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AN EXAMPLE CONCERNING VALDIVIA COMPACT SPACES

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Abstract. We prove that the dual unit ball of the space $\mathcal{C}_0[0,\omega_1)$ endowed with the weak* topology is not a Valdivia compact. This answers a question posed to the author by V. Zizler and has several consequences. Namely, it yields an example of an affine continuous image of a convex Valdivia compact (in the weak* topology of a dual Banach space) which is not Valdivia, and shows that the property of the dual unit ball being Valdivia is not an isomorphic property. Another consequence is that the space $\mathcal{C}_0[0,\omega_1)$ has no countably 1-norming Markuševič basis.

The classes of Corson and Valdivia compact spaces play an important role in study of both topological and linear properties of Banach spaces. In particular, they are closely related with projectional resolutions of the identity

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and (countably) norming Markuševič bases (see [8, 9, 2, 3]). While it is well-known (see [1, Corollary IV.3.15], original proofs are due to S. P. Gul'ko and to E. Michael and M. E. Rudin) that every continuous image of a Corson compact space is again Corson, just recently M. Valdivia [10] found an example of a continuous image of a Valdivia compact space which is not Valdivia. We present here another example, which is a dual unit ball of a Banach space. This strengthens the example of [10] and has several interesting consequences.

Let us start with definitions.

Definitions and notation

- If Γ is a set, we put
$$\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma \mid \{\gamma \in \Gamma \mid x(\gamma) \neq 0\} \text{ is countable}\}.$$
- A compact Hausdorff space K is called *Corson* if K is homeomorphic to a subset of $\Sigma(\Gamma)$ for some set Γ.
- A compact Hausdorff space K is called *Valdivia* if there is a homeomorphism h of K into some $[0,1]^\Gamma$ such that $h(K) \cap \Sigma(\Gamma)$ is dense in $h(K)$.
- We say that a topological space X is *a Fréchet-Urysohn space* (or, shortly an *FU-space*) if, whenever $A \subset X$ and $x \in \overline{A}$ there is a sequence $x_n \in A$ with $x_n \to x$.
- Let X be a topological space and $A \subset X$. We say that A is *countably closed* in X if, whenever $C \subset A$ is countable then $\overline{C} \subset A$.
- Let $(X, \|\cdot\|)$ be a Banach space and X^* the dual space endowed with the dual norm. The system $(x_\alpha, f_\alpha)_{\alpha\in\Lambda} \subset X \times X^*$ is called *Markuševič basis* if the following three conditions are satisfied.

  (i) $f_\alpha(x_\beta) = \delta_{\alpha\beta}$ for every $\alpha, \beta \in \Lambda$, where $\delta_{\alpha\beta}$ is the Kronecker symbol;

  (ii) $\overline{span}\{x_\alpha \mid \alpha \in \Lambda\} = X$;

  (iii) $(\forall x \in X, x \neq 0)(\exists \alpha \in \Lambda)(f_\alpha(x) \neq 0)$.

  This basis is called *norming (1-norming)* if $span\{f_\alpha \mid \alpha \in \Lambda\}$ is norming (1-norming) subspace of X^*. It is called *countably norming (countably 1-*

norming) if $\{f \in X^* \mid \{\alpha \in \Lambda, f(x_\alpha) \neq 0\}$ is countable$\}$ is norming (1-norming) subspace of X^*.

Recall that a subspace $Y \subset X^*$ is called *norming (1-norming)* if the norm on X defined by the formula $|x| = \sup\{f(x) \mid f \in Y, \|f\| \leq 1\}$ is equivalent (equal) to the original norm $\|\cdot\|$.

We will deal namely with $\mathcal{C}[0,\omega_1]$, the space of all continuous functions on the compact ordinal segment $[0,\omega_1]$, and $\mathcal{C}_0[0,\omega_1)$, the space of all continuous functions on the locally compact ordinal segment $[0,\omega_1)$ that vanish at infinity. These spaces are both considered with the supremum norm. We will use some well-known properties of the space $[0,\omega_1]$, namely the fact that each continuous function on this space is constant on some neighborhood of ω_1, and that every finite Radon measure on $[0,\omega_1]$ is supported by a countable set.

Now we state our main result and its consequences.

