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ORIGINAL PAPER

Mathematics competitions: an integral part of the educational process

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Abstract
For a century and a half, the scene of mathematics competitions underwent a remarkable transformation from isolated and geographically scattered events to a full-scale and a full-featured vibrant global ecosystem comprising an impressive variety of competitions, school students, university students, teachers, mentors, scientists, schools, universities, research institutions, journals, websites, civil society organizations, educational authorities, parents, etc. The evolution, the current state, the functioning of this ecosystem as well as its role for the identification and development of talent, its impact on the educational process, and the institutions involved with it, is briefly reflected on. Some relatively new online competitions are presented that cultivate the use of dynamic geometry software systems for a deeper understanding of mathematical facts and phenomena, and for finding approximate numerical solutions to problems that are not part of a typical school curriculum, but often arise from real-life practice.

Keywords Mathematics competitions · Online competitions · Dynamic geometry software · Learning by inquiry

1 The world of mathematics competitions

The existence of a large number of mathematical competitions and challenges for school student is a phenomenon characteristic of the second half of the twentieth century. The primary goal of this paper is to presents several findings related to this phenomenon. Namely, its evolution from the origins to our days; its contemporary quantitative characteristics with respect to scope and diversity; the driving forces behind it and some of the pros and cons related to competitions.

The secondary goal of the paper is to present some problems that have been used in a new type of online competitions based on the use of dynamic geometry software systems or CAS calculators.

The current earliest mention of mathematics competition is from 1885. On May 21, 1885, a competition for primary school students took place in Bucharest, Romania (Berinde, 2004; Duca, 2015). The number of participants was 70 and 2 girls and 9 boys received prizes. The information about this competition concerning its organization, conduct, and the type of problems given, is scarce. The *Eötvös competition*[1] in Hungary that took place in 1894 is much better documented (Connelly Stockton, 2012; Kürschák, 1963; Wieschenberg, 1990). Koichu and Andzans (2009) regarded the Eötvös competition as "the first mathematical Olympiad of the modern world" (p. 287). The participants in this competition had to solve three problems within a timeframe of four hours. Year after year the problems in this competition were composed with the aim of testing creativity and mathematical thinking, not just technical skills. The participants had to provide a rigorous proof for the validity of the solution to many of the problems. Rich collections of problems appearing in the yearly issues of this competition can be found in John Scholes's collection (Scholes, 2003)[2] and in Ercole Suppa's collection (Suppa, 2007). This competition is considered to be a forerunner of contemporary mathematics and physics competitions for high school students. For many decades it served and still serves as a model for preparing and conducting many such competitions. Its role in the development and spread of mathematics competitions all over the world is difficult to overestimate. The Eötvös

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[1] Named after the prominent physicist Loránd Eötvös, who served his country, at different times, as President of the Hungarian Academy of Sciences, Founding member of the Hungarian Physics and Mathematics Society, Minister of Education.

[2] All online resources mentioned in the text have been accessed on 22.06.2021.

competition has taken place every year since its inception except for some years when the world was at war. According to Péter Vankó (Vankó, 2018), the competition was renamed the *Kürschák competition* while the name *Eötvös Competition* was retained for a similar competition in physics.

At the very beginning of 1894 (on January 1st) another significant event took place in Hungary. Dániel Arany, a teacher from Györ, began publishing a journal with the goal "of giving a wealth of examples to students and teachers" (https://www.komal.hu/info/miazakomal.e.shtml)[3]). The name of the journal under which it is widely known today is "KöMaL"—an abbreviation from the Hungarian name "Középiskolai Matematikai Lapok" (High School Mathematics Journal). Editors of the journal organized a competition for high school students by publishing problems in each issue of the journal. The student readers were invited to solve some of the problems and send their solutions to the journal. The best solutions together with the names of their authors were published in subsequent issues of the journal. Each correct solution brought points to the participant and a list of people who had collected the most points was published at the end of the year. The response to this initiative was remarkable. G. Berzsenyi (Oláh, 1999) noted that about 120–150 problems were published in KöMaL per year while the solutions received were 2500–3000. This type of competition, requiring effort all through the year on the part of the students, turned out to be very effective. Climbing the 'ladder of problems' during the whole year, many young people improved their knowledge of mathematics, developed their mathematical abilities, and took their first steps in research. Quite a number of the participants in the early years of this competition later became world-famous scientists who made important contributions to the development of science and society. It would be unfair not to mention here the names of some people, other than Arany, who managed the journal over the years and contributed to its brilliant success. László Rátz edited the journal in the period 1896–1914. World War I interrupted the publishing of the journal. The publication was renewed again in 1925 by Andor Faragó. He edited the journal until the beginning of World War II. In this period the name of the journal was changed to Középiskolai Matematikai és Fizikai Lapok (High School Mathematics and Physics Journal). After the war the journal was edited by János Surányi who became the first editor of its New Series. Very detailed information about KöMaL can be found at (https://www.komal.hu/lap/archivum.e.shtml). At present KöMaL publishes 9 issues per year. It is maintained by the Hungarian High School Mathematics and Physics Foundation, the János Bolyai Mathematical Society, and the Roland Eötvös Physics Society with the financial support of the Hungarian Ministry of Education. New problems have appeared, both in Hungarian and in English, starting more than 40 years ago, and are still accessible at Problems of KöMaL from the previous years (komal.hu). This is an essential contribution of Hungarian mathematicians to the development of mathematics worldwide. The model of the KöMaL competition has spread all over the world and today virtually every journal for teachers and school students has a section devoted to problem solving.

Similar activities took place, at approximately the same time, in Romania, another East-European country. The first issue of the monthly Gazeta Matematică was published on September 15th, 1895. A competition for school students was organized in its pages, giving rise to a very effective national system of competitions in Romania. Moreover, the journal laid the base for the establishment of the Romanian Mathematical Society. Dorel Duca (2015) mentions a competition for secondary school students in Romania conducted on June 25th, 1898. Soon other countries started to organize mathematics competitions as well. In Georgia, then a part of the Soviet Union (USSR), a mathematics competition was held in 1933[4]. The same year in the USA a mathematics competition took place between 10 students from Harvard and 10 students from the Military Academy at West Point. This competition is considered a predecessor of the famous Putnam Competition that started in 1938 (Gallian, 2017). A competition named 'Mathematical Olympiad' was organized in 1934 in Leningrad, USSR, (now St. Petersburg, Russian Federation). Somewhat later a similar event took place in Moscow and soon after that in other places in the USSR. A Mathematical Contest with 6000 participants from 238 schools in the New York area took place on May 11th, 1950 (Turner, 1985). It was organized by the Metropolitan New York Section of the Mathematical Association of America. In 1955 the number of schools participating in this competition rose to 881. Bulgaria had its first National Mathematics Olympiad in 1949, Serbia in 1959, the German Democratic Republic in 1960, Spain in 1965, Austria in 1969, and the USA in 1972. The first International Mathematics Olympiad (IMO) gave a strong impulse to the development of the system of mathematics competition worldwide. It was initiated and organized in Romania in 1959. Jainta (2000) wrote that "…IMO, the pinnacle of competitions among individuals, was the brainchild of Romania's Tiberiu Roman, an educator of monumental vision…" (p. 20). The first IMO had 52 participants (contestants) from seven countries, namely, Bulgaria, Czechoslovakia, German Democratic Republic, Hungary, Poland, Romania, and the Soviet Union (USSR).

