CONTROL PROPERTIES FOR HEAT EQUATION WITH DOUBLE SINGULAR POTENTIAL

Nikolai Kutev, Tsviatko Rangelov#

Dedicated to the 75th anniversary of Academician Petar Popivanov

Abstract

The aim of this article is to study the noncontrollability of the heat equation with double singular potential at an interior point and on the boundary of the domain.

Key words: singular parabolic equations, noncontrollability, Hardy inequalities

2020 Mathematics Subject Classification: 35K20, 35Q93

1. Introduction. We suppose that Ω is a star-shaped domain with respect to a ball centred at the origin, i.e.,

$$(1) \qquad \Omega = \{x \in \mathbb{R}^n, n \geq 3, |x| < \varphi(x)\}, \quad \partial\Omega = \{x\colon |x| = \varphi(x)\},$$

where $\varphi(x)$ is a positive homogeneous function of 0-th order, $\varphi(x) \in C^{0,1}(\Omega)$.

Let us consider the singular parabolic problem

$$(2) \quad \begin{cases} u_t - \Delta u - \mu \Psi(x)u = f(t,x), & f(t,x) \in L^2((0,T) \times \Omega), \\ u(t,x) = 0 \text{ for } (t,x) \in (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x), & u_0(x) \in L^2(\Omega), \end{cases}$$

where the potential

$$(3) \qquad \Psi(x) = |x|^{-2}\left[1 - |x|^{n-2}\varphi^{2-n}(x)\right]$$

is singular at the origin of the domain Ω and on the whole boundary $\partial\Omega$.

In the pioneering paper [1] it is proved that for $\Omega \subset \mathbb{R}^n$, $0 \in \Omega$, $n \geq 3$, problem (2) with $\Psi(x) = |x|^{-2}$ is well-posed for $\mu \leq \left(\dfrac{n-2}{2}\right)^2$ and has a global solution. However, for $\mu > \left(\dfrac{n-2}{2}\right)^2$, $u_0 > 0$ and $f \geq 0$, problem (2) is ill-posed, i.e., there is complete instantaneous blow-up, see [2].

The motivation for the investigations of the above problem is the applications in quantum mechanics, for example in [3], where this model is derived to analyze the confinement of neutral fermions, see also [4]. Other applications appear in molecular physics [5], in quantum cosmology [6], electron capture problems [7], porous medium of fluid [8].

The results in [1] are extended in different directions, for example for general positive singular potentials, equations with variable coefficients, the asymptotic behaviour of the solutions, semilinear equations, etc., see [9–13].

Most of the studies of controllability theory are in the case of interior singularities, see [13–15]. The threshold for controllability or noncontrollability of (2) is the optimal constant $\left(\dfrac{n-2}{2}\right)^2$ for $\Psi(x) = |x|^{-2}$ in the corresponding Hardy inequality. The boundary controllability of (2) in [16] is proved for $\mu \leq \dfrac{n^2}{4}$, where $\dfrac{n^2}{4}$ is the optimal constant in the Hardy inequality when the potential is singular at a boundary point, i.e., $0 \in \partial\Omega$.

Finally, let us mention the result in [17] for the potential

$$\Psi(x) = d^{-2}(x), \quad d(x) = \text{dist}(x, \partial\Omega), \quad \Omega \subset \mathbb{R}^n, \quad n \geq 3$$

which is singular on the whole boundary $\partial\Omega$. The authors prove existence of a unique global weak solution of (2) for $\mu \leq 1/4$, where $1/4$ is the optimal Hardy constant. When $\mu > 1/4$, then there is no control which means that the blow-up, phenomena cannot be prevented, see Theorem 5.1 in [17].

In the present paper we consider the case of potential (3) singular at an interior point and on the whole boundary of the domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$. We prove

existence of a global weak solution for $\mu < \left(\dfrac{n-2}{2}\right)^2$ and boundary noncontrolla-

bility of (2) for $\mu > \left(\dfrac{n-2}{2}\right)^2$, where $\left(\dfrac{n-2}{2}\right)^2$ is the optimal constant in Hardy inequality, see (4) below.

2. Preliminaries. We recall Hardy inequality for the double singular potential (3).

Theorem 1. *Suppose $\Omega \subset \mathbb{R}^n$, $n \geq 3$, $0 \in \Omega$ and Ω is a star-shaped domain with respect to a ball centred at the origin satisfying (1). For every $u(x) \in H_0^1(\Omega)$ the inequality*

(4)
$$\int_\Omega |\nabla u|^2 \, dx \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{|u|^2}{|x|^2 |1 - |x|^{n-2}\varphi^{2-n}(x)|^2} \, dx$$

holds. The constant $\left(\dfrac{n-2}{2}\right)^2$ is optimal.

Proof. The proof follows from Theorem 1.1 in [18] for special choice of the parameters $\alpha = 1$, $\beta = 1$, $p = 2$ and hence $\gamma = \dfrac{1}{2}$, $k = n-2$, $g(s) = \dfrac{1}{n-2}\left(1 - s^{n-2}\right)$, $v(x) = 1$, $w(x) = (n-2)|x|^{-1}\left(1 - |x|\varphi^{-1}(x)\right)^{-1}$. The optimality of the constant $\left(\dfrac{n-2}{2}\right)^2$ is proved in Theorem 1.2 in [18] which means that it cannot be replaced with a greater one. However, equality in (4) is not achieved for any $u(x) \in H_0^1(\Omega)$, except in the trivial case $u(x) = 0$. $\qquad\square$

For $\mu < \left(\dfrac{n-2}{2}\right)^2$ problem (2) with right-hand side $f(t,x) \in L^2((0,T) \times \Omega)$ has a global solution for every $t > 0$ by means of Hardy inequality (4).