Theorem. *The dual unit ball $\left(B_{\mathcal{C}_0[0,\omega_1)^*}, w^*\right)$ is not a Valdivia compact.*

Corollary 1. *There exist a Banach space X and a closed subspace $Y \subset X$ such that B_{X^*} is a Valdivia compact in the weak* topology but B_{Y^*} is not.*

This corollary shows that our theorem yields a strengthening of the example of [10]. Indeed, if we denote by i the injection of Y into X, the adjoint mapping i^* maps B_{X^*} onto B_{Y^*}, and is weak* to weak* continuous. Hence we get even a linear continuous image of a convex Valdivia compact (in the weak* topology of a dual Banach space) which is not Valdivia.

Corollary 2. *There exists a Banach space X with two equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_2$ such that $\left(B_{(X,\|\cdot\|_1)^*}, w^*\right)$ is a Valdivia compact but $\left(B_{(X,\|\cdot\|_2)^*}, w^*\right)$ is not.*

This proves that the property of the dual unit ball being Valdivia compact is not an isomorphic property. This is another difference from other classes of compact spaces (Corson etc.). (This fact was independently observed (using several deep theorems) in [2, Example 1]. But our proof is essentially simpler.)

Corollary 3. *The space $\mathcal{C}_0[0,\omega_1)$ has no countably 1-norming Markuševič basis.*

It is easy to see that the space $\mathcal{C}_0[0,\omega_1)$ has a countably norming Markuševič basis. But this basis cannot be countably 1-norming. This is related so some questions studied by A. N. Plička (see [7] and other papers).

Now we proceed to proofs. To prove Theorem we need two simple lemmas.

Lemma 1. *The space $\Sigma(\Gamma)$ is an FU-space and is countably closed in $\mathbb{R}^\Gamma$ for any set Γ.*

Proof. For every $x \in \Sigma(\Gamma)$ the set $\mathit{supp}\, x = \{\gamma \in \Gamma \mid x(\gamma) \neq 0\}$ is countable, so we can fix an enumeration $\mathit{supp}\, x = \{\gamma_1(x), \gamma_2(x), \dots\}$. If $\mathit{supp}\, x$ is finite, we fill up the sequence $\gamma_k(x)$ with some elements of Γ. Now let $A \subset \Sigma(\Gamma)$, $x \in \Sigma(\Gamma)$, $x \in \overline{A}$. We can construct by induction a sequence of $x_n \in A$ such that $|x_n(\gamma_k(x_l)) - x(\gamma_k(x_l))| < \frac{1}{n}$ for $0 \leq l < n$ and $1 \leq k \leq n$, where $x_0 = x$. Then clearly $x_n \to x$ (since the convergence in the product topology is the coordinatewise convergence).

The second assertion is easy to see. $\square$

Lemma 2. *Let K and L be topological spaces and $f : K \to L$ be a continuous mapping. Then the following hold.*

(i) If K is compact, then $\overline{f(A)} = f(\overline{A})$ for every $A \subset K$.

(ii) $f^{-1}(A)$ is countably closed in K whenever A is countably closed in L.

Proof. (i) The inclusion $f(\overline{A}) \subset \overline{f(A)}$ holds for every continuous map between topological spaces. The inverse one follows from the fact that $f(\overline{A})$ is a closed set containing $f(A)$.

(ii) Let $C \subset f^{-1}(A)$ be countable. Then $f(C)$ is countable and contained in A, hence $\overline{f(C)} \subset A$, and therefore $\overline{C} \subset f^{-1}(\overline{f(C)}) \subset f^{-1}(A)$. $\square$

Proof of Theorem. By Riesz theorem we identify $\mathcal{C}_0[0,\omega_1)^*$ with the space of signed Radon measures on $[0,\omega_1)$ and similarly for $\mathcal{C}[0,\omega_1]^*$. We put

$$\begin{aligned} B &= \{\mu \in \mathcal{C}[0,\omega_1]^* \mid \|\mu\| \leq 1\}, & B' &= \{\mu \in \mathcal{C}_0[0,\omega_1)^* \mid \|\mu\| \leq 1\}, \\ M &= \{\mu \in B \mid \mu(\{\omega_1\}) = 0\}, & M' &= \{\mu \in B' \mid \mu([0,\omega_1)) = 0\}. \end{aligned}$$

Consider the natural injection $i : \mathcal{C}_0[0,\omega_1) \to \mathcal{C}[0,\omega_1]$. Since i is an isometric isomorphism, the adjoint mapping i^* maps B onto B' and, of course, i^* is $w^* \to w^*$ continuous. It is easy to check that

$$i^*(\mu) = \mu \restriction [0,\omega_1) \text{ for every } \mu \in B, \text{ and}$$

i^* maps M onto B' in an one-to-one manner.