[3] However, at (https://www.komal.hu/a_lap.e.shtml) it is stated that "The first issue of KöMaL appeared in 1893.".

[4] (https://ru.wikipedia.org/wiki/) (Search term: Математическая олимпиада).

The number of participants gradually increased. In 1988 IMO was organized in Canberra, Australia. There were 268 participants from 49 countries. In 2005, IMO took place in Mexico. There were 513 participants from 93 countries. In 2019, in Bath, United Kingdom, the number of participants was 621 (coming from 112 countries). Because of COVID'19, the 2020 IMO was conducted online.[5] It was hosted in St. Petersburg, Russian Federation and included 616 competitors from 105 countries.

The IMO, like the Olympic Games, is a competition for individuals. Competitors are ranked according to their score in points. Multiple individual medals are awarded on this basis. By totaling the points obtained by the participants from a certain country and comparing it with the total of other countries, one gets an idea about how well the competitors from this country are prepared. Even more, this unofficial ranking of the countries is used (without proper justification) as an informal indicator of the level of the educational system in a given country. This generates, almost automatically, a competition between countries for better performance at IMO. Preliminary preparation of the competitors is needed in order to get better results at IMO. This has sparked countless competitions, competition-like activities, and other mathematical enrichment events in many countries. Soon it became clear that the selection of gifted students and the preparation for IMO should be based on larger participation and should start much earlier. This was one of the reasons for the appearance of mass competitions with participants from early school years.

Today the world of mathematics competitions, considered together with all the mathematics enrichment activities, is truly large. To get a rough quantitative idea about all mathematics competitions a search with Google Chrome was undertaken at the end of 2020. The search term *mathematics competition* returned 150 million entries! The same search term in the Spanish language (*competencia matematicas*) produced 50 million hits. The search results in some other widely spoken languages were also in the area of tens of millions. These spectacular numbers are not quite reliable and should be interpreted with a degree of caution. One and the same competition can be commented on and posted on the Web by many participants, teachers, parents, organizations, etc. There were entries in the list in which the word *mathematics* was in one sentence and the word *competition* in another. A few days later the same search 'produced' different numbers, but still of the same magnitude. Despite these shortcomings, the search does provide a kind of upper bound for the size of the *world of mathematics competitions*. To get a lower bound, a more restrictive search was conducted with the search term *"mathematics competitions"* (pay attention to the quotes which make the search machine consider the search term *"mathematics competitions"* as one word, not as a union of two words). This search gave the results exhibited in Table 1.

These numbers leave no space for doubts about the size of the *world of mathematics competitions*. It is huge, spans globally, and does not consist of competitions only. It comprises thousands of mathematical enrichment events such as summer schools, training camps, mathematics circles, mathematics leagues, journals, online resources, preparatory materials, and much more. People engaged with such activities are teachers, university professors, researchers, as well as people for whom this is a professional and social realization. The activities are organized by schools, educational authorities, universities, non-governmental organizations (for instance, national mathematical societies), and private enterprises. People and organizations working in this area collaborate. They develop and exchange ideas, problem-solving techniques, resources, organizational know-how, etc. There are organizations (for instance, the World Federation of National Mathematics Competitions, (http://www.wfnmc.org/), and the mathematical societies in many countries) that facilitate this collaboration. In analogy with the use of the term "business ecosystem" as *a group of businesses or business activities that affect each other and work well together* (see (https://dictionary.cambridge.org/)), it is proper to say that mathematics competitions and accompanying activities, taken together with the organizations and people behind

Table 1 Results of a search for "mathematics competitions"

Language	Search terms	Number of hits in thousands	Search terms	Number of hits in thousands
English	Mathematics competition	707	Mathematics challenges	84.2
Spanish	competencia de matematicas	526	desafíos matemáticos	408
French	concours de mathématiques	797	défis mathématiques	44.5
German	Wettbewerb mathematik	22.5	mathematische Herausforderungen	45.3
Chinese	数学比赛 (math competition)	508	数学挑战 (math challenges)	43.8

[5] See (https://www.imo-official.org/).

them, form a unique and dynamic global ecosystem. Interesting new forms/events for attracting young people to learn mathematics and to develop their mathematical abilities constantly appear, grow, involve more and more students and, sometimes, disappear. Students come into the ecosystem, get mathematics enrichment and leave the system to enter university or the world of work. The involvement of teachers, scientists, and organizations is also dynamic. The diversity of the ecosystem is dynamic as well. The changes are mainly towards increasing diversity.

The above numbers generated by Google Chrome searches give an impression of the volume of the activities but not about the content. Today there exists a truly rich variety of different competitions and mathematical enrichment activities. There are 'open for all' (inclusive) events that are oriented to all students. There are 'by invitation only' (exclusive) events targeting talented students only (as is the case with IMO and many national events related to the selection and preparation of the best math students). The majority of the competitions are for individuals but there is an increasing trend of conducting competitions for teams. There are 'mixed type' competitions that comprise both 'individual' and 'team' parts. There are 'multiple-choice' tests as well as classic style competitions in which students have to present proofs for the solutions of the problems. The competitions organized by KöMaL and Gazeta Matematiča fall in the group of 'non-presence' competitions, in which the students do not necessarily meet each other. Until 2020 the most frequent competitions were of the type 'presence competitions' in which the participants were sitting together at a place where the competition (or part of it) was conducted. In the last decade, some online mathematics competitions appeared. The experience gathered with these online competitions turned out to be very useful in the time of the pandemic caused by COVID-19. In order to circumvent the pandemic restrictions, and, sometimes, to survive, many mathematics competitions had to go online in 2020. The Google Chrome search with the term *online math competition* returned in January 2021 more than 9.27 million hits.

In order to get an approximate idea of what the ratio is between the numbers of different types of competitions, profiles of 110 pseudo-randomly selected competitions were compiled with respect to the following indicators:

- Established since (year).
- Type of competition (yearly event, several events—rounds—per year).
- Type of competition (international, regional, national).
- Competition for (individuals, teams, both).
- Age of participants (school years 1–4, school years 5–7, school years 8–12, all school years, school years 1–7).
- Organizer (international body, state educational authority, university, private organization, civil society organization—NGO, other).
- Number of participants (1–500, 501–1000, 1001–2000, 2001–10,000, above 10,000, not known).
- Type of answers (classic solutions with proofs, multiple choice answers, mixed type).

The selection of competitions was amongst those presented in the world wide web in the English language. Despite the relatively small size of the sample (110 competitions only) and the incomplete data for some of them, the features of the whole set of competitions do leave traces in the sample and, looking at the features of the sample we can derive conclusions for the whole set of competitions (all public opinion polls rely on such an assumption). For instance, in Fig. 1 one can see how many competitions from the sample were started in a decade beginning in a specific year.

