}(y) &= 2^{k-1} \\ -\mathrm{wt}(x) + \mathrm{wt}(y) &= T_{k^-} - T_{k^\mp} \text{ or } T_{k^+} - T_{k^\mp} \end{aligned}$$

It follows that $\mathrm{wt}(y) = t_{k^-}$ or t_k if $\mathrm{wt}(v) = T_{k^+}$, and $\mathrm{wt}(y) = t_k$ or t_{k^+} if $\mathrm{wt}(v) = T_{k^-}$. This proves that the residual codes of $C \in \Psi'_{S_k}$ with respect to a codeword of weight T_{k^-} belonging to the family Φ'_{k^+}, and the residual codes with respect to a codeword of weight T_{k^+} belong to Φ'_{k^-}. □

5. Construction of Codes from Φ_k Using Codes from Φ_{k-2}

We use two constructions to generate a linear code form Φ_k by using a code from Φ_{k-2}. The first construction is presented in this section. It can give us recursively infinite families of codes. We name this construction *self-complementary lifting (SCL)*. The second construction is computational and uses residual codes. We present the results in the next section.

Proposition 6. *Let $A \in \Phi_{k^\pm}$ be a code with a generator matrix G_A and B be the code generated by the matrix*

$$G_B = \begin{pmatrix} G_A & G_A & G_A & \overline{G_A} \\ 1\ldots1 & 1\ldots1 & 0\ldots0 & 0\ldots0 \\ 0\ldots0 & 1\ldots1 & 1\ldots1 & 0\ldots0 \end{pmatrix}.$$

Then $B \in \Phi'_{(k+2)^\pm}$.

Proof. Obviously, B is a projective code of length $2T_{k^\pm} + 2^k - 1 = 2(2^{k-1} \pm 2^{s-1}) + 2^k - 1 = 2^{k+1} \pm 2^s - 1 = T_{(k+2)^\pm} - 1$. Let us calculate the weights of the codewords in B. If $v \in B$, then $v = (w,w,w,\overline{w})$, $(\mathbf{1}+w,\mathbf{1}+w,w,\overline{w})$, $(w,\mathbf{1}+w,\mathbf{1}+w,\overline{w})$, or $(\mathbf{1}+w,w,\mathbf{1}+w,\overline{w})$, where $w \in A$ and $(w,\overline{w}) \in \mathcal{S}_k$. Since

$$\mathrm{wt}(\mathbf{1}+w,\mathbf{1}+w,w,\overline{w}) = \mathrm{wt}(w,\mathbf{1}+w,\mathbf{1}+w,\overline{w}) = \mathrm{wt}(\mathbf{1}+w,w,\mathbf{1}+w,\overline{w}),$$

we calculate only the weights of $(w,w,w,\overline{w})$ and $(\mathbf{1}+w,\mathbf{1}+w,w,\overline{w})$. For the first vector, we have $\mathrm{wt}(w,w,w,\overline{w}) = 2\mathrm{wt}(w) + 2^{k-1}$. Hence

$$\mathrm{wt}(w,w,w,\overline{w}) = \begin{cases} 2t_k + 2^{k-1} = 2^k = t_{k+2}, & \text{if } \mathrm{wt}(w) = t_k \\ 2t_{k^\pm} + 2^{k-1} = 2^k \pm 2^s = t_{(k+2)^\pm}, & \text{if } \mathrm{wt}(w) = t_{k^\pm} \end{cases}$$

For the second vector, we have

$$\begin{aligned}\mathrm{wt}(\mathbf{1}+w,\mathbf{1}+w,w,\overline{w}) &= 2(T_{k^\pm}-\mathrm{wt}(w))+2^{k-1} = 2^k \pm 2^s + 2^{k-1} - 2\mathrm{wt}(w)\\ &= 3.2^{k-1} \pm 2^s - 2\mathrm{wt}(w).\end{aligned}$$