To prove the assertion we will need the following four claims.

Claim 1. *There exists a homeomorphic embedding h of (B, w^*) in some $\mathbb{R}^\Gamma$ such that $h(M) = h(B) \cap \Sigma(\Gamma)$.*

Claim 2. *The set $\widetilde{M} = \{\mu \in M \mid \|\mu\| = 1\}$ is weak*-dense in B.*

Claim 3. *The set $M_\varepsilon = \{\mu \in B \mid |\mu(\{\omega_1\})| \geq \varepsilon\}$ is weak*-nowhere dense in B for every $\varepsilon > 0$.*

Claim 4. *The set M' is weak*-countably closed but not weak*-closed in B'.*

Suppose that these claims hold and that B' is a Valdivia compact. Hence there is a homeomorphism $h : B' \to \mathbb{R}^\Gamma$ such that $h(B') \cap \Sigma(\Gamma)$ is dense in $h(B')$. Put $A' = h^{-1}(h(B') \cap \Sigma(\Gamma))$. Then A' is dense and countably closed in B'. By Lemma 2(ii) we get that $A = (i^*)^{-1}(A')$ is countably closed in B. Moreover, A is dense in B. Suppose not. Then, by Lemma 2(i), $i^*(\overline{A}) = B'$ and, by Claim 2, there is $\mu \in M$ of norm 1 such that $\mu \notin \overline{A}$. But we have $(i^*)^{-1}(i^*(\mu)) = \{\mu\}$, hence $i^*(\mu) \notin i^*(\overline{A})$, a contradiction. So the density of A in B is proved.

Further, since A is countably closed in the compact space B, we get that A is countably compact (and, of course, regular) and hence a Baire space (it is easy to see that every regular countably compact space is even "α-favorable"). It follows from the facts that M is residual (by Claim 3) and A a dense Baire subspace of B that $A \cap M$ is dense in B (see [6, I. 10 IV]). But M is an FU-space (by Claim 1 and Lemma 1), and A is countably closed, so $M \subset A$. But then $i^*(A) = B'$, thus $A' = B'$ and therefore B' is a Corson compact, which contradicts Claim 4 and Lemma 1.

Proof of Claim 1. We define $h : B \to \mathbb{R}^{[-1,\omega_1)}$ by the formula

$$h(\mu)(\alpha) = \mu([\alpha + 1, \omega_1]), \qquad \mu \in B, \alpha \in [-1, \omega_1).$$

Using the well-known fact that every Radon measure on $[0, \omega_1]$ is supported by a countable set, it is easy to check that h is a homeomorphism and that $h(\mu) \in \Sigma([-1, \omega_1))$ if and only if $\mu(\{\omega_1\}) = 0$, which was to be shown.

Proof of Claim 2. Choose a nonempty open subset U of B. Pick $\mu_1 \in U$. By the definition of the w^*-topology, there are $f_1, \dots, f_n \in \mathcal{C}[0, \omega_1]$ and $I_1, \dots, I_n$ open intervals of reals such that

$$\mu_1 \in U_1 = \{\nu \in B \mid (\nu, f_i) \in I_i, i = 1, \dots, n\} \subset U.$$

Since each f_i is constant on a neighborhood of ω_1, there is some $\alpha_1 < \omega_1$ such that f_i is constant on $[\alpha_1, \omega_1]$ for every i. Put

$$\mu_2 = \mu_1 - \mu_1(\{\omega_1\})\delta_{\omega_1} + \mu_1(\{\omega_1\})\delta_{\alpha_1+1}.$$

Then $\mu_2 \in U_1$ (since $(\mu_2, f_i) = (\mu_1, f_i)$ for every i) and, moreover, $\mu_2 \in M$. Hence there is some $\alpha_2 \in (\alpha_1, \omega_1)$ such that $\mu_2 \restriction [\alpha_2, \omega_1] = 0$. Choose two different ordinals $\beta, \gamma \in (\alpha_2, \omega_1)$ and put

$$\mu_3 = \mu_2 + \frac{1}{2}(1 - \|\mu_2\|)(\delta_\beta - \delta_\gamma).$$

Then clearly $\mu_3 \in U \cap \widetilde{M}$.