The diagram suggests that a real increase in the number of new competitions began in the 60s of the last century and reached its highest levels in the decades between 1980 and 2010. This confirms the opinion expressed earlier that IMO contributed to the increase of the number of competitions worldwide. The last column in the diagram indicates a decrease in the number of new competitions in the decade beginning in 2010. Further and more detailed research

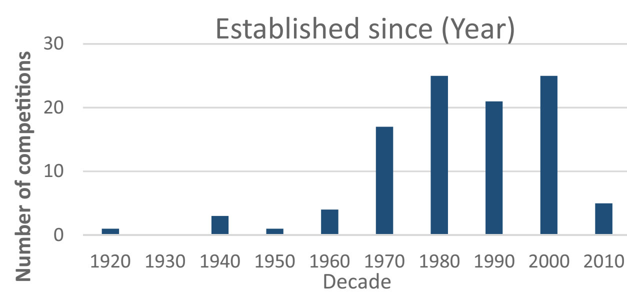


Fig. 1 Competitions starting in a decade beginning with a specific year

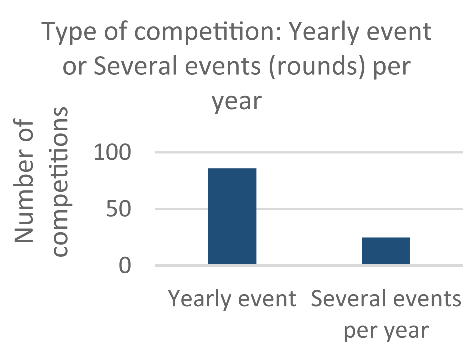


Fig. 2 Rounds per year

(with larger samples of competitions) is needed to see if this decrease is a permanent trend.

The diagram (Fig. 2) gives an idea of the number of competitions conducted once a year and the competitions which have several runs (rounds) per year. The relation between them is approximately 4:1 in favor of 'once a year' events.

The other diagrams refer to the other indicators in the profiles of the competitions from the sample. Most of them are easy to explain. For instance, Fig. 3 shows that the majority of competitions are national (for students from one country only), while Fig. 4 demonstrates that competitions, where students compete as individuals, are most widely spread. Figure 5 shows that the competitions for students from grades 8 to 12 are dominating the competition scene. Similarly, Fig. 6 shows that competitions organized by civil society organizations (NGO's) are

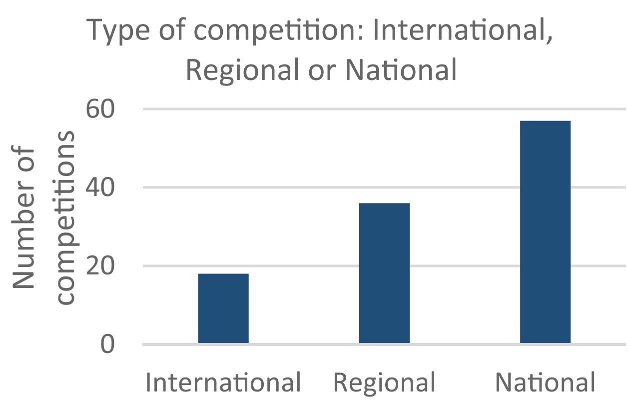


Fig. 3 Scope

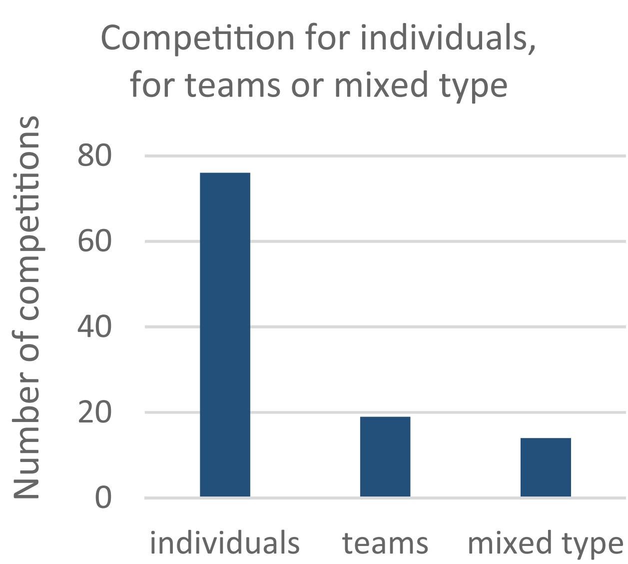


Fig. 4 Type

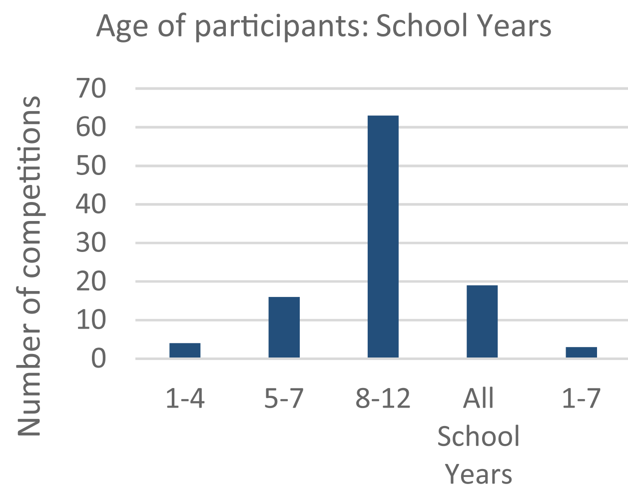


Fig. 5 Age of participants

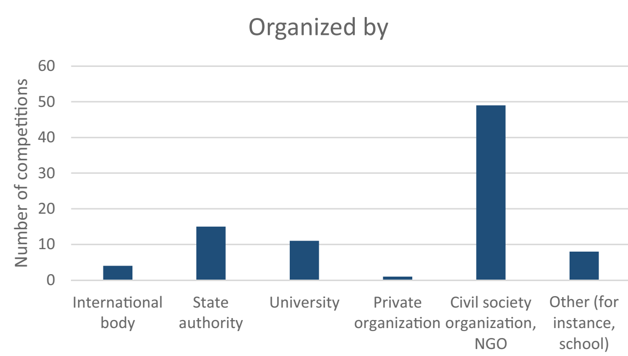


Fig. 6 Organization

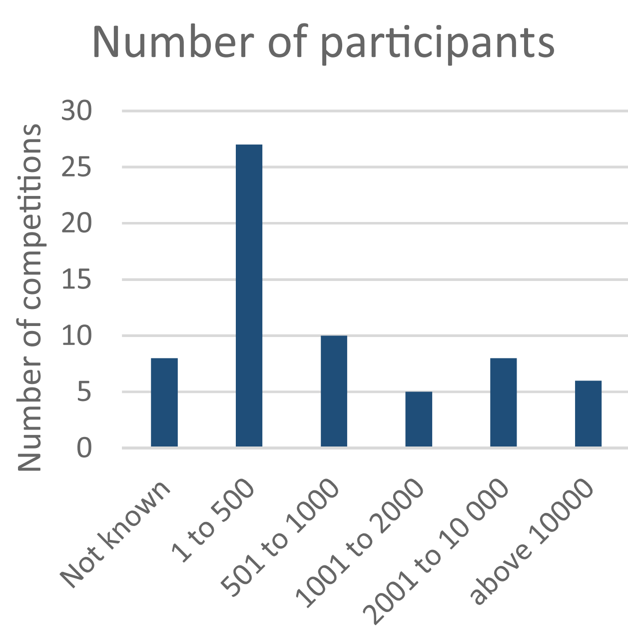


Fig. 7 Participation

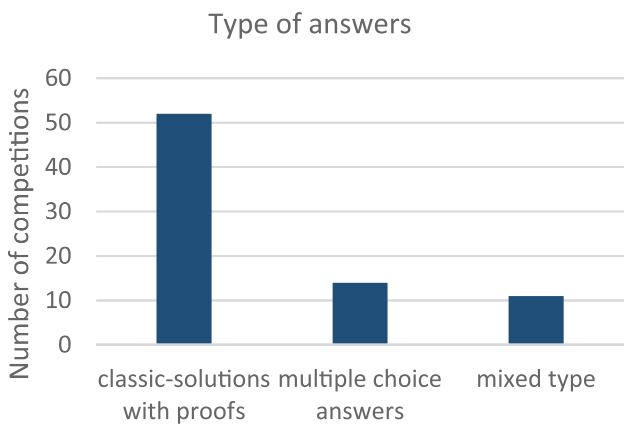


Fig. 8 Type of answers

most frequent. This confirms the almost evident fact that national mathematical societies are heavily involved with competitions. Figure 7 suggests that most often there are up to 500 participants in a competition.

