mathematics

Article

Taylor Series for the Mittag–Leffler Functions and Their Multi-Index Analogues

Jordanka Paneva-Konovska

Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria; jpk@math.bas.bg

Abstract: It has been obtained that the n-th derivative of the 2-parametric Mittag–Leffler function is a 3-parametric Mittag–Leffler function, with exactness to a constant. Following the analogy, the author later obtained the n-th derivative of the $2m$-parametric multi-index Mittag–Leffler function. It turns out that this is expressed via the $3m$-parametric Mittag–Leffler function. In this paper, upper estimates of the remainder terms of these derivatives are found, depending on n. Some asymptotics are also found for "large" values of the parameters. Further, the Taylor series of the 2 and $2m$-parametric Mittag–Leffler functions around a given point are obtained. Their coefficients are expressed through the values of the corresponding n-th order derivatives at this point. The convergence of the series to the represented Mittag–Leffler functions is justified. Finally, the Bessel-type functions are discussed as special cases of the multi-index ($2m$-parametric) Mittag–Leffler functions. Their Taylor series are derived from the general case as corollaries, as well.

Keywords: Mittag–Leffler functions; multi-index Mittag–Leffler functions; integer order derivatives; estimates; asymptotics; Taylor series

MSC: 33E12; 33C10; 30E15; 30A10; 41A58

check for updates

Citation: Paneva-Konovska, J. Taylor Series for the Mittag–Leffler Functions and Their Multi-Index Analogues. *Mathematics* **2022**, *10*, 4305. https://doi.org/10.3390/math10224305

Academic Editor: Snezhana Hristova

Received: 22 September 2022
Accepted: 14 November 2022
Published: 17 November 2022

1. Introduction

The Mittag–Leffler function is given by the following series:

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\qquad z\in\mathbb{C};\ \alpha,\beta\in\mathbb{C},\ Re(\alpha)>0. \tag{1}$$

Arising in the beginning of the 20th century (initially for $\beta=1$), the function remained unremarked upon and unused for a long time (for almost 50 years). Recently, interest in this function and its generalizations has risen due to their important role in fractional calculus and related fractional order integral and differential equations (as their solutions) and applications [1]; for example, to model some evolution processes [2], fractional diffusion processes [3–8], nonlinear waves, etc. It is worth pointing out that the Mittag–Leffler function and its various generalizations are widely used in the field of anomalous diffusion, non-exponential relaxation, etc. For some other properties and applications, see, e.g., the recent papers [9–11].

Among the well-known generalizations of $E_{\alpha,\beta}$ is the Mittag–Leffler function $E^{\gamma}_{\alpha,\beta}$ with three parameters, namely

$$E^{\gamma}_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{(\gamma)_k}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!},\qquad z\in\mathbb{C};\ \alpha,\beta,\gamma\in\mathbb{C},\ Re(\alpha)>0, \tag{2}$$

where $(\gamma)_k$ is the Pochhammer symbol ([12] [Section 2.1.1])

$$(\gamma)_k=\gamma(\gamma+1)\dots(\gamma+k-1),\ k\in\mathbb{N};\quad (\gamma)_0=1.$$

This function was introduced by Prabhakar [13] and is also known as the Prabhakar function.

Other generalizations are the multi-index Mittag–Leffler functions with $2m$ and $3m$ parameters. First of them is this one with $2m$ parameters:

$$E_{(\alpha_i),(\beta_i)}(z) = E^{m}_{(\alpha_i),(\beta_i)}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha_1 k + \beta_1)\dots\Gamma(\alpha_m k + \beta_m)}. \tag{3}$$

The function $E_{(\alpha_i),(\beta_i)}$ was introduced by Luchko and Yakubovich [14] and Kiryakova [15] and studied in detail by Kiryakova [15–17]. Initially defined for $\alpha_i > 0$ and β_i arbitrary real (complex) numbers (see e.g., [15]), its definition was later extended to complex parameters with $Re(\alpha_i) > 0$ [18,19]. For the applications in the solutions of fractional order equations and models, see Kiryakova and Luchko [20]. In their survey [21], Kilbas, Koroleva, and Rogosin describe the historical development of the theory of these multi-index functions as a subclass of the Wright-generalized hypergeometric functions ${}_p\Psi_q(z)$. The method of Mellin–Barnes-type integral representations allowed them to extend the function (3) and to study it in the general case of parameters. The other multi-index Mittag–Leffler function (with $3m$ parameters),

$$E^{(\gamma_i)}_{(\alpha_i),(\beta_i)}(z) = E^{(\gamma_i),m}_{(\alpha_i),(\beta_i)}(z) = \sum_{k=0}^{\infty} \frac{(\gamma_1)_k \dots (\gamma_m)_k}{\Gamma(\alpha_1 k + \beta_1)\dots\Gamma(\alpha_m k + \beta_m)} \frac{z^k}{(k!)^m}, \tag{4}$$

$$z \in \mathbb{C};\ \alpha_i,\ \beta_i,\ \gamma_i \in \mathbb{C},\ Re(\alpha_i) > 0,$$

was introduced by the author in [22,23], generalizing both the Prabhakar functions (2) and (3) and has been studied in a series of works (see, e.g., the recent papers by Paneva-Konovska [24], Paneva-Konovska, and Kiryakova [25], Ali et al. [26] and Kiryakova [27]).

In connection with the following considerations, we recall the well-known denotation for the operator for n-tuple differentiation

$$D^n = \frac{d^n}{dz^n} = \left(\frac{d}{dz}\right)^n, \quad n \in \mathbb{N}_0;$$

it is understood conventionally that $D^n f(z) = f(z)$ when $n = 0$, i.e., $D^0 f(z) = f(z)$.

It has been obtained that the n-th derivative of the 2-parametric Mittag–Leffler function is a 3-parametric Mittag–Leffler function (with exactness to a constant), namely [28]

$$D^n\left[E_{\alpha,\beta}(z)\right] = E^{(n)}_{\alpha,\beta}(z) = n!E^{n+1}_{\alpha,\beta+n\alpha}(z). \tag{5}$$

Exactly the same form, for the n-th derivative of the two-parameter Mittag–Leffler function, can be found in the book [29] [Equation 1.8.22]; see also the paper [30]. The analogical-type relation is obtained between the functions (4) and the n-th derivative of (3) in Paneva-Konovska [31], namely

$$D^n\left[E_{(\alpha_i),(\beta_i)}(z)\right] = \frac{d^n}{dz^n}\left[E_{(\alpha_i),(\beta_i)}(z)\right] = \Gamma(n+1)\, E^{(\gamma_i),m}_{(\alpha_i),(\beta_i+n\alpha_i)}(z), \tag{6}$$

with $\gamma_1 = n+1$, $\gamma_2 = \dots = \gamma_m = 1$.

In this paper, we find upper estimations for the modules of remainder terms of the functions $E^{n+1}_{\alpha,\beta+n\alpha}$ and $E^{(\gamma_i),m}_{(\alpha_i),(\beta_i+n\alpha_i)}$ with the relevant conditions imposed on the parameters. We also give their asymptotics for "large" values of n (that means simultaneously for two "large" parameters). Finally, we represent the functions (1) and (3) in Taylor series around a given point in the complex plane, providing their convergences reduce.

In general, inequalities and asymptotic formulae, as well Taylor series, can be used when studying different problems, both theoretical and applied in nature, such as numerical methods, approximations, convergence, monotonicity, the modelling of physical processes, etc. That is why these types of results are very useful and widely investigated.