Proof of Claim 3. Let $\varepsilon > 0$ be arbitrary. Choose a nonempty open subset U of B. We will find a nonempty open subset V of U not intersecting M_ε which will yield that M_ε is nowhere dense.

Pick $\mu_0 \in U$. By the definition of the w^*-topology, there are $f_1, \dots, f_n \in \mathcal{C}[0, \omega_1]$ and $I_1, \dots, I_n$ open intervals of reals such that

$$\mu_0 \in U_1 = \{\nu \in B \mid (\nu, f_i) \in I_i, i = 1, \dots, n\} \subset U.$$

By Claim 2 there is $\mu \in \widetilde{M} \cap U_1$. Now, since μ is supported by a countable set, $\|\mu\| = 1$ and $\mu(\{\omega_1\}) = 0$, there are some $\beta_1, \dots, \beta_k < \omega_1$ such that

$$\sum_{i=1}^{k} |\mu(\{\beta_i\})| > 1 - \frac{\varepsilon}{2}.$$

Choose a continuous function $g : [0, \omega_1] \to [-1, 1]$ such that

$$g(\beta_i) = sgn\, \mu(\{\beta_i\}), \quad i = 1, \dots, k, \qquad \text{and} \qquad g(\omega_1) = 0,$$

and put

$$U_2 = \{\nu \in U_1 \mid (\nu, g) > 1 - \varepsilon\}.$$

Then U_2 is open and, moreover, $\mu \in U_2$. Indeed,

$$\begin{aligned}(\mu, g) &= \sum_{\eta<\omega_1} g(\eta)\mu(\{\eta\}) \geq \sum_{i=1}^{k} g(\beta_1)\mu(\{\beta_1\}) - \left| \sum_{\substack{\eta\neq\beta_i,\\ i=1,\dots,k}} g(\eta)\mu(\{\eta\}) \right| \\ &> 1 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} = 1 - \varepsilon.\end{aligned}$$

Suppose that there is some $\nu \in U_2 \cap M_\varepsilon$. Then we can choose $\alpha < \omega_1$ such that $\nu \restriction [\alpha, \omega_1] = \nu(\{\omega_1\}) \cdot \delta_{\omega_1}$, $g \restriction [\alpha, \omega_1] = 0$ and each f_i is constant on $[\alpha, \omega_1]$. Then we have

$$\|\nu\| \geq \big(\nu, g + sgn(\nu(\{\omega_1\})) \cdot \chi_{[\alpha+1,\omega_1]}\big) > 1 - \varepsilon + \varepsilon = 1,$$

which is a contradiction.

Proof of Claim 4. Let us define a mapping $F : B' \to [0,1]^{[0,\omega_1)}$ by putting

$$F(\mu)(\alpha) = \mu([0, \alpha]), \qquad \mu \in B', \alpha < \omega_1.$$

It is easy to check that F is a homeomorphism onto its image and that $F(M') = F(B') \cap \Sigma([0, \omega_1))$, so M' is countably closed in B'. However, the measure $\frac{1}{2}\delta_0$ which does not belong to M', is the limit of the net $\frac{1}{2}\delta_0 - \frac{1}{2}\delta_\alpha$, $1 \leq \alpha < \omega_1$, hence belongs to the closure of M'.

This completes the proof. $\square$

Proof of Corollary 1. Put $X = \mathcal{C}[0, \omega_1]$ and $Y = \{x \in X \mid x(\omega_1) = 0\}$. Then Y is isometric to $\mathcal{C}_0[0, \omega_1)$. By Theorem, B_{Y^*} is not a Valdivia compact. However, B_{X^*} is Valdivia due to Claims 1 and 2 in the proof of Theorem. $\square$

Proof of Corollary 2. It is enough to observe that $\mathcal{C}[0, \omega_1]$ and $\mathcal{C}_0[0, \omega_1)$ are isomorphic. Indeed, the mapping $F : \mathcal{C}[0, \omega_1] \to \mathcal{C}_0[0, \omega_1)$ defined by the formula

$$F(x)(\alpha) = \begin{cases} x(\omega_1) & \alpha = 0 \\ x(\alpha - 1) - x(\omega_1) & 0 < \alpha < \omega \\ x(\alpha) - x(\omega_1) & \omega \leq \alpha < \omega_1 \end{cases}$$

is clearly a linear bijection and $\|F\| = \|F^{-1}\| = 2$. $\square$

To prove Corollary 3 we need also the following lemma.