Finally, Fig. 8 shows that the classical style of competitions requiring justification of the solutions is still well represented in the ecosystem of competitions.

The World Compendium of Mathematics Competitions compiled by Peter O'Halloran (1992) gives a good overview of the world of mathematics competitions in 1992. Information about the situation in 2007 in a single country (Bulgaria) can be retrieved from the site of the Union of Bulgarian Mathematicians (UBM, 2007). It contains a description (in English and in Bulgarian) of all national and international competitions in which Bulgarian school students participate: Mathematics (20 competitions), Informatics (5 competitions), Mathematical Linguistics (5 competitions), and Information Technologies (5 competitions).

The branch of mathematics that is the scientific base for the functioning of this ecosystem is classical mathematics, known also under the somewhat misleading name 'elementary mathematics'. Here is what the world-famous mathematician Terrence Tao, a former successful participant in IMO, wrote in his blog (Tao, 2014):

> Also, the "classical" type of mathematics you learn while doing Olympiad problems (e.g. Euclidean geometry, elementary number theory, etc.) can seem dramatically different from the "modern" mathematics you learn in undergraduate and graduate school, though if you dig a little deeper you will see that the classical is still hidden within the foundation of the modern. For instance, classical theorems in Euclidean geometry provide excellent examples to inform modern algebraic or differential geometry, while classical number theory similarly informs modern algebra and number theory, and so forth.

One of the many side effects of the existence of the mathematics competitions ecosystem is that this branch of mathematics is kept alive and popular among the young generations.

2 Why and how did mathematics competitions appear?

Competition, understood in a broader sense, is one of the essential characteristics of life. Plants compete for more light and moisture. Animals compete for more food, more favorable environment, and procreation. The modern economy is based on competition between economic actors. Countries are competing for more investment, for stronger influence in the world. People compete with each other for a higher position in the formal and/or informal social hierarchy. Competition plays an important role in evolution as well. The desire to have more, to be stronger, to be better seems deeply rooted in the 'mind' of every living creature. Because of this, competitions have long been used as a tool for managing social processes and improving the results thereof. Education, seen as a social process, is no exception. It was observed, already in ancient times, that through competition between learners better educational results could be achieved. Verhoeff (1997), who reflected on the role of competitions in education, wrote as follows:

> Marcus Verrius Flaccus, a Roman teacher famous in the late 1st century BC, is credited to have introduced the principle of competition among his students as a pedagogical aid. He awarded attractive books as prizes. The Italian scholar Battista Guarino (1434–1513) writes in his account of proper educational techniques, De ordine docendi et studendi, that teachers should refrain from physically punishing pupils, and that students are stimulated best by competition, which can be intensified by pairing them off (p. 4).

Grading/marking during the school year and at final examinations is perhaps the oldest widely used tool for generating competition between learners. However, it took a long time before school assessment was transformed into a mathematical competition in the modern sense of the term. The proper conditions appeared and matured only in the second half of the nineteenth century. They were the result of the rapid development of mathematics in the eighteenth and nineteenth centuries and the evolving understanding of the important role mathematics would play in science and public life. This is one of the reasons for the emergence of mathematical societies and associations in different countries.

Table 2 Years of appearance of mathematical societies

Name of the society/association	Established since
London Mathematical Society	1865
Finland Mathematical Society Suomen matemaatinen yhdistys	1868
French Mathematical Society Société mathématique de France	1872
Mathematical Society in Denmark Matematisk Forening	1873
German Mathematical Society Deutsche Mathematiker-Vereinigung	1890
Mathematical Association of America	1888
Palermo Mathematical Circle Circolo matematico di Palermo	1884
Romanian Mathematical Society Societatea de Stiinte Matematice din Romania	1895
Austrian Mathematical Society Österreichische Mathematische Gesellschaft	1903
Greek Mathematical Society Ελληνικ Μαθηματικ Εταιρεα	1918
Hungarian Mathematics and Physics Society Matematikai és Fizikai Társulat	1891
Physics and Mathematics Society in Bulgaria Физико-математическото дружество в София	1898

They, in turn, supported the development of mathematics education and prepared the ground for the emergence of mathematical competitions. The following Table 2, taken in part from Ramskov (2000), shows the appearance of some mathematical societies/associations.

There is a remarkable similarity in the goals and activities of these organizations. All set the development of education and science as their main and primary goals. Most of them started publishing a journal as an instrument to reach these goals. For some of these associations, the emergence of the journal preceded, and to a large extent provoked, the emergence of the organization itself. Such is the case, for example, with the Romanian Mathematical Society. The date September 15, 1895, of the publication of the first issue of Gazeta Matematică, is considered as the birthday of the organization. The Mathematical Association of America was officially registered in 1915, but its American Mathematical Monthly, one of the most respected and widely read journals, appeared in 1894. Once it was realized that the future of mathematics as science depends on attracting talented students, then there was only one step left to organizing a competition: the talent had to be discovered and developed before it could be attracted to mathematics as a profession. Also, there was another, even more important, reason for the appearance of mathematics competitions. Starting from the beginning of the twentieth century a much larger portion of the young population got access to higher education. Accordingly, the educational standards and requirements, the educational process itself, had to be tuned to the needs of the majority of students, i.e., to the students with average abilities. Bright students covered easily the standard requirements and had no incentives to work hard and reach higher results. This had a negative effect on the development of their abilities and talent. Unlike natural resources (such as ore, oil, etc.), which, if not discovered and developed, can still be utilized in the future, a young person's talent and abilities disappear forever as a social resource, if not discovered and developed in time. Competitions were a good remedy against this shortcoming of the educational system. They got an important role in the educational process and established themselves as an integral part of the educational system.