For example, other interesting inequalities connected to the discussed functions can be seen, e.g., in [23,26]. For inequalities referring to other classes of functions, see [32,33]. Co-ordinated convex functions are considered in Tunç, Sarıkaya, and Yaldız [32], and some Hermite–Hadamard type inequalities, expressed via Riemann–Liouville fractional integrals, are obtained. They are further used in proving integral inequalities for the left-hand side of the fractional Hermite–Hadamard type inequality on the coordinates. Some concepts of the relative strongly preinvex functions and relative strongly monotone operators with respect to the auxiliary non-negative function and bi-function are considered in Noor and Noor [33].

Currently, the Taylor series are actively studied and widely used not only from a purely theoretical point of view. Various problems connected to them are considered in different aspects. For example, describing Monte Carlo simulations with mathematical convolutions of frequency and severity distributions in the operational risk capital model in the Basel accords, Mun [34] uses the Taylor expansion series. Luchko [35] discusses the generalized Taylor series in the form of convolution series and deduces the formulae for its coefficients, involving the n-th order general sequential fractional derivatives. For the applications in the fields of the approximations and numerical methods, see, e.g., [36,37]. Sunday, Shokri, and Marian use Taylor series in their variable step hybrid block method for the approximation of the Kepler problem [36]. The connection between closed Newton–Cotes formulae, trigonometrically-fitted methods, symplectic integrators, and the efficient integration of the Schrödinger equation is studied in Shokri, Saadat, and Khodadadi [37]. For matrix functions and their applications to condition number estimation, see Deadman and Relton [38].

2. Inequalities and Asymptotics

In this section, we consider the 3-parametric function $E^{n+1}_{\alpha,\beta+n\alpha}$ and multi-index function $E^{(\gamma_i),m}_{(\alpha_i),(\beta_i+n\alpha_i)}$, involved in the formulae (5) and (6), with positive parameters and $n \in \mathbb{N}_0$. We find some estimates, connected to their remainder terms, beginning with the representation below.

Lemma 1. *Let $\alpha, \beta, z \in \mathbb{C}$ and let $Re(\alpha) > 0$. Then, the following equality holds true for all the values of $n \in \mathbb{N}_0$:*

$$E^{n+1}_{\alpha,\beta+n\alpha}(z) = \frac{1}{\Gamma(\alpha n+\beta)}(1+\vartheta_n(z)), \tag{7}$$

where

$$\vartheta_n(z) = \Gamma_{0,n}\left[\sum_{k=1}^{\infty} \Gamma_{k,n}\,\frac{z^k}{k!}\right], \tag{8}$$

is an entire function with

$$\Gamma_{0,n} = \frac{(n+1)\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta+\alpha)}, \quad \Gamma_{1,n} = 1, \tag{9}$$

and

$$\Gamma_{k,n} = \frac{(n+2)\dots(n+k)\Gamma(\alpha n+\beta+\alpha)}{\Gamma(\alpha n+\beta+\alpha k)}, \text{ for } k = 2,3,\dots. \tag{10}$$

Proof. The relation (7) immediately follows due to (2), applied with $\gamma = n+1$ and $\alpha n+\beta$ instead of β, along with (8)–(10). □

The next assertion refers to the coefficients $\Gamma_{k,n}$ involved in the above formulae.

Lemma 2. *Let $\alpha \geq 1$, $\beta > 0$, and $\Gamma_{k,n}$ be defined by (9) and (10). Then, the following inequalities hold true for all the values of $n \in \mathbb{N}_0$:*

$$\Gamma_{k,n} \leq k, \quad \text{for } k = 1, 2, 3, \dots, \tag{11}$$

and

$$\Gamma_{0,n} \leq \frac{n+1}{n+\beta}, \quad \text{for } \alpha = 1. \tag{12}$$

Moreover, if $\alpha \geq 1$ and n is "large enough", then

$$\Gamma_{0,n} \sim \frac{(n+1)}{(\alpha n+\beta)^{\alpha}}. \tag{13}$$

Proof. Letting $\alpha \geq 1$ and $k \geq 2$, we write down

$$\Gamma_{k,n} = \frac{(n+2)\dots(n+k)\Gamma(\alpha n+\beta+\alpha)}{\Gamma(\alpha n+\beta+\alpha k)} \leq \frac{(n+2)\dots(n+k)\Gamma(\alpha n+\beta+\alpha)}{\Gamma(\alpha n+\beta+\alpha+k-1)}.$$

This inequality leads to

$$\Gamma_{k,n} \leq \frac{(n+1)\dots(n+k-1)}{(\alpha n+\beta+\alpha)\dots(\alpha n+\beta+\alpha+k-2)}\frac{n+k}{n+1} \leq 1.k,$$

which, along with $\Gamma_{1,n} = 1$, proves (11).

Now, let $\alpha = 1$. Then,

$$\Gamma_{0,n} = \frac{(n+1)\Gamma(n+\beta)}{\Gamma(n+\beta+1)} = \frac{n+1}{n+\beta},$$

that is, (12).

In order to prove (13), we take $\alpha \geq 1$ and $k = 0$, and using (9) and the Γ-functions' quotient property [23] [Remark 6.5]

$$\frac{\Gamma(z)}{\Gamma(z+\alpha)} = \left(\frac{1}{z^{\alpha}}\right), \quad |\arg z| < \pi, \ |\arg(z+\alpha)| < \pi, \tag{14}$$

with $z = \alpha n + \beta$, we affirm the validity of the relation (13). □

We set $\alpha > 1$ in (13) and let $n \to \infty$ lead to a corollary, as follows.

Corollary 1. *Let $\alpha > 1$, $\beta > 0$ and $\Gamma_{0,n}$ be defined by (9). Then, the following relation holds true:*

$$\lim_{n\to\infty} \Gamma_{0,n} = \lim_{n\to\infty} \frac{(n+1)\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta+\alpha)} = 0. \tag{15}$$

Proof. The equality (15) immediately follows, due to formula (13). Indeed, it can be written that

$$\lim_{n\to\infty} \Gamma_{0,n} = \lim_{n\to\infty} \frac{n+1}{(\alpha n+\beta)^{\alpha}} = 0,$$

which is the desired equality. □

The results obtained referring to the coefficients $\Gamma_{k,n}$ allow estimates to be found for $|\vartheta_n|$ and an asymptotic for ϑ_n when $n \to \infty$.

Theorem 1. *Let $\alpha \geq 1$, $\beta > 0$, ϑ_n be defined by (8) and $z \in \mathbb{C}$. Let K be a nonempty compact subset of $\mathbb{C}$. Then, the following inequality holds true:*

$$|\vartheta_n(z)| \leq \frac{(n+1)\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta+\alpha)} |z| \exp |z|, \quad \text{for } z \in \mathbb{C}, \tag{16}$$

and there exists a constant $C = C(K)$ such that

$$|\vartheta_n(z)| \leq C \frac{(n+1)\Gamma(\alpha n+\beta)}{\Gamma(\alpha n+\beta+\alpha)}, \quad \text{for } z \in K. \tag{17}$$

Moreover, if $\alpha > 1$, then

$$\lim_{n\to\infty} \vartheta_n(z) = 0, \tag{18}$$

for $z \in \mathbb{C}$, and the convergence is uniform on the compact subsets of $\mathbb{C}$.