Lemma 3. *Let X be a Banach space having a countably 1-norming Markuševič basis. Then the dual unit ball B_{X^*} endowed with the weak* topology is a Valdivia compact.*

Proof. countably 1-norming Markuševič basis of X. Put

$$Y = \{f \in X^* \mid \{\alpha \in \Lambda, f(x_\alpha) \neq 0\} \text{ is countable}\}.$$

Then Y is 1-norming, i.e. $\|x\| = \sup\{f(x) \mid f \in Y \cap B_{X^*}\}$. Now it follows easily by Hahn-Banach theorem that $Y \cap B_{X^*}$ is weak* dense in B_{X^*}. Further the mapping $h : B_{X^*} \to \mathbb{R}^\Lambda$ defined by the formula

$$h(f)(\alpha) = f(x_\alpha), \qquad \alpha \in \Lambda,\ f \in B_{X^*}$$

is a homeomorphism onto its image (by the condition (ii) of definition of Markuševič basis) and $h(B_{X^*}) \cap \Sigma(\Lambda) = h(Y \cap B_{X^*})$ by the definition of Y, which yields the result. $\square$

Proof of Corollary 3. This follows immediately from Theorem and Lemma 3. $\square$

Remark 1. It follows from Lemma 3 and [9, Corollary 2.2] that $(B_{\mathcal{C}(K)^*}, w^*)$ is a Valdivia compact whenever K is Valdivia. However it seems not to be known whether the converse statement holds too. It is shown in [5] that it holds in certain special cases.

Remark 2. Using the general result of [5] mentioned in Remark 1 we can prove Corollary 1 and 2 directly from the example of [10]. In fact, in [10] there is constructed a non-Valdivia compact space L which is a continuous image of $[0, \omega_1]$. Now it is a standard fact that $B_{\mathcal{C}(L)^*}$ is a continuous image of $B_{\mathcal{C}[0,\omega_1]^*}$. By [5] the space $B_{\mathcal{C}(L)^*}$ is not Valdivia as L is scattered and not Valdivia. Further, $\mathcal{C}(L)$ is isometric to the subspace of $\mathcal{C}[0, \omega_1]$ of the form $\{x \in \mathcal{C}[0, \omega_1] \mid x(\omega_1) = x(\omega)\}$, which is also isomorphic to the whole space $\mathcal{C}[0, \omega_1]$. But the example in our Theorem seems to be more natural. (It can be easily shown that the space $\mathcal{C}_0[0, \omega_1)$ is not isometric to $\mathcal{C}(L)$. In fact, $\mathcal{C}_0[0, \omega_1)$ is not isometric to $\mathcal{C}(K)$ for any compact Hausdorff space K as the set of extreme points of $B_{\mathcal{C}_0[0,\omega_1)^*}$ is not weak* closed.)

Questions. It is proved in [4] that a compact space K is Corson provided every continuous image of K is Valdivia. In view of this natural questions arise.

(1) Suppose that $(X, \|\cdot\|)$ is a Banach space such that (B_{Y^*}, w^*) is a

Valdivia compact for every closed subspace Y of X. Is then (B_{X^*}, w^*) necessarily Corson?

(2) Suppose that X is a Banach space such that $(B_{(X,\|\cdot\|)^*}, w^*)$ is a Valdivia compact for every equivalent norm $\|\cdot\|$ on X. Is then (B_{X^*}, w^*) necessarily Corson?

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Added in proof. The question (2) was positively answered by the author in a forthcoming paper *Valdivia compacta and equivalent norms.* A partial positive answer to question (1), for the case $X = C(K)$ where K is a continuous image of a Valdivia compact, was obtained by the author, and will be included in the paper *Valdivia compacta and subspaces of* $C(K)$ *spaces.*

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