Hence

$$\mathrm{wt}(\mathbf{1}+w,\mathbf{1}+w,w,\overline{w}) = \begin{cases} 3.2^{k-1} \pm 2^s - 2t_k = 2^k \pm 2^s = t_{(k+2)^\pm}, & \text{if } \mathrm{wt}(w) = t_k \\ 3.2^{k-1} \pm 2^s - 2t_{k^\pm} = 2^k \pm 2^s = t_{(k+2)^\pm}, & \text{if } \mathrm{wt}(w) = t_{k^\pm} \end{cases}$$

Moreover,

$$\mathrm{wt}(\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0}) = \mathrm{wt}(\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{0}) = \mathrm{wt}(\mathbf{0},\mathbf{1},\mathbf{1},\mathbf{0}) = 2^k = t_{k+2}.$$

It follows that the nonzero weights in B are t_{k+2} and $t_{(k+2)^\pm}$, and therefore $B \in \Phi'_{(k+2)^\pm}$. □

Proposition 7. *Let $A \in \Phi'_{k^\pm}$ has generator matrix G_A, and B has generator matrix*

$$\begin{pmatrix} G_A\,0 & G_A\,0 & G_A\,0 & \overline{G_A} \\ 1\ldots1 & 1\ldots1 & 0\ldots0 & 0\ldots0 \\ 0\ldots0 & 1\ldots1 & 1\ldots1 & 0\ldots0 \end{pmatrix}.$$

Then, $B \in \Phi_{(k+2)^\pm}$.

Proof. Here, we can apply the same logic as that in the proof of Proposition 6. Codes in $\Phi_{(k+2)^\pm}$ have parameters $[T_{(k+2)^\pm}, k+2, \{2^{2s}, 2^{2s} \pm 2^s\}]$. Since $A \in \Phi'_{k^\pm}$, we need to extend the matrix G_A with a zero column to obtain the needed length. This does not change the weights; since we add vectors $(\mathbf{1},\mathbf{1},\mathbf{0},\mathbf{0})$ and $(\mathbf{0},\mathbf{1},\mathbf{1},\mathbf{0})$ to the generator matrix, B is a projective code. All other calculations are analogous to these in the proof of Proposition 6. Thus, for $A \in \Phi'_{k^+}$, the resulting code B is in $\Phi_{(k+2)^+}$ and for $A \in \Phi'_{k^-}$, the code B is in $\Phi_{(k+2)^-}$. □

The next theorem follows directly from Propositions 2, 6, and 7.

Theorem 4. *From any code $C \in \Phi_k$, codes in Φ_{k+2l} for all integers $l \geq 1$ can be constructed.*

6. Computational Results

The classification problem is defined as constructing a generator matrix of exactly one representative of any equivalence class. This problem has two subproblems. One is constructive to build a generator matrix that generates a code with desired parameters and properties. The other concerns the construction of inequivalent representatives only. In the classification, full or partial, the restrictions that follow from the properties of the studied codes and their use at each step in the construction process are of particular importance. For example, the characteristics of the codes from the considered type are: (1) they have two or three fixed weights divisible by 2^m for a given integer $m \geq 1$, and (2) they have a dual distance at least 3. When we study linear codes with a significantly greater length than the dimension, it is very appropriate to use already classified residual codes. In most of the cases, we use the residual codes with respect to a codeword with minimum weight. This is because, in that case, the known part of the matrices that we want to construct is the largest for the parameters.