Proof. Due to (9) and (11), the module of $\vartheta_n(z)$ is estimated in the following way in the complex plane:

$$|\vartheta_n(z)| \leq \Gamma_{0,n} \left[\sum_{k=1}^{\infty} \Gamma_{k,n} \frac{|z|^k}{k!} \right] \leq \Gamma_{0,n} \left[\sum_{k=1}^{\infty} k \frac{|z|^k}{k!} \right] = \Gamma_{0,n} |z| \exp |z|,$$

which shows that the inequality (16) holds true. The inequality (17) automatically follows in the set K. The equality (18) is also fulfilled, due to (15)–(17). □

Restricting the parameter α to 1 yields the following corollary.

Corollary 2. *Let $\alpha = 1$, $\beta > 0$, ϑ_n be defined by (8) and $z \in \mathbb{C}$. Let K be a nonempty compact subset of $\mathbb{C}$. Then, the inequalities (16) and (17) are reduced to*

$$|\vartheta_n(z)| \leq \frac{n+1}{n+\beta} |z| \exp |z|, \quad \text{for } z \in \mathbb{C}, \tag{19}$$

respectively

$$|\vartheta_n(z)| \leq C \frac{n+1}{n+\beta}, \quad \text{for } z \in K, \tag{20}$$

with a constant $C = C(K)$.

Analogical statements can be formulated and proved for the multi-index Mittag–Leffler function $E^{(\gamma_i),m}_{(\alpha_i),(\beta_i+n\alpha_i)}$ with positive parameters α_i, β_i, and $n \in \mathbb{N}_0$.

Lemma 3. *Let $\alpha_i, \beta_i, z \in \mathbb{C}$ and let $Re(\alpha_i) > 0$. Then, the following equality holds true for all the values of $n \in \mathbb{N}_0$:*

$$E^{(\gamma_i),m}_{(\alpha_i),(\beta_i+n\alpha_i)}(z) = \frac{1}{\Gamma(\alpha_1 n+\beta_1)\ldots\Gamma(\alpha_m n+\beta_m)}(1+\vartheta_n(z)), \tag{21}$$

with $\gamma_1 = n+1$ and $\gamma_2 = \cdots = \gamma_m = 1$.

The function ϑ_n is an entire function, defined as follows:

$$\vartheta_n(z) = \Gamma_{0,n} \left[\sum_{k=1}^{\infty} \Gamma_{k,n} \frac{z^k}{k!} \right], \tag{22}$$

with

$$\Gamma_{0,n} = \frac{(n+1)\Gamma(\alpha_1 n+\beta_1)\ldots\Gamma(\alpha_m n+\beta_m)}{\Gamma(\alpha_1 n+\beta_1+\alpha_1)\ldots\Gamma(\alpha_m n+\beta_m+\alpha_m)}, \quad \Gamma_{1,n} = 1, \tag{23}$$

and

$$\Gamma_{k,n} = \frac{(n+2)\dots(n+k)\Gamma(\alpha_1 n+\beta_1+\alpha_1)\dots\Gamma(\alpha_m n+\beta_m+\alpha_m)}{\Gamma(\alpha_1 n+\beta_1+k\alpha_1)\dots\Gamma(\alpha_m n+\beta_m+k\alpha_m)}, \text{ for } k = 2,3,\dots. \tag{24}$$

Proof. The relation (21) immediately follows due to (4), applied with $\gamma_1 = n+1$, $\gamma_2 = \dots = \gamma_m = 1$ and $\alpha_i n + \beta_i$ instead of β_i $(i = 1, = \dots, m)$, along with (22)–(24). □

The next assertion refers to the coefficients $\Gamma_{k,n}$ involved in the above formulae.

Lemma 4. *Let* $\alpha_i > 0$, $\beta_i > 0$ $(i = 1,\dots,m)$, $\alpha_{i_0} \geq 1$,

$$\alpha_{i_0} = \max_{i=1\div m}(\alpha_i), \tag{25}$$

and $\Gamma_{k,n}$ *be defined by* (23) *and* (24)*. Then, the following inequalities hold true for all the values of* $n \in \mathbb{N}_0$ *with* $\alpha_i(n+1)+\beta_i \geq 2$*:*

$$\Gamma_{k,n} \leq k, \quad \text{for } k = 1,2,3,\dots, \tag{26}$$

and

$$\Gamma_{0,n} \leq \frac{n+1}{n+\beta_{i_0}}, \text{ for } \alpha_{i_0} = 1; \quad \Gamma_{0,n} \leq \frac{(n+1)\Gamma(\alpha_{i_0}+\beta_{i_0})}{\Gamma(\alpha_{i_0} n+\beta_{i_0}+\alpha_{i_0})}, \text{ for } \alpha_{i_0} > 1. \tag{27}$$

Moreover, if $\alpha_{i_0} \geq 1$ *and* n *is "large enough", then*

$$\Gamma_{0,n} \sim \frac{(n+1)}{(\alpha_{i_0} n+\beta_{i_0})^{\alpha_{i_0}}}. \tag{28}$$

Proof. Letting $\alpha_{i_0} \geq 1$ and $k \geq 2$, we write

$$\begin{aligned}\Gamma(\alpha_{i_0} n+\beta_{i_0}+k\alpha_{i_0}) &= \Gamma(\alpha_{i_0} n+\beta_{i_0}+\alpha_{i_0}+(k-1)\alpha_{i_0}) \geq \Gamma(\alpha_{i_0} n+\beta_{i_0}+\alpha_{i_0}+k-1)\\ &= (\alpha_{i_0} n+\beta_{i_0}+\alpha_{i_0})\dots(\alpha_{i_0} n+\beta_{i_0}+k-2)\Gamma(\alpha_{i_0} n+\beta_{i_0}+\alpha_{i_0}).\end{aligned}$$

If $i_0 \neq i$ and $\alpha_i(n+1)+\beta_i \geq 2$, then

$$\frac{\Gamma(\alpha_i n+\beta_i+\alpha_i)}{\Gamma(\alpha_i n+\beta_i+k\alpha_i)} \leq 1.$$

The last two inequalities lead to

$$\Gamma_{k,n} \leq \frac{(n+1)\dots(n+k-1)}{(\alpha_{i_0} n+\beta_{i_0}+\alpha_{i_0})\dots(\alpha_{i_0} n+\beta_{i_0}+\alpha_{i_0}+k-2)}\frac{n+k}{n+1} \leq 1.k,$$

which, along with $\Gamma_{1,n} = 1$, proves (26).

The proof of (27) proceeds in a similar way, using the fact that the inequality

$$\frac{\Gamma(\alpha_i n+\beta_i)}{\Gamma(\alpha_i n+\beta_i+\alpha_i)} \leq 1,$$

holds true for $i_0 \neq i$. The details are omitted.

In order to prove (28), we take $\alpha_{i_0} \geq 1$ and $k = 0$, and using (27) and the Γ-functions' quotient property (14) with $z = \alpha_{i_0} n + \beta_{i_0}$, we affirm the validity of the relation (28). □

We set $\alpha_{i_0} > 1$ in (28) and let $n \to \infty$ lead to a corollary, as follows.