3 Mathematics competitions. Pros and cons.

Debate 'for' and 'against' competitions has accompanied the competition ecosystem from its very beginning. There is an extensive body of literature on this debate, and hardly anything substantial can be added to what has already been said or written (see, for instance, Georgiev et al., 2008; Rusczyk, 2005; Dilcher, 2013; Swaminathan, 2013). At the moment, the following points seem to have general agreement:

- Competitions are a good motivating tool for independent work and in-depth study of mathematics on the part of students (and, sometimes, on the part of teachers).
- The preparation of students for the competition has a significant educational impact. Solving difficult tasks not only generates better knowledge, but also cultivates skills for dealing with problems of all kinds, not only mathematical ones.
- Through competitions, the mathematical abilities of young people are discovered and developed.
- From their inception up to today, competitions perform the function of a 'social elevator'. Through them, children coming from lower socio-economic backgrounds can receive attention, financial support, and better chances to enroll in a prestigious university.
- The universities themselves are also interested in getting higher ability students. Such students serve as 'role models' and 'reference points'. They help other students in the process of education. In addition, having such students in the group provokes more effort and dedication on the part of the teachers. This raises the level of the educational process at the corresponding university and contributes to the improvement of its image. Moreover, after completing their studies the graduates from such a university make good impressions on the society and this elevates the prestige of the institution (Kenderov, 2006).

- Competitions may help to teach participants how to deal with poor performance and even failure. One cannot expect always to be very successful in solving competition problems. Knowing how to accept and learn from a poor performance is useful for future life.
- Last but not least, competitions provide an opportunity to test the reactions of students when confronted with unknown material, new educational approaches, technologies, and other things that have to be implemented in the educational process.

Proponents of the thesis that competitions are harmful also have solid arguments:

- Competitions create a stressful atmosphere, because the problems have to be solved in a short time and in the presence of other competitors.
- Since many competitions are conducted once a year, their educational impact is limited in time—only to the period during the intensive preparation for the competition and the few hours of the competition itself.
- Excessive striving to win a prize at any cost is counterproductive. It gives rise to individualism, and today it is known that teamwork is an important skill that largely determines the social realization of the young person in the future.
- Competitions at school level convey an inadequate impression that mathematics is a collection of problems.
- Competitions cultivate the ability to answer questions and tasks posed by others (for instance, by the jury). In research, however, it is very important to ask questions that are relevant to the object studied and can be answered with existing knowledge and available technologies.
- Many competitions prohibit the use of aids and technology. In real life, however, any available means can be used to solve the problem at hand. Moreover, the vast majority of contemporary applications of mathematics rely on numerical approximations and the use of computers. This is still not adequately reflected by the system of competitions.

The ecosystem of mathematics competitions responds to these criticisms by permanently improving and diversifying its activities. Now there are many mathematical knowledge-enriching activities that are based on participation and games which are not stressful. Students are not fighting with peers but with the set of tasks or challenges.

Many competitions today have several rounds per year. Preparation for them requires long-term engagement on the part of students. Correspondingly, the educational impact is higher.

In response to critics, many 'individual type' competitions incorporated also a 'team phase'. Many purely team competitions have also appeared.

In numerous enrichment activities (summer schools, special classes, etc.) the emphasis is not only on problem solving but also on problem composition and on acquiring theoretical knowledge, which gives students the opportunity to get acquainted with the real face of mathematics. Moreover, project-based competitions (in which students report, in front of a jury and public, on the results obtained by working on a research or research-like project) offer the opportunity to identify students with *inclination toward research*—those who can *pose and answer the right questions.* The theme of the project is usually determined by the student or by the mentor who supervises the work on the project. The roots of these competitions go back to the very beginning of the 20th century when Science Fairs were organized on a regular basis in the USA on many occasions. Several such competitions were briefly described by Kenderov (2006). One of the pinnacles of such competitions is the famous INTEL International Science and Engineering Fair (ISEF) which comprises more than 1500 participants every year. Since 2020 the name of this event is REGENERON ISEF (https://www.societyforscience.org/isef/).

4 What additional competitions are needed today?

One of the essential characteristics of our time is the penetration of computers in all spheres of public life. Education is not an exception. With the advent of powerful computers and the corresponding software for handling and processing mathematical objects, a gradual transformation of mathematics education began at all levels. We are witnessing how the use of CAS calculators and/or software systems such as Logo, GeoGebra, Cabri, Geonext, Cinderella, Matlab, Maple, Mathematica, and many other systems change the way we teach and learn mathematics. CAS calculators and the mentioned software systems are, in fact, full-featured mathematical laboratories. They equip students with opportunities not only to visualize and play with mathematical objects but also to discover some of their basic properties through experimental exploration. This facility makes teaching and learning mathematics similar, at least to some extent, to teaching and learning other scientific disciplines—through experimentation and inquiry. Simultaneously, the use of such systems cultivates digital skills needed for the World of Work (Maass & Engeln, 2019) and computational/algorithmic thinking which gradually become an important part of contemporary literacy (Freiman et al., 2009;

Stephens & Kadijevich, 2020). With the help of technology, it is possible to offer to students much more demanding mathematical content and interesting applications (Hoyles & Lagrange, 2009). Clark-Wilson et al. (2020) gave a very informative and meaningful analysis of trends in the use of modern technologies in mathematics education, with an emphasis on the work of teachers.

This transformation of mathematics education toward the use of contemporary technology can be supported by organizing new competitions in which digital devices are used in the process of exploring and solving competition problems. Such competitions would attract students with a natural inclination to 'construction and calculation'. We should not forget that the major role of education is to prepare youngsters for future life in society. Society and its members have at their disposal today ubiquitous computational resources and mathematical laboratories by means of which many tasks arising in everyday life can be solved. It makes sense to acquire skills for using such mathematical laboratories at school age as well. Such skills will increase the chances for young people to reach more noteworthy social and professional achievements in the future. It will be highly beneficial also for those who continue their education in STEM and AI disciplines at the university level, as computational aspects of science and AI become more and more important.

Let us consider some simple examples of problems that could be given at such competitions. They are mainly variations of known mathematical problems and illustrate the difference between a traditional mathematical solution of a problem and a solution using software. The latter can be grasped and implemented by larger groups of students at a much younger age.

Problem 1. An Exercise Book for 5^{th} grade contains 100 problems numbered from 1 to 100. One day, at the end of the classroom session, the teacher told his students: "For homework, you have to solve three consecutive problems from the exercise book. The product of their numbers is 21924. Find out which those problems are and solve them."

Thus, the first task for the student is to figure out which those three consecutive problems are. From a mathematical point of view, this task reduces to solving in positive integers the equation $x(x+1)(x+2) = 21924$. Solving cubic equations 'by hand' is not an easy task even for students in the last years of upper secondary school. It is not difficult, however, to find the solution using a CAS calculator. Also, a 5^{th} grader with very initial skills in using GeoGebra would construct a slider for the integer n (ranging from 1 to 50) and write in the input field (the command field) the expression $n(n+1)(n+2)$. The execution of this command