Our efforts are focused on the classification of projective two-weight codes with parameters $[119, 8, \{56, 64\}]$. These codes belong to the family Φ'_{8^-}. Their residual codes with respect to a codeword of weight 56 have parameters $[63, 7, \{28, 32, 36\}]$ and belong to Ψ'_6. Taking one more step with residual codes, we obtain codes with parameters $[35, 6, \{14, 16, 18, 20, 22\}]$. Using the algorithm of [21], we found that there are 267,370 inequivalent projective codes with such parameters. Only seven of them are codes in the family Φ'_{6+} with parameters $[35, 6, \{16, 20\}]$. Using these seven codes as residuals, we obtained the total number of $3,220,339$ inequivalent $[63, 7, \{28, 32, 36\}]$ codes. Therefore, we restricted our computations to the case when the residual codes were from the family Ψ'_{S_6} (containing the simplex code as a subcode). This family consists of four codes that we extended. The results are presented in Table 3. In the second column, we give generator matrices for these four $[63, 7, \{28, 32, 36\}]$ codes. In the third column, we present the number of inequivalent $[119, 8, \{56, 64\}]$ codes constructed from the matrix in the same row. The total number of the constructed $[119, 8, \{56, 64\}]$ codes was $91,397$, and $91,337$ of them were inequivalent. Adding a zero column and then the all-ones vector as a row to the generator matrices of all these codes, we obtained exactly 2946 inequivalent self-complementary $[120, 9, \{56, 64, 120\}]$ codes. These 2946 codes contain, as subcodes of dimension 8, exactly $175,213$ codes with effective length 120, and $156,763$ codes with effective length 119. Hence, the set of all inequivalent $331,976$ subcodes with dimension 8 was partitioned into 2946 classes of SC-equivalence.

Table 3. The [119,8,{56,64}] codes constructed using residuals.

No	[63,7,{28,32,36}] *	[119,8,{56,64}]
1	$\begin{pmatrix} 111111111111111111111111111100000000000000000000000000000000000 \\ 000000000000000111111111111000000000001111111111111111111100000 \\ 000000000111111000000111111001111111110000000001111111111010000 \\ 000001111100000110000011001111010000001111100000011111111001000 \\ 000110001011100001101010011000001101110011100110011000111000100 \\ 011000010101100010110100101000110010110101101010101001011000010 \\ 101010100000101110000111010011010100011000100110101011101000001 \end{pmatrix}$	63,836
2	$\begin{pmatrix} 111111111111111111111111111100000000000000000000000000000000000 \\ 000000000000000111111111111000000000001111111111111111111100000 \\ 000000000111111000000111111001111111110000000001111111111010000 \\ 000001111100000110000011001111010000001111100000011111111001000 \\ 000110001011100001101010011000001101110011100110011000111000100 \\ 011000010101100010110100101000110010110101101010101001011000010 \\ 101010100000101010010011101011010100011010100101011011001000001 \end{pmatrix}$	4406
3	$\begin{pmatrix} 111111111111111111111111111100000000000000000000000000000000000 \\ 000000000000000111111111111000000000001111111111111111111100000 \\ 000000000111111000000111111001111111110000000001111111111010000 \\ 000001111100000110000011001111010000001111100000011111111001000 \\ 001110001000101000101010111010011100010011100010111000111000100 \\ 010010010011100011010100011000100101110101100111001001011000010 \\ 100100100101100101010100101001001010111010101011010010101000001 \end{pmatrix}$	23,075
4	$\begin{pmatrix} 111111111111111111111111111100000000000000000000000000000000000 \\ 000000000000000111111111111000000000001111111111111111111100000 \\ 000000000111111000000111111001111111110000000001111111111010000 \\ 000001111100000110000011001111010000001111100000011111111001000 \\ 001110001000101000101010111010011100010011100010111000111000100 \\ 110010010001001001001011011011100100101100100101011011001000010 \\ 010110111010001010001100001010101101110101101111101101010000001 \end{pmatrix}$	80
	Total:	91,397
	Inequivalent codes:	91,337

* These codes were constructed from the four [27,6,{12,16}] codes that were in different SC-equivalence classes.

In Table 4, we present summarized results for the $[28, 7, \{12, 16, 28\}]$ self-complementary codes. There were exactly four inequivalent codes with these parameters [16,17]. Therefore, their subcodes of dimension 6 were partitioned into four classes of SC-equivalence. The first column in the table shows a generator matrix of the self-complementary code. The second and third columns present the number of inequivalent $[27, 6, \{12, 16\}]$ and $[28, 6, \{12, 16\}]$ codes that are subcodes of the corresponding self-complementary code.

Table 4. Self-complementary $[28, 7, \{12, 16, 28\}]$ codes.