Corollary 3. *Let $\alpha_i > 0$, $\beta_i > 0$, α_{i_0} be defined by (25), $\alpha_{i_0} > 1$, and $\Gamma_{0,n}$ be defined by (23). Then, the following relation holds true:*

$$\lim_{n\to\infty} \Gamma_{0,n} = \lim_{n\to\infty} \frac{(n+1)\Gamma(\alpha_1 n+\beta_1)\ldots\Gamma(\alpha_m n+\beta_m)}{\Gamma(\alpha_1 n+\beta_1+\alpha_1)\ldots\Gamma(\alpha_m n+\beta_m+\alpha_m)} = 0. \tag{29}$$

The results obtained referred to the coefficients $\Gamma_{k,n}$ allowing estimates to be found for $|\vartheta_n|$ and an asymptotic for ϑ_n when $n \to \infty$.

Theorem 2. *Let $\alpha_i > 0$, $\beta_i > 0$ $(i = 1, \ldots, m)$, α_{i_0} be defined by (25), $\alpha_{i_0} \geq 1$, ϑ_n be defined by (22) and $z \in \mathbb{C}$. Let K be a nonempty compact subset of $\mathbb{C}$. Then, the following inequality holds true, if additionally $\alpha_i(n+1) + \beta_i \geq 2$:*

$$|\vartheta_n(z)| \leq \frac{(n+1)\Gamma(\alpha_{i_0} n+\beta_{i_0})}{\Gamma(\alpha_{i_0} n+\beta_{i_0}+\alpha_{i_0})} |z| \exp|z|, \quad \textit{for } z \in \mathbb{C}, \tag{30}$$

and there exists a constant $C = C(K)$ such that

$$|\vartheta_n(z)| \leq C \frac{(n+1)\Gamma(\alpha_{i_0} n+\beta_{i_0})}{\Gamma(\alpha_{i_0} n+\beta_{i_0}+\alpha_{i_0})} \quad \textit{for } z \in K. \tag{31}$$

Moreover, if $\alpha_{i_0} > 1$, then

$$\lim_{n\to\infty} \vartheta_n(z) = 0, \tag{32}$$

for $z \in \mathbb{C}$, and the convergence is uniform on the compact subsets of $\mathbb{C}$.

Proof. Due to (23) and (26), the module of $\vartheta_n(z)$ is estimated in the following way in the complex plane:

$$|\vartheta_n(z)| \leq \Gamma_{0,n} \left[\sum_{k=1}^{\infty} \Gamma_{k,n} \frac{|z|^k}{k!} \right] \leq \Gamma_{0,n} \left[\sum_{k=1}^{\infty} k \frac{|z|^k}{k!} \right] = \Gamma_{0,n} |z| \exp|z|,$$

which shows that the inequality (30) holds true. The inequality (31) automatically follows in the set K. The equality (32) is also fulfilled, due to (29)–(31). □

In particular, taking $\alpha_{i_0} = 1$, the following corollary can be produced.

Corollary 4. *Let $\alpha_i > 0$, $\beta_i > 0$ $(i = 1, \ldots, m)$, α_{i_0} be defined by (25), $\alpha_{i_0} = 1$, ϑ_n be defined by (22) and $\alpha_i(n+1) + \beta_i \geq 2$. Let K be a nonempty compact subset of $\mathbb{C}$. Then, the inequalities (30) and (31) are reduced to*

$$|\vartheta_n(z)| \leq \frac{n+1}{n+\beta_{i_0}} |z| \exp|z|, \quad \textit{for } z \in \mathbb{C}, \tag{33}$$

respectively,

$$|\vartheta_n(z)| \leq C \frac{n+1}{n+\beta_{i_0}}, \quad \textit{for } z \in K, \tag{34}$$

with a constant $C = C(K)$.

3. Taylor Series

It is well known that a given function f, holomorphic in an open disk D, can be represented with a Taylor series, i.e., a series of the form

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \qquad (z, z_0 \in D). \tag{35}$$

In this section, we consider the Mittag–Leffler type functions (1) and (3) with complex indices, which are entire functions when $Re(\alpha) > 0$, respectively, $Re(\alpha_i) > 0$ for $i = 1, \dots, m$, and give their Taylor series. However, before that, we state the following lemma.

Lemma 5. *Let z, z_0, $\zeta \in \mathbb{C}$, $\rho > 0$, and n be a nonnegative integer. Let C_ρ and $D(z_0, \rho)$ be the curve and open disk, centred at the point z_0, as follows:*

$$C_\rho : \ |\zeta - z_0| = \rho, \quad D(z_0, \rho) : \ |z - z_0| < \rho. \tag{36}$$

Then, the function

$$R_{m,n}(z) = \frac{1}{2\pi i} \int_{C_\rho} \frac{E_{(\alpha_i),(\beta_i)}(\zeta)}{(\zeta - z)} \frac{(z - z_0)^{n+1}}{(\zeta - z_0)^{n+1}} \, d\zeta, \tag{37}$$

with $E_{(\alpha_i),(\beta_i)}$ defined in (3), satisfies the following relation:

$$\lim_{n\to\infty} R_{m,n}(z) = 0, \quad z \in D(z_0, \rho). \tag{38}$$

Proof. Denoting

$$M(\rho) = \max_{|\zeta - z_0| = \rho} |E_{(\alpha_i),(\beta_i)}(\zeta)|, \tag{39}$$

and bearing in mind that $E_{(\alpha_i),(\beta_i)}$ is an entire function, $R_{m,n}(z)$ is estimated modulo in the following way:

$$|R_{m,n}(z)| < \frac{\rho}{2\pi} \frac{M(\rho)}{(\rho - |z - z_0|)} \left(\frac{|z - z_0|}{\rho} \right)^n.$$

Now, taking into account that $|z - z_0|/\rho < 1$, the equality (38) automatically follows. □

Remark 1. *In particular, if $m = 1$, the function (3) is reduced to (1). Thus, in this case, Lemma 5 is referred to the function (1).*

The Taylor series of the function (1) is given with the theorem below.

Theorem 3. *Let α, $\beta \in \mathbb{C}$, $Re(\alpha) > 0$, and let z, $z_0 \in \mathbb{C}$. Then, the Mittag–Leffler function (1) has the following Taylor series:*

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} E^{n+1}_{\alpha,\beta+n\alpha}(z_0)\,(z - z_0)^n. \tag{40}$$

Proof. Since the function (1) is a holomorphic function in the whole complex plane, it can be represented there in a series of the kind (35). Taking $f(z) = E_{\alpha,\beta}(z)$ and bearing in mind (5), we can write that its n-th derivative at the point z_0 is equal to

$$f^{(n)}(z_0) = n! E^{n+1}_{\alpha,\beta+n\alpha}(z_0).$$

That means that the function (1) is represented in a Taylor series of the kind (35) with coefficients $\frac{f^{(n)}(z_0)}{n!} = E^{n+1}_{\alpha,\beta+n\alpha}(z_0)$. The convergence of the series in (40) to the function $E_{\alpha,\beta}(z)$ is provided with Lemma 5. Indeed, it is well known that the remainder of (35)

$$R_k(z) = f(z) - \sum_{n=0}^{k} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n,$$

can be expressed in terms of a contour integral as follows:

$$R_k(z) = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(\zeta)}{(\zeta - z)} \frac{(z - z_0)^{k+1}}{(\zeta - z_0)^{k+1}} \, d\zeta, \quad |z - z_0| < \rho,$$

where the circle C_ρ is such as in (35) (centred at z_0 and with an arbitrary radius $\rho > 0$). For details and proof of the above representation, see, e.g., [39] (Volume 1, Chapter 4, (4.4:3)). Now, taking $f(\zeta) = E_{\alpha,\beta}(\zeta)$, and in view of Lemma 5 (applied with $m = 1$) and Remark 1, the remainder $R_k(z)$ tends to zero, when $k \to \infty$ in the whole open disk $D(z_0, \rho) : |z - z_0| < \rho$. Since ρ is an arbitrary radius, the convergence of the remainder is in the whole complex plane. Therefore, the series in (40) converges to the Mittag–Leffler function $E_{\alpha,\beta}(z)$ in the whole complex plane, which completely proves the theorem. □