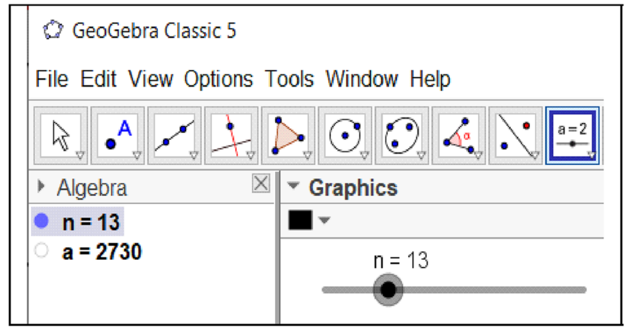


Fig. 9 The GeoGebra screen

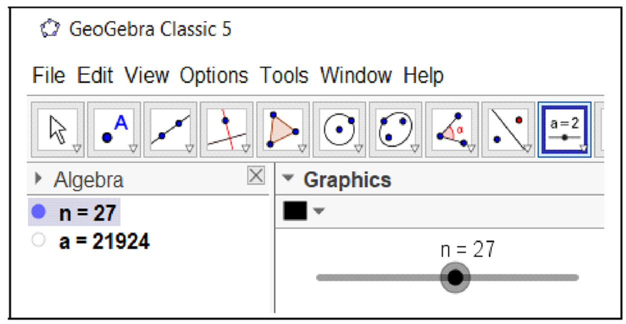


Fig. 10 The solution

calculates the product $n(n+1)(n+2)$ for the value of n which is currently on the slider and immediately inscribes the result in the algebraic window of GeoGebra. In Fig. 9 the case $n = 13$ is presented. The result of the calculation is $a = 2730(= 13 \times 14 \times 15)$. It is written in the left (algebraic) window of GeoGebra. Changing n, by dragging the point on the slider, one finds that for $n = 27$ the product is $a = 21924(= 27 \times 28 \times 29)$ (Fig. 10). Hence, the homework assigned was to solve the problems numbered 27, 28, and 29 in the exercise book.

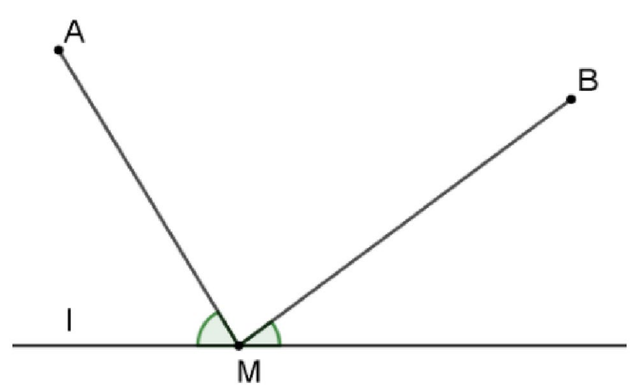


Fig. 11 Original problem

The solution we just presented is based on a ‘computer model’ of the problem (calculating $n(n+1)(n+2)$ for a given n), which automatizes the experimentation with different values of n.

As another example, we use a modification of the well-known problem when two points A and B as well as a line l are given in the plane (as in Fig. 11) and the task is to find a point M on l such that the sum $MA + MB$ is minimal. It is known that this sum will be minimal when the angles between l and each of the segments MA and MB are equal. Equivalently, the sum will be minimal when M is the intersection of l with the segment AB', where B' is the mirror image of B with respect to l(not shown in Fig. 11).

Here is the modified problem.

Problem 2. Given a point A, a line l, and a circle k in the plane, find a point M on l such that the sum MA + MC is minimal, where the segment MC is tangent to k (Fig. 12).

The first idea that comes to mind is based on the analogy with the original, unmodified, problem: the point M we are looking for should coincide with the intersection of l and the tangent from A to the mirror image of the circle k

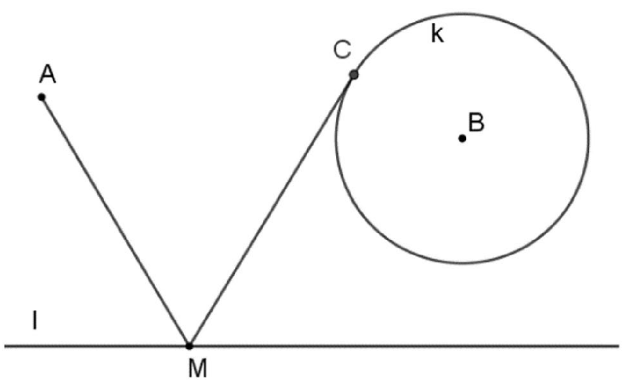


Fig. 12 Modified problem

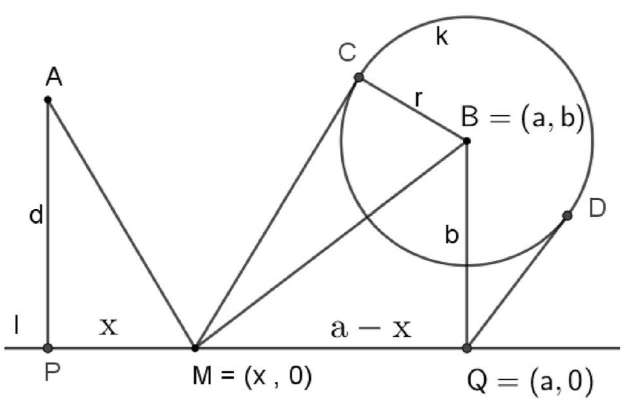


Fig. 13 A minimizer x

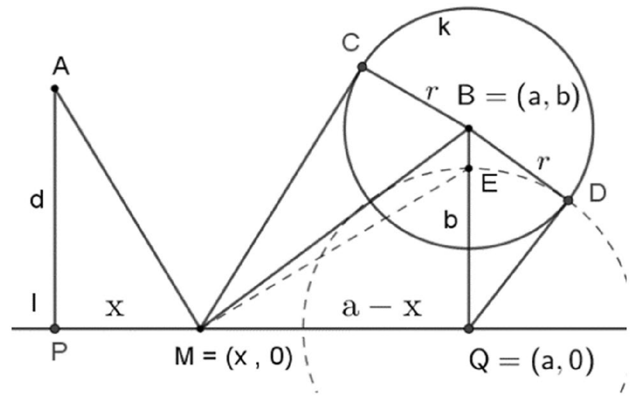
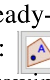
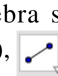
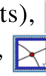


Fig. 14 Further analysis

with respect to l. This is, as we will see, misleading. With the notation from Fig. 13, the mathematical model of this problem is:

Find x for which the expression $AM + MC = \sqrt{d^2 + x^2} + \sqrt{(a-x)^2 + b^2 - r^2}$ is minimal.

One can find the minimizer x by means of calculus (using derivatives). There is also a simple geometric strategem which reduces Problem 2 to the original problem with two points. To present this, let D (Fig. 14) be the common point of k and a tangent from point Q to circle k. Then $QD^2 = QB^2 - BD^2 = b^2 - r^2$. If E is a point on the segment QB such that $QE = QD$, then $MC = \sqrt{(a-x)^2 + b^2 - r^2} = \sqrt{(a-x)^2 + QE^2} = ME$ for every M. Hence the sum $AM + MC = AM + ME$ will attain its minimum when M is the intersection of l and the segment AE', where E' is the mirror image of E with respect to l (not shown in Fig. 14). The mathematical essence of this strategem is the fact that the line l is the radical axis (Coxeter & Greitzer, 1967) of the circle k and the point E (considered as a degenerated circle).

Calculus, geometric strategem presented here, or the notion ‘radical axis’ are hardly widely known in the lower secondary school. On the other hand, a student from such a school who knows the basics of GeoGebra could make a ‘computer model’ of Problem 2 by using the ready-made instruments (buttons) of GeoGebra such as:  (for selecting a point M from the line l),  (for drawing and measuring segments),  (for construction of tangents from M to circle$k$),  (for intersecting two lines). The only hand-written command is to calculate the sum $MA + MC$. The result is automatically inscribed in the algebraic window. By dragging point M along l and observing the change of the sum $AM + MC$ one can find, with satisfactory precision, the location of M for which $AM + MC$ is minimal. If $a = 10, b = 5, d = 6$, and $r = 3$, then the minimal value will be attained for $M = (6, 0)$.