G	$[27, 6, \{12, 16\}]$	$[28, 6, \{12, 16\}]$	Total:
$\begin{pmatrix} 1111111111111111111111111111 \\ 0000000111111111111111100000 \\ 0011111000000000111111010000 \\ 0100011000001111001111001000 \\ 1001100000110011110011000100 \\ 0110001011010001011101000010 \\ 1010111101000101000101000001 \end{pmatrix}$	1	1	2
$\begin{pmatrix} 1111111111111111111111111111 \\ 0000000111111111111111100000 \\ 1111111000000011111111010000 \\ 0000111000011100011111001000 \\ 0001011001100101100111000100 \\ 0110001010001110111001000010 \\ 1010001100100111101010000001 \end{pmatrix}$	1	2	3
$\begin{pmatrix} 1111111111111111111111111111 \\ 0000000111111111111111100000 \\ 1111111000000011111111010000 \\ 0000111000011100011111001000 \\ 0001011001100101100111000100 \\ 0110001010001110111001000010 \\ 1010001110100001101011000001 \end{pmatrix}$	2	2	4
$\begin{pmatrix} 1111111111111111111111111111 \\ 0000000111111111111111100000 \\ 1111111000000011111111010000 \\ 0000111000011100011111001000 \\ 0011001001100111100011000100 \\ 0100011010101001100111000010 \\ 1001100100011010101101000001 \end{pmatrix}$	1	2	3

7. Conclusions

In this paper, we defined and studied four families of two-weight codes and two families of three-weight codes that were closely connected to the self-complementary codes of even length meeting the Grey–Rankin bound. Furthermore, we presented the relations between these six special families of binary linear codes and showed how one could construct from a two-weight code in one of the families codes of the same dimension in the other three families of two-weight codes. Similarly, we showed how to construct a three-weight code in one of the families of three-weight codes from a code from the other family with the same dimension. We also presented a construction of a two-weight code of dimension $k + 2$ in one of the families from a two-weight code of dimension k. Lastly, we classified codes on the basis of the theoretical results and some other well-known properties. Our efforts were focused on the classification of projective two-weight codes with parameters $[119, 8, \{56, 64\}]$. Since the number of inequivalent linear codes that meet the Grey–Rankin bound grows exponentially [15], we presented a partial classification for the codes with these parameters; more precisely, we gave the classification of the projective two-weight $[119, 8, \{56, 64\}]$ codes constructed from the seven $[35, 6, \{16, 20\}]$ codes.

In conclusion, we mention some open problems related to our research:

- Is it possible to obtain equivalent codes from inequivalent ones by self-complementary lifting, and under which conditions? We applied this construction to codes with parameters $[119, 8, \{56, 64\}]$ to obtain codes with parameters $[496, 10, \{240, 256\}]$, and then to codes with parameters $[2015, 12, \{992, 1024\}]$. In these cases, inequivalent codes led to inequivalent ones.
- Is reverse construction possible?
- What is the relationship between the automorphism groups of codes and their extensions?
- How does this construction reflect on other related combinatorial objects?

Author Contributions: Conceptualization, I.B.; methodology, I.B., S.B. and M.P.-G.; software, I.B.; formal analysis, S.B.; resources, I.B. and M.P.-G.; data curation, I.B. and M.P.-G.; writing—original draft preparation, I.B., S.B. and M.P.-G.; writing—review and editing, I.B., S.B. and M.P.-G.;project administration, I.B.; funding acquisition, I.B., S.B. and M.P.-G. All authors have read and agreed to the published version of the manuscript.

Funding: S. Bouyuklieva was supported by the Bulgarian National Science Fund, grant number KP-06-N32/2-2019; I, Bouyukliev was supported by the Bulgarian National Science Fund, grant number KP-06-Russia/33/17.12.2020; M. Pashinska-Gadzeva was supported by the Bulgarian Ministry of Education and Science, grant no. DO1-168/28.07.2022 for NCHDC, a part of the Bulgarian National Roadmap on RIs.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Generator matrices of the constructed codes are sent upon request by the authors.