Further, we deal with the multi-index function (3). For convenience, we introduce the denotation

$$\widetilde{E}^{\gamma,m}_{(\alpha_i),(\beta_i)}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_k}{\Gamma(\alpha_1 k + \beta_1) \ldots \Gamma(\alpha_m k + \beta_m)} \frac{z^k}{k!}, \tag{41}$$

which is the particular case of the function (4) with $\gamma_1 = \gamma, \gamma_2 = \cdots = \gamma_m = 1$ (mentioning that $\widetilde{E}^{\gamma,1}_{(\alpha_i),(\beta_i)} = E^{\gamma}_{\alpha_1,\beta_1}$). Then, the relation (6) takes the form

$$D^n\left[E_{(\alpha_i),(\beta_i)}(z)\right] = \frac{d^n}{dz^n}\left[E_{(\alpha_i),(\beta_i)}(z)\right] = \Gamma(n+1)\, \widetilde{E}^{n+1,m}_{(\alpha_i),(\beta_i+n\alpha_i)}(z). \tag{42}$$

Theorem 4. *Let $\alpha_i, \beta_i \in \mathbb{C}$, $Re(\alpha_i) > 0$ for $i = 1, \ldots, m$, and let $z, z_0 \in \mathbb{C}$. Then, the multi-index Mittag–Leffler function (3) is represented by the following Taylor series:*

$$E_{(\alpha_i),(\beta_i)}(z) = \sum_{n=0}^{\infty} \widetilde{E}^{n+1,m}_{(\alpha_i),(\beta_i+n\alpha_i)}(z_0)\,(z - z_0)^n, \tag{43}$$

with $\widetilde{E}^{n+1,m}_{(\alpha_i),(\beta_i+n\alpha_i)}$ defined by (41).

Proof. Since the function (3) is a holomorphic function in the whole complex plane, it can be represented there in a series of the kind (35). Taking $f(z) = E_{(\alpha_i),(\beta_i)}(z)$ and bearing in mind (6), (41), and (42), we can write that its n-th derivative at the point z_0 is equal to

$$f^{(n)}(z_0) = \Gamma(n+1)\, \widetilde{E}^{n+1,m}_{(\alpha_i),(\beta_i+n\alpha_i)}(z_0).$$

That means that the function (3) is represented in a Taylor series of the kind (35) with coefficients $\frac{f^{(n)}(z_0)}{n!} = \widetilde{E}^{n+1,m}_{(\alpha_i),(\beta_i+n\alpha_i)}(z_0)$. The convergence of the series in (43) to the multi-index function $E_{(\alpha_i),(\beta_i)}(z)$ is provided by Lemma 5. It goes analogously to the proof of Theorem 3. The details are omitted. □

Remark 2. *Naturally, one might expect that if $z_0 = 0$, then the Taylor series (40) and (43) coincide respectively with the series (1) and (3). Indeed, the value of $E^{n+1}_{\alpha,\beta+n\alpha}$ at the point 0 is*

$$E^{n+1}_{\alpha,\beta+n\alpha}(0) = \frac{1}{\Gamma(\alpha n + \beta)},$$

due to Definition (2). Then, the Taylor series (40) of the two-parametric Mittag–Leffler function $E_{\alpha,\beta}$ is reduced to the series (1), defining this function. Analogously, in view of (41),

$$\widetilde{E}^{n+1,m}_{(\alpha_i),(\beta_i+n\alpha_i)}(0) = \frac{1}{\Gamma(\alpha_1 n + \beta_1) \ldots \Gamma(\alpha_m n + \beta_m)},$$

which means that, in this case, the series (43) produces the series (3), defining the $2m$-parametric multi-index Mittag–Lefler function $E_{(\alpha_i),(\beta_i)}$.

Let us note that the multi-index Mittag–Leffler functions have many interesting and useful special cases. Among them are for example both the classical Bessel functions of the first kind (up to a power function) and the closely related Bessel–Clifford functions. The first ones are defined by the series

$$J_\nu(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+\nu}}{k!\,\Gamma(k+\nu+1)}, \quad z \in \mathbb{C} \setminus (-\infty, 0];\ \nu \in \mathbb{C}. \tag{44}$$

The second ones are defined by the power series

$$C_\nu(z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(k+\nu+1)}, \quad z \in \mathbb{C};\ \nu \in \mathbb{C}. \tag{45}$$

Arising from specific problems in mechanics and astronomy, these functions have various applications. That is why they both have numerous generalizations with more indices (parameters), or Bessel type functions. They are also connected with the multi-index Mittag–Leffler functions. Naturally, all of them have a Taylor series of the kind (43). Below, in this section, several Bessel type functions are considered, and their Taylor series are given in the relevant forms.

Case 1. A special case of (3) (for $m \geq 2$) is the generalized Lommel–Wright function $J_{\nu,\lambda}^{\mu,q}$ with four parameters, introduced by de Oteiza, Kalla, and Conde (for more details, see, e.g., [23]):

$$J_{\nu,\lambda}^{\mu,q}(z) = (z/2)^{\nu+2\lambda} \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k}}{(\Gamma(k+\lambda+1))^q\, \Gamma(\mu k + \lambda + \nu + 1)}, \tag{46}$$
$$z \in \mathbb{C} \setminus (-\infty, 0]; \quad \nu, \lambda \in \mathbb{C}, \quad \mu > 0, \quad q \in \mathbb{N}.$$

The sum in (46) is an example of the multi-index Mittag–Leffler function with an arbitrary $m = q + 1$ ($q \in \mathbb{N}$). Further on, if the parameters are as follows,

$$\alpha_1 = \mu,\ \alpha_2 = 1,\ \ldots,\ \alpha_{q+1} = 1,\ \beta_1 = \lambda + \nu + 1,\ \beta_2 = \lambda + 1,\ \ldots,\ \beta_{q+1} = \lambda + 1,$$

the generalized Lommel–Wright function (46) can be expressed by the multi-index Mittag–Leffler functions (3). Setting $2\sqrt{z}$ instead of z, the relation (46) produces the following:

$$J_{\nu,\lambda}^{\mu,q}(2\sqrt{z}) = z^{\nu/2+\lambda}\, \widetilde{J}_{\nu,\lambda}^{\mu,q}(z), \quad z \in \mathbb{C} \setminus (-\infty, 0];\ \nu, \lambda \in \mathbb{C},\ \mu > 0\ q \in \mathbb{N}, \tag{47}$$

with $\widetilde{J}_{\nu,\lambda}^{\mu,q}$ being the entire function

$$\widetilde{J}_{\nu,\lambda}^{\mu,q}(z) = E^{q+1}_{(\mu,1,\ldots,1),\,(\lambda+\nu+1,\lambda+1,\ldots,\lambda+1)}(-z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{(\Gamma(k+\lambda+1))^q\, \Gamma(\mu k + \lambda + \nu + 1)}; \tag{48}$$

defined for $\nu, \lambda \in \mathbb{C}$, $\mu > 0$, $q \in \mathbb{N}$ and $z \in \mathbb{C}$.