Theorem 2. *Suppose $\Omega = \{|x| < \varphi(x)\} \subset \mathbb{R}^n$, $n \geq 3$, is a star-shaped domain with respect to a small ball centred at the origin. Then if $\mu < \left(\dfrac{n-2}{n}\right)^2$, problem (2) with $\Psi(x)$ given by (3) has a global solution $u(t,x)$, such that*

(5)
$$u(t,x) \in L^\infty\left([0,\tau), L^2(\Omega)\right) \cup L^2\left((0,\tau), W^{1,2}(\Omega)\right) \quad \text{for all} \ \ \tau > 0.$$

Proof. For the reader's convenience we sketch the proof. We consider the truncated function $\Psi_N(x) = \min\{N, \Psi(x)\}$, $N = 1, 2, \ldots$. Let $u_N(t,x)$ be the solution of the truncated problem

(6)
$$\begin{cases} u_{N,t} - \Delta u_N = \mu \Psi_N(x) u_N + f(t,x), & t > 0, \ \ x \in \Omega, \\ u_N(t,x) = 0 \ \text{ for } \ t > 0, \ \ x \in \partial\Omega, \\ u_N(0,x) = u_0(x) \ \text{ for } \ x \in \Omega. \end{cases}$$

Multiplying the equation in (6) with u_N and integrating by parts we get from Hardy's inequality (4) the following estimates for every $T > 0$, see Theorem 4.1

C. R. Acad. Bulg. Sci., **75**, No 2, 2022

189

in [9]

$$(7) \quad \int_\Omega |u_N(T,x)|^2\,dx + \int_0^T \int_\Omega |\nabla u_N(t,x)|^2\,dx\,dt$$

$$= \int_\Omega u_0^2(x)\,dx + \frac{\mu+C}{2} \int_0^T \int_\Omega \Psi_N(x) u_N^2(t,x)\,dx\,dt$$

$$- \frac{C-\mu}{2} \int_0^T \int_\Omega \Psi_N(x) u_N^2(t,x)\,dx\,dt + \int_0^T \int_\Omega f(t,x) u_N(t,x)\,dx\,dt$$

$$\leq \int_\Omega u_0^2(x)\,dx + \frac{\mu+C}{2C} \int_0^T \int_\Omega |\nabla u_N(x,t)|^2\,dx\,dt - \frac{m(C-\mu)}{2} \int_0^T \int_\Omega u_N^2(t,x)\,dx\,dt$$

$$+ \frac{m(C-\mu)}{2} \int_0^T \int_\Omega u_N^2(t,x)\,dx\,dt + \frac{1}{2m(C-\mu)} \int_0^T \int_\Omega f^2(t,x)\,dx\,dt,$$

where $m = \inf\limits_{x\in\Omega} \Psi(x) > 0$ and $C = \left(\dfrac{n-2}{2}\right)^2$. Since $\mu < C$ we get from (7) the energy estimate

$$\int_\Omega |u_N(x,T)|^2\,dx + \frac{C-\mu}{2C} \int_0^T \int_\Omega |\nabla u_N(x,t)|^2\,dx\,dt$$

$$\leq \int_\Omega |u_0(x)|^2\,dx + \frac{1}{2m(C-\mu)} \int_0^T \int_\Omega f^2(t,x)\,dx\,dt.$$

From the comparison principle $u_N(t,x)$ is a nondecreasing sequence of functions because $\Psi_N(x) \geq \Psi_M(x)$ for every $x \in \Omega$, $t > 0$ and $N \geq M$. We can pass to the limit $N \to \infty$ by using Theorem 4.1 in [19]. Thus the global solution $u(t,x)$ of (2) is defined as a limit of the solution $u_N(t,x)$ of the truncated problem (6) and $u(t,x)$ has the regularity properties given in (5). $\qquad\square$

Thus the natural question is whether $\left(\dfrac{n-2}{2}\right)^2$ is the sharp constant for global existence of the solutions to (2). In the present paper we give more precise answer. When $\mu > \left(\dfrac{n-2}{2}\right)^2$ we prove null-noncontrollability of (2), i.e., it is not possible for given $u_0(x) \in L^2(\Omega)$ one to find a control function $f(t,x) \in L^2((0,T)\times\Omega)$ localized in $(0,T)\times\omega$, $\omega \Subset \Omega \setminus \{0\}$ such that there exists a solution of (2). In this way we can not prevent the blow-up phenomena acting on a subset for $\mu > \left(\dfrac{n-2}{2}\right)^2$.

We recall also the classical Hardy inequality

$$(8) \qquad\qquad \int_\Omega |\nabla u|^2\,dy \geq \left(\frac{n-2}{2}\right)^2 \int_\Omega \frac{u^2}{|y|^2}\,dy$$

for every $u \in H_0^1(\Omega)$ in a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 3$ and optimal constant $\left(\dfrac{n-2}{2}\right)^2$, see [20].

3. Main result In Theorem 3 below we prove that problem (2) cannot be stabilized due to the explosive modes concentrated around