Acknowledgments: We are greatly indebted to the anonymous referees for their useful suggestions.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. McGuire, G. Quasi-symmetric designs and codes meeting the Grey-Rankin bound. *J. Combin. Theory Ser. A* **1997**, *78*, 280–291. [CrossRef]
2. Grey, L.D. Some bounds for error-correcting codes. *IEEE Trans. Inform. Theory* **1962**, *8*, 200–202. [CrossRef]
3. Rankin, R.A. The closest packing of spherical caps in n-dimensions. *Proc. Glasg. Math. Assoc.* **1995**, *2*, 139–144. [CrossRef]
4. Rankin, R.A. On the minimal points of positive definite quadratic forms. *Mathematika* **1956**, *3*, 15–24. [CrossRef]
5. Bouyukliev, I.; Fack, V.; Willems, W.; Winne, J. Projective two-weight codes with small parameters and their corresponding graphs. *Des. Codes Cryptogr.* **2006**, *41*, 59–78. [CrossRef]
6. Calderbank, R.; Kantor, W.M. The geometry of two-weight codes. *Bull. Lond. Math. Soc.* **1986**, *18*, 97–122. [CrossRef]
7. Delsarte, P. Weights of linear codes and strongly regular normed spaces. *Discret. Math.* **1972**, *3*, 47–64. [CrossRef]
8. Tonchev, V.D. The uniformly packed binary [27, 21, 3] and [35, 29, 3] codes. *Discret. Math.* **1996**, *149*, 283–288. [CrossRef]
9. Ward, H. Divisible codes—Survey. *Serdica Math. J.* **2001**, *27*, 263–278.
10. Bouyukliev, I.; Bouyuklieva, S.; Dodunekov, S. On binary self-complementary $[120, 9, 56]$ codes having an automorphism of order 3 and associated SDP designs. *Probl. Inf. Transm.* **2007**, *43*, 89–96. [CrossRef]
11. Carlet, C.; Mesnager, S. Four decades of research on bent functions. *Des. Codes Cryptogr.* **2016**, *78*, 5–50. [CrossRef]
12. Dillon, J.F.; Schatz, J.R. Block designs with the symmetric difference property. *Proc. NSA Math. Sci. Meet.* **1987**, 159–164.
13. Ding, C.; Munemasa, A.; Tonchev, V. Bent vectorial functions, codes and designs. *IEEE Trans. Inform. Theory* **2019**, *65*, 7533–7541. [CrossRef]
14. Gulliver, T.A.; Harada, M. Codes of lengths 120 and 136 meeting the Grey-Rankin bound and quasi-symmetric designs. *IEEE Trans. Inform. Theory* **1999**, *45*, 703–706. [CrossRef]
15. Jungnickel, D.; Tonchev, V.D. Exponential number of quasi-symmetric SDP designs and codes meeting the Grey-Rankin bound. *Des. Codes Cryptogr.* **1991**, *1*, 247–253. [CrossRef]
16. Dodunekov, S.M.; Encheva, S.B.; Kapralov, S.N. On the $[28, 7, 12]$ binary self-complementary codes and their residuals. *Des. Codes Cryptogr.* **1994**, *4*, 57–67. [CrossRef]
17. Parker, C.; Spence, E.; Tonchev, V.D. Designs with the symmetric difference property on 64 points and their groups. *J. Combin. Theory Ser. A* **1994**, *67*, 23–43. [CrossRef]

18. Huffman, W.C.; Pless, V. *Fundamentals of Error-Correcting Codes*; Cambridge University Press: Cambridge, UK, 2003.
19. Kurz, S. Divisible codes. *arXiv* **2021**, arXiv:2112.11763. https://doi.org/10.48550/arXiv.2112.11763
20. Pless, V.S. Power moment identities on weight distributions in error correcting codes. *Inform. Control* **1963**, *6*, 147–152. [CrossRef]
21. Bouyukliev, I. The program GENERATION in the software package QEXTNEWEDITION. In *Mathematical Software—ICMS 2020. Lecture Notes in Computer Science*; Bigatti, A., Carette, J., Davenport, J., Joswig, M., de Wolff, T., Eds.; Springer: Cham, Switzerland, 2020; Volume 12097, pp. 181–189.