Since the function (48) satisfies the conditions of Theorem 4 and

$$D^n\, \widetilde{J}_{\nu,\lambda}^{\mu,q}(z) = (-1)^n \Gamma(n+1)\, \widetilde{E}^{n+1,q+1}_{(\alpha_i),\,(\beta_i+n\alpha_i)}(-z),$$

with parameters

$$\alpha_1 = \mu, \alpha_2 = \cdots = \alpha_{q+1} = 1, \quad \beta_1 = \lambda + \nu + 1, \beta_2 = \cdots = \beta_{q+1} = \lambda + 1, \tag{49}$$

in view of (41) and (42), the series (43) is reduced to

$$\widetilde{J}_{\nu,\lambda}^{\mu,q}(z) = \sum_{n=0}^{\infty} (-1)^n\, \widetilde{E}^{n+1,q+1}_{(\alpha_i),\,(\beta_i+n\alpha_i)}(-z_0)\, (z - z_0)^n, \quad z, z_0 \in \mathbb{C}, \tag{50}$$

with parameters as in (49).

Case 2. Further, for $q = 1$, $\alpha_1 = \mu$, $\alpha_2 = 1$, $\beta_1 = \lambda + \nu + 1$, $\beta_2 = \lambda + 1$, the equalities (47) and (48) give the generalized Bessel–Maitland (or Bessel–Wright) function $J^{\mu}_{\nu,\lambda}$ with three indices, introduced by R.S. Pathak:

$$J^{\mu}_{\nu,\lambda}(2\sqrt{z}) = z^{\nu/2+\lambda}\widetilde{J}^{\mu}_{\nu,\lambda}(z), \quad z \in \mathbb{C} \setminus (-\infty, 0]; \quad \nu, \lambda \in \mathbb{C}, \quad \mu > 0, \tag{51}$$

with $\widetilde{J}^{\mu}_{\nu,\lambda}$ as follows:

$$\widetilde{J}^{\mu}_{\nu,\lambda}(z) = E_{(\mu,1),(\lambda+\nu+1,\lambda+1)}(-z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma(k+\lambda+1)\Gamma(\mu k+\lambda+\nu+1)}, \tag{52}$$

for $z \in \mathbb{C} \setminus (-\infty, 0]; \quad \nu, \lambda \in \mathbb{C}, \quad \mu > 0$.

Then, the series (50) is reduced to

$$\widetilde{J}^{\mu}_{\nu,\lambda}(z) = \sum_{n=0}^{\infty} (-1)^n \, \widetilde{E}^{n+1,2}_{(\alpha_i),(\beta_i+n\alpha_i)}(-z_0)\,(z-z_0)^n, \quad z, z_0 \in \mathbb{C}, \tag{53}$$

with $\alpha_1 = \mu$, $\alpha_2 = 1$, $\beta_1 = \lambda + \nu + 1$, $\beta_2 = \lambda + 1$.

Case 3. Additionally, if $\lambda = 0$, i.e., for parameters $q = 1$, $\alpha_1 = \mu$, $\alpha_2 = 1$, $\beta_1 = \nu + 1$, $\beta_2 = 1$, the relations (48) and (52) produce the Bessel–Maitland (or Bessel–Wright) function J^{μ}_{ν} with two parameters:

$$J^{\mu}_{\nu}(z) = E_{(\mu,1),(\nu+1,1)}(-z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(\mu k+\nu+1)}, \quad z \in \mathbb{C};\ \nu \in \mathbb{C} \text{ and } \mu > 0. \tag{54}$$

This function was introduced by the great British mathematician E. M. Wright and is known as the Bessel–Maitland function (after his second name).

Applying (42), we obtain consecutively

$$\begin{aligned} D^n J^{\mu}_{\nu}(z) &= (-1)^n n!\, E^{n+1}_{(\mu,1),(\mu n+\nu+1,n+1)}(-z) \\ &= (-1)^n \sum_{k=0}^{\infty} \frac{n!\,(n+1)_k}{\Gamma(k+n+1)\Gamma(\mu k+\mu n+\nu+1)} \frac{(-z)^k}{k!} \\ &= (-1)^n \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(\mu k+\mu n+\nu+1)} = (-1)^n J^{\mu}_{\nu+\mu n}(z). \end{aligned}$$

Thus, in this case, the formula (43) is reduced to the following:

$$J^{\mu}_{\nu}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n J^{\mu}_{\nu+\mu n}(z_0)}{n!}(z-z_0)^n, \quad z, z_0 \in \mathbb{C}. \tag{55}$$

Case 4. Finally, if $\mu = 1$, the formula (54) is reduced to the Bessel–Clifford function C_ν, and Bessel function J_ν (up to a power function), given respectively by (45) and (44), as follows:

$$C_\nu(z) = \widetilde{J}_\nu(z) = E_{(1,1),(\nu+1,1)}(-z), \quad z \in \mathbb{C};\ \nu \in \mathbb{C}, \tag{56}$$

where $\widetilde{J}_\nu$ is the entire function defined by

$$J_\nu(2\sqrt{z}) = z^{\nu/2}\widetilde{J}_\nu(z), \quad z \in \mathbb{C};\ \nu \in \mathbb{C}. \tag{57}$$

The Taylor series (43) is now reduced to both series

$$\widetilde{J}_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \widetilde{J}_{\nu+n}(z_0)}{n!}(z-z_0)^n, \quad z, z_0 \in \mathbb{C}, \tag{58}$$

and

$$C_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n\, C_{\nu+n}(z_0)}{n!}\,(z-z_0)^n, \quad z, z_0 \in \mathbb{C}. \tag{59}$$

Case 5. There is one more interesting function of the Bessel type. It is the so called hyper-Bessel function, defined as follows:

$$J^{(m)}_{\nu_1,\ldots,\nu_m}(z) = \left(\frac{z}{m+1}\right)^{\sum\limits_{i=1}^{m}\nu_i} \sum_{k=0}^{\infty} \frac{(-1)^k\left(\frac{z}{m+1}\right)^{k(m+1)}}{\Gamma(k+\nu_1+1)\ldots\Gamma(k+\nu_m+1)}\,\frac{1}{k!}, \tag{60}$$

for $z \in \mathbb{C} \setminus (-\infty, 0]$ and $Re(\nu_i+1) > 0 \quad (i = 1, \ldots, m)$.

The hyper-Bessel function was introduced in 1953 by Delerue [40] as a generalization of the Bessel function J_ν of the first type with vector index $\nu = (\nu_1, \nu_2, \ldots, \nu_m)$. Later on, this function was also studied by other authors; for example, by Marichev, Kljuchantcev, Dimovski, Kiryakova, etc.