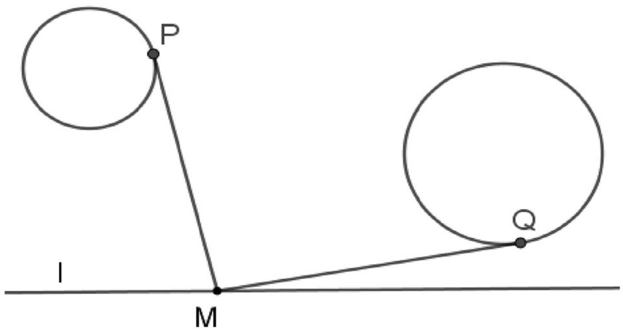


Fig. 15 A derived problem

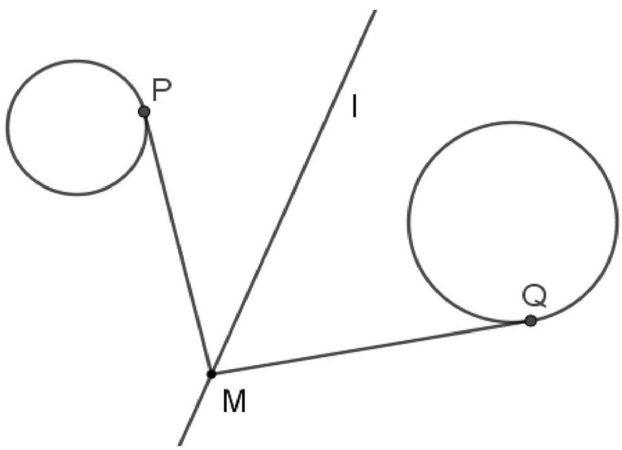


Fig. 16 Another variation

One can solve in a similar way the natural generalizations of Problem 2 represented in Figs. 15 and 16.

There is at least one more reason for introducing new competitions based on mathematical laboratories such as GeoGebra. For many problems related to real life, there is a suitable mathematical description (i.e., *mathematical model*) based on school mathematics. Solving this mathematical model within the framework of school mathematics, however, is not always possible. This is the reason for not considering such problems in school. This situation conveys the wrong impression that mathematics studied at school is not applicable. Very often, however, the mathematical model of the problem can be solved easily and rapidly with satisfactory precision by means of software such as GeoGebra. This opportunity can be demonstrated and exploited by means of a competition where digital devices are allowed. This will enlarge considerably the set of problems with practical significance that could be considered at school age. In this way, the appeal of mathematics among young people would increase and the eyes of many of them would be opened to the beauty of STEM studies. In addition, considering such problems at school age can, at least to some extent, counteract the trend observed in STEM education to understate the role of mathematics in it (Maass et al., 2019). The following is one such problem that is related to optimization. Many other similar problems can be found in the papers by Kenderov (2018), Kenderov et al. (2015) and Kenderov and Chehlarova (2015).

Construction of a bowl A truncated cone-shaped bowl (Fig. 17) with a circular base of radius $r > 0$ has to be produced from a circular plastic sheet of radius $l > r$ by cutting and gluing (sticking). The construction steps are as follows:

1. Cut from the plastic sheet a concentric circle of radius r (Fig. 18). It will serve as a base for the bowl;

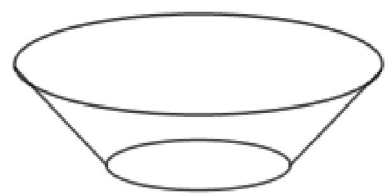


Fig. 17 The bowl to be constructed

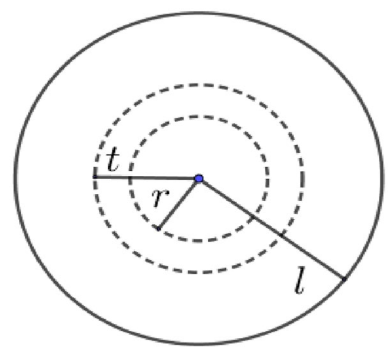


Fig. 18 Steps 1 and 2 from the construction of the bowl

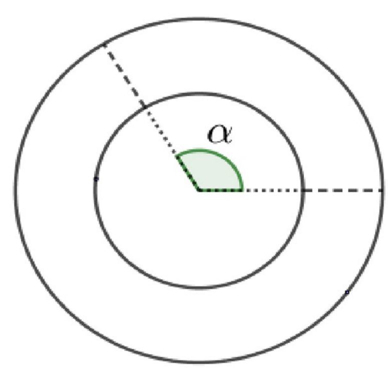


Fig. 19 Step 3 from the construction of the bowl

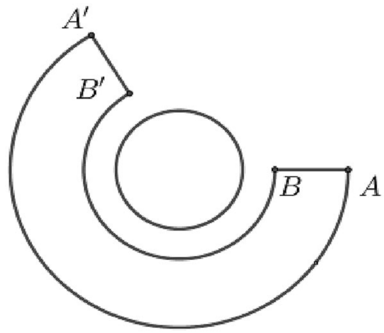


Fig. 20 Gluing the segments AB and A'B'

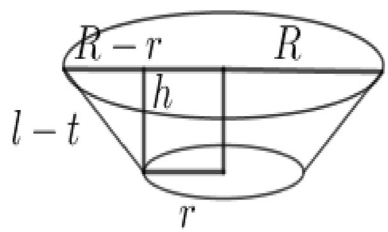


Fig. 21 Mathematical model of the problem

2. Take a number t such that $r < t < l$ and cut from the remaining piece of plastic sheet a concentric circle of radius t as shown in Fig. 18;
3. Cut a circular sector of measure $\alpha = 360\left(\frac{t-r}{t}\right)$ (in degrees) from the remaining circular strip of width $(r - t)$ (as in Fig. 19).

What we are left with is depicted in Fig. 20. By twisting the non-circular part and gluing (sticking) the segments AB and$A'B'$, we get the truncated cone from Fig. 17. Because of the special choice of α, we have $t\left(1 - \frac{\alpha}{360}\right) = r$. Note that the length of arc BB' (clockwise) is equal to $2\pi t\left(1 - \frac{\alpha}{360}\right) = 2\pi r$. This means that the circle cut in step 1) can serve as a base for the just constructed truncated cone. In this way, we constructed, for every t such that $r < t < l$, a bowl with a circular base of radius r.

Problem 3 What is the largest possible volume of the bowl thus constructed if $l = 4\,dm$ and $r = 1.5\,dm$?

The radius R of the upper base of this truncated cone can be determined from the equation $2\pi l\left(1 - \frac{\alpha}{360}\right) = 2\pi R$ which is equivalent to $R = \frac{lr}{t}$. Using the notation from Fig. 21, the altitude h of the truncated cone is $h = \sqrt{(l-t)^2 - (R-r)^2} = (l-t)\sqrt{1 - \left(\frac{r}{t}\right)^2}$. The formula for the volume V of a truncated cone is $V = \frac{\pi}{3}\left(R^2 + Rr + r^2\right)h = \frac{\pi}{3}\frac{r^2}{t^2}\left(l^3 - t^3\right)\sqrt{1 - \frac{r^2}{t^2}}$. Finding the maximum of this function of t is discouraging even for university students and is hardly suitable for considering in school. At the same time, putting $l = 4$, $r = 1.5$, and experimenting with GeoGebra or another similar system, one finds that the largest possible volume is $23.3833\ dm^3$ which is obtained when $t = 1.7968\,dm$.