The hyper-Bessel function of Delerue is closely related to the *hyper-Bessel differential operator* of arbitrary order $m > 1$, which was *introduced by Dimovski* in [41]. This is a singular linear differential operator, appearing very often in problems of mathematical physics as a generalization of the second-order Bessel operator. It can be represented in the following alternative forms:

$$B = z^{\alpha_0}\frac{d}{dz}z^{\alpha_1}\ldots\frac{d}{dz}z^{\alpha_m} = z^{-\beta}\prod_{k=1}^{m}\left(z\frac{d}{dz}+\beta\gamma_k\right) \tag{61}$$

$$= z^{-\beta}\left(z^m\frac{d^m}{dz^m} + a_1 z^{m-1}\frac{d^{m-1}}{dz^{m-1}} + \ldots + a_{m-1}z\frac{d}{dz} + a_m\right),$$

with $0 < z < \infty$, and sets of $(m+1)$ parameters

$$\{\alpha_0, \alpha_1, \ldots, \alpha_m\}, \quad \text{or} \quad \{\beta > 0,\ \gamma_k \text{ real},\ k = 1, \ldots m\}, \quad \text{or} \quad \{\beta > 0,\ a_1, \ldots, a_m\}.$$

The details can also be seen in Dimovski and Kiryakova [42,43], and Kiryakova [44] (Chapter 3). In her book [44] (Theorem 3.4.3 and Correction 3.4.4), Kiryakova showed that the fundamental system of solutions of the m-th order *hyper-Bessel differential equation*

$$By(z) = \lambda\, y(z),\ \lambda \neq 0,$$

consists of the set of hyper-Bessel functions

$$J^{(m-1)}_{1+\gamma_1-\gamma_k,\ldots,*,\ldots,1+\gamma_m-\gamma_k}\left[(-\lambda)^{1/m}(m/\beta)\, z^{\beta/m}\right],\ k = 1, \ldots, m.$$

This assertion was proved under the condition of the formal arrangement of the parameters as $\gamma_1 < \gamma_2 < \ldots < \gamma_m < \gamma_1 + 1$, where $*$ means to omit the index γ_k. Under this assumption, the solutions of hyper-Bessel ODEs $By(z) = \lambda\, y(z) + f(z)$ can be written explicitly in terms of hyper-Bessel functions, series in hyper-Bessel functions, or series in integrals of them ([44]).

Replacing z with $(m+1)z^{1/(m+1)}$, the relation (60) yields

$$J^{(m)}_{\nu_1,\ldots,\nu_m}\left((m+1)z^{1/(m+1)}\right) = z^{\frac{1}{m+1}\sum\limits_{i=1}^{m}\nu_i}\,\widetilde{J}^{(m)}_{\nu_1,\ldots,\nu_m}(z), \tag{62}$$

with $\widetilde{J}^{(m)}_{\nu_1,\ldots,\nu_m}$ being the entire function

$$\widetilde{J}^{(m)}_{\nu_1,\ldots,\nu_m}(z) = \sum_{k=0}^{\infty}\frac{(-z)^k}{\Gamma(k+\nu_1+1)\ldots\Gamma(k+\nu_m+1)}\,\frac{1}{k!}, \quad z \in \mathbb{C}.$$

Further, considering the last function with parameters ν_i $(i = 1, \dots, m-1)$ and denoting for convenience $\nu_m = 0$, we express this via the multi-index Mittag–Leffler function of the kind (3). Namely, the following relation holds true (with $\nu_m = 0$):

$$\widetilde{J}^{(m-1)}_{\nu_1,\dots,\nu_{m-1}}(z) = E^{m}_{(1)(\nu_i+1)}(-z) = \sum_{k=0}^{\infty} \frac{(-z)^k}{\Gamma(k+\nu_1+1)\dots\Gamma(k+\nu_{m-1}+1)\,\Gamma(k+1)}. \tag{63}$$

Applying (42), the following relations chain is obtained:

$$\begin{aligned} D^n\,\widetilde{J}^{(m-1)}_{\nu_1,\dots,\nu_{m-1}}(z) &= D^n\,E^{m}_{(1)(\nu_i+1)}(-z) = (-1)^n\,n!\,\widetilde{E}^{n+1}_{(1),(n+\nu_i+1)}(-z) \\ &= (-1)^n \sum_{k=0}^{\infty} \frac{n!\,(n+1)_k}{\Gamma(k+n+\nu_1+1)\dots\Gamma(k+n+\nu_{m-1}+1)\Gamma(k+n+1)} \frac{(-z)^k}{k!} \\ &= (-1)^n \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\,\Gamma(k+n+\nu_1+1)\dots\Gamma(k+n+\nu_{m-1}+1)} \\ &= (-1)^n\,\widetilde{J}^{(m-1)}_{n+\nu_1,\dots,n+\nu_{m-1}}(z). \end{aligned}$$

Thus, in this case the formula (43) is reduced to the following:

$$\widetilde{J}^{(m-1)}_{\nu_1,\dots,\nu_{m-1}}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n\,\widetilde{J}^{(m-1)}_{n+\nu_1,\dots,n+\nu_{m-1}}(z_0)}{n!}(z-z_0)^n, \quad z, z_0 \in \mathbb{C}, \tag{64}$$

i.e., (64) gives the Taylor series for the function $\widetilde{J}^{(m-1)}_{\nu_1,\dots,\nu_{m-1}}(z)$.

4. Conclusions

In conclusion, we emphasize that if $z_0 = 0$, then the Taylor series (40) and (43) of $E_{\alpha,\beta}$ and $E_{(\alpha_i),(\beta_i)}$ coincide respectively with the series (1) and (3), defining these functions. It is also worth summarizing that all the considered functions of the Bessel type, discussed in this section, are expressed by the multi-index Mittag–Leffler functions (3). For that reason, using the formula (42) for the n-th derivative of (3) and Theorem 4, they can be represented by Taylor series. The coefficients in the series obtained are not always expressed only by the values of the corresponding more complicated multi-index Mittag–Leffler functions at the given point z_0. In some of the considered cases, the coefficients in the Taylor series include the values at z_0 of the represented Bessel type functions with "translated" parameters.

Funding: This research received no financial funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: This paper was completed under the working program of the bilateral collaboration contract between the Bulgarian Academy of Sciences and Serbian Academy of Sciences and Arts.

Conflicts of Interest: The author declares no conflict of interest.

References

1. Rogosin, S. The role of the Mittag-Leffler function in fractional modeling. *Mathematics* **2015**, *3*, 368–381. [CrossRef]
2. Gorenflo, R.; Mainardi, F. On Mittag-Leffler function in fractional evolution processes. *J. Comput. Appl. Math.* **2000**, *118*, 283–299.
3. Mainardi, F.; Pagnini, G.; Gorenflo, R. Mellin transform and subordination laws in fractional diffusion processs. *Fract. Calc. Appl. Anal.* **2003**, *6*, 441–459.
4. Klafter, J.; Sokolov, I.M. Anomalous diffusion spreads its wings. *Phys. World* **2005**, *18*, 29–32. [CrossRef]

5. Metzler, R.; Jeon, J.-H.; Cherstvya, A.G.; Barkai, E. Anomalous diffusion models and their properties: Non-stationarity, non-ergodicity, and ageing at the centenary of single particle tracking. *Phys. Chem. Chem. Phys.* **2014**, *16*, 24128–24164. [CrossRef] [PubMed]
6. Chechkin, A.; Sokolov, I.M. Relation between generalized diffusion equations and subordination schemes. *Phys. Rev. E* **2021**, *103*, 032133–032143. [CrossRef] [PubMed]
7. Dieterich, P.; Lindemann, O.; Moskopp, M.L.; Tauzin S.; Huttenlocher A.; Klages R.; Chechkin, A.; Schwab, A. Anomalous diffusion and asymmetric tempering memory in neutrophil chemotaxis. *PLoS Comput. Biol.* **2022**, *18*, e1010089. [CrossRef]
8. Sandev, T.; Iomin, A. *Special Functions of Fractional Calculus: Applications to Diffusion and Random Search Processes*; World Scientific: Singapore, 2022. [CrossRef]
9. Ghanim, F.; Al-Janaby, H.; Bazighifan, O. Some new extensions on fractional differential and integral properties for Mittag–Leffler confluent hypergeometric function. *Fractal Fract.* **2021**, *5*, 143. [CrossRef]
10. Andrić, M. Fractional integral inequalities of Hermite–Hadamard type for $(h, g; m)$-convex functions with extended Mittag–Leffler function. *Fractal Fract.* **2022**, *6*, 301. [CrossRef]
11. Ghanim, F.; Al-Janaby, H.F.; Al-Momani, M.; Batiha, B. Geometric studies on Mittag–Leffler type function involving a new integrodifferential operator. *Mathematics* **2022**, *10*, 3243. [CrossRef]
12. Erdélyi, A. (Ed.) *Higher Transcendental Functions*; McGraw-Hill: New York, NY, USA; Toronto, ON, Canada; London, UK, 1953.
13. Prabhakar T.R. A singular integral equation with a generalized Mittag–Leffler function in the kernel. *Yokohama Math. J.* **1971**, *19*, 7–15.
14. Yakubovich, S.; Luchko, Y. *The Hypergeometric Approach to Integral Transforms and Convolutions*; Kluwer Acad. Publ.: Dordrecht, The Netherlands; Boston, MA, USA; London, UK, 1994.
15. Kiryakova, V. Multiple (multiindex) Mittag–Leffler functions and relations to generalized fractional calculus. *J. Comput. Appl. Math.* **2000**, *118*, 241–259. [CrossRef]
16. Kiryakova, V. From the hyper-Bessel operators of Dimovski to the generalized FC. *Fract. Calc. Appl. Anal.* **2014**, *17*, 977–1000. [CrossRef]
17. Kiryakova, V. Fractional calculus operators of special functions?—The result is well predictable! *Chaos Solitons Fractals* **2017**, *102*, 2–15. [CrossRef]
18. Kilbas, A.A.; Koroleva, A.A. Generalized Mittag-Leffler function and its extension. *Tr. Inst. Mat. Minsk* **2005**, *13*, 43–52. (In Russian)
19. Kilbas, A.A.; Koroleva, A.A. Integral transform with the extended generalized Mittag-Leffler function. *Math. Model. Anal.* **2006**, *11*, 173–186. [CrossRef]
20. Kiryakova, V.; Luchko, Y. The multi-index Mittag-Leffler functions and their applications for solving fractional order problems in applied analysis. *AIP Conf. Proc.* **2010**, *1301*, 597–613. [CrossRef]
21. Kilbas A.A.; Koroleva, A.A.; Rogosin, S.V. Multi-parametric Mittag-Leffler functions and their extension. *Fract. Calc. Appl. Anal.* **2013**, *16*, 378–404. [CrossRef]
22. Paneva-Konovska, J. Multi-index (3m-parametric) Mittag-Leffler functions and fractional calculus. *Compt. rend. Acad. Bulg. Sci.* **2011**, *64*, 1089–1098.
23. Paneva-Konovska, J. *From Bessel to Multi-Index Mittag Leffler Functions: Enumerable Families, Series in them and Convergence*, 1st ed.; World Scientific Publ.: London, UK, 2016. [CrossRef]
24. Paneva-Konovska, J. A survey on Bessel type functions as multi-index Mittag-Leffler functions: Differential and integral relations. *Int. J. Appl. Math.* **2019**, *32*, 357–380. [CrossRef]
25. Paneva-Konovska, J.; Kiryakova, V. On the multi-index Mittag-Leffler functions and their Mellin transforms. *Intern. J. Appl. Math.* **2020**, *33*, 549–571. [CrossRef]
26. Ali, M.; Paneva-Konovska, J.; Pogány, T. On multi-index Whittaker function and related integrals. *J. Math. Inequal.* **2022**, *16*, 513–527. [CrossRef]
27. Kiryakova, V. A Guide to Special Functions in Fractional Calculus. *Mathematics* **2021**, *1*, 106. [CrossRef]
28. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
29. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006.
30. Bazhlekova, E.; Dimovski, I. Exact solution for the fractional cable equation with nonlocal boundary conditions. *Cent. Eur. J. Phys.* **2013**, *11*, 1304–1313. [CrossRef]
31. Paneva-Konovska, J. Differential and integral relations in the class of multi-index Mittag-Leffler functions. *Fract. Calc. Appl. Anal.* **2018**, *21*, 254–265. [CrossRef]
32. Tunç, T.; Sarıkaya, M.Z.; Yaldız, H. Fractional Hermite Hadamard's type inequality for the co-ordinated convex functions. *TWMS J. Pure Appl. Math.* **2020**, *11*, 3–29.
33. Noor, M.A.; Noor, K.I. Some new classes of strongly generalized preinvex functions. *TWMS J. Pure Appl. Math.* **2021**, *12*, 181–192.
34. Mun, J. Monte Carlo simulation with mathematical convolution of frequency and severity distributions in operational risk capital model in the Basel accords. *J. Adv. Math. Comput. Sci.* **2022**, *37*, 8–21. [CrossRef]
35. Luchko, Y. Convolution series and the generalized convolution Taylor formula. *Fract. Calc. Appl. Anal.* **2022**, *25*, 207–228. [CrossRef]

36. Sunday, J.; Shokri, A.; Marian, D. Variable step hybrid block method for the approximation of Kepler problem. *Fractal Fract.* **2022**, *6*, 343. [CrossRef]
37. Shokri, A.; Saadat, H.; Khodadadi, A.R. A new high order closed Newton-Cotes trigonometrically-fitted formulae for the numerical solution of the Schrödinger equation. *Iran. J. Math. Sci. Inform.* **2018**, *13*, 111–129.
38. Deadman, E.; Relton, S.D. Taylor's theorem for matrix functions with applications to condition number estimation. *Linear Algebra Its Appl.* **2016**, *504*, 354–371. [CrossRef]
39. Markushevich, A. *Theory of Analytic Functions*; Nauka: Moscow, Russia, 1967. (In Russian)
40. Delerue, P. Sur le calcul symbolic *à* n variables et fonctions hyperbesséliennes (II). *Annales Soc. Sci. Bruxelle Ser. 1* **1953**, *3*, 229–274.
41. Dimovski, I. Operational calculus for a class of differential operators. *Compt. Rend. Acad. Bulg. Sci.* **1966**, *19*, 1111–1114.
42. Dimovski, I.; Kiryakova, V. Generalized Poisson transmutations and corresponding representations of hyper-Bessel functions. *Compt. Rend. Acad. Bulg. Sci.* **1986**, *39*, 29–32.
43. Dimovski, I.; Kiryakova, V. Generalized Poisson representations of hypergeometric functions ${}_pF_q$, $p < q$, using fractional integrals. In Proceedings of the 16th Spring Conference Union Bulg. Math., Sofia, Bulgaria, 6–10 April 1987; pp. 205–212.
44. Kiryakova, V. *Generalized Fractional Calculus and Applications*; Longman & J. Wiley: Harlow, UK; New York, NY, USA, 1994.