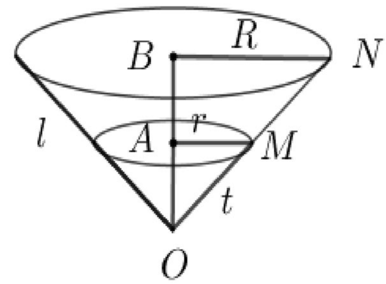


Fig. 22 Computer model of the problem

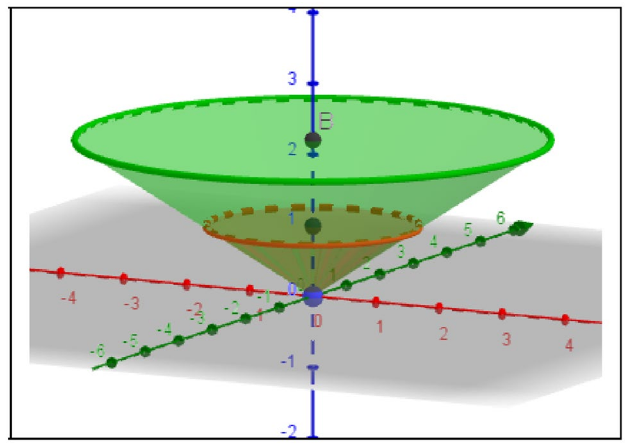


Fig. 23 Visualization of the optimal bowl

In order to see what the optimal bowl looks like, we can make another model of the problem by using the 3D version of GeoGebra. Given the numbers $r < l$ one makes a slider for the variable t (with endpoints r and l). Next, we have to construct two cones (Fig. 22). The first one has vertex O, a circular base centered at B of radius $R = \frac{lr}{t}$ and altitude $OB = \sqrt{ON^2 - NB^2} = \sqrt{l^2 - R^2} = l\sqrt{1 - \frac{r^2}{t^2}}$. The second cone is similar to the first one. It has vertex O, a circular base centered at A with radius r, and altitude $OA = \sqrt{OM^2 - MA^2} = \sqrt{t^2 - r^2} = t\sqrt{1 - \frac{r^2}{t^2}}$. The execution of the command Cone(B,O,R) exhibits the first cone and automatically calculates its volume $V1$. Analogously, the command Cone(A,O,r) exhibits the second cone and calculates its volume $V2$. The volume of the truncated cone is $V = V1 - V2$. Setting $l = 4$, $r = 1.5$, and dragging t along the slider, one gets again that the optimal truncated cone (depicted in Fig. 23) has volume $V = 23.3833\, dm^3$.

In order to test the reaction of school students to problems like the ones considered above, two online contests have been designed and were launched in Bulgaria in 2014 with the financial support of the telecommunication company VIVACOM. The names of the contests are *Theme of the month* and *VIVA Mathematics with computer*. The *Theme of the month* was a Kömal-type competition for high school students (grades 8–12). A worksheet with five problems related to one and the same mathematical or practical problem/situation was published every month in the *Vivacognita* platform (http://www.vivacognita.org). The problems were ordered in increasing difficulty. Their solution required exploration with a computer. Some of the problems were accompanied by auxiliary GeoGebra files which facilitated the process of problem solving. To explore and solve the more difficult problems the participant had to modify the auxiliary file by changing its parameters and, sometimes, by adding new commands. The students had one month to solve the problems, enter the answers obtained in the answer fields located after each problem and submit the worksheet back to *Vivacognita*. The number of points obtained for each answer depended on how close the student's answer was to the correct one (provided by the author of the problem or by the jury). A lot of information about the ideology behind this contest and its first four issues can be found in the paper by Kenderov, Chehlarova, Sendova (2015). The goal of this contest was to popularize the exploration of mathematical ideas and problems by software and to generate a collection of related educational materials. *Theme of the month* was conducted 42 times (till the middle of 2018) when it was decided that it had fulfilled its role. By that time the *Virtual Math Laboratory* (http://cabinet.bg/) of the Institute of Mathematics and Informatics at the Bulgarian Academy of Sciences already comprised more than 1100 applets allowing the exploration of mathematical objects and phenomena.

The other contest *VIVA Mathematics with computer* was started in December 2014 and is still running. It has two rounds per year. The first round is for students from grades 3 to 12 while the second round is only for students from upper secondary school (grades 7–12). Admission to the second round is based on good performance in the first round. On a previously announced day and hour, the participants enter the platform *Vivacognita* and are given access to a worksheet with 10 problems for 60 min altogether. Some of the problems are of the type 'multiple-choice'. To find the correct answer (or answers) the participant has to explore some mathematical situation. The rest of the problems have a real number as an answer which has to be found again by exploring a mathematical situation. There are no restrictions concerning what tools and auxiliary materials can be used—just as in real life when one has to solve some problem. The number of participants in recent years has varied between 800 and 1300. More information about this competition and the degree to which the students cope with it can be found in the papers by Chehlarova and Kenderov (2014) and Kenderov (2018).

The general conclusion from running these two contests is that students are not only capable of solving such problems but do so with enthusiasm and pleasure. Performance improves from year to year indicating that it makes sense to try new designs of such competitions and to increase their number.

5 COVID -19 and the mathematics competitions[6]

The pandemic caused by the coronavirus-induced disease (COVID-19) drastically changed many aspects of society's life. It influenced heavily some of the most essential parts of the competition ecosystem. One of the major pillars of many competitions—to put participants in the same controlled place (to ensure 'equal opportunities' for all participants)—had to be abandoned. Another pillar—getting the jury together at one place in order to design the tasks, to control the conduct of the competition, and to ensure proper marking of the students' works—had to be abandoned as well. The opening and closing ceremonies as well as the social events aiming at networking and exchange of experience had to be cancelled too. The mere existence of team competitions was under question. Had this pandemic happened 20 years ago, many integral parts of the ecosystem probably would have ceased to exist. Meanwhile, however, the competition ecosystem developed in the last decades some 'distributed forms' of competitions (such as the Asian Pacific Mathematics Olympiad (https://www.apmo-official.org/)) and collected experience with the conduct of online competitions (such as, for instance, the Australian Mathematics Competition (https://www.amt.edu.au/australian-mathematics-competition), Caribou Contests (https://caribouTests.com/) and the contest *Mathematics with Computer* discussed in the previous section). All this, combined with contemporary technology and with the innovational spirit of the competition ecosystem, helped find a way to deal with the unusual situation. 'Going online' and 'become distributed' were the major remedies.

[6] This section of the article is included upon a suggestion and inspiration by Maria Falk de Losada.

It is too early to analyze and reflect on the whole impact of COVID -19 on the functioning of the mathematics competition ecosystem. A separate study would be needed to do so. One thing is however already clear and beyond doubt. The reaction of the system was adequate and proper. This is something the entire mathematical community can be proud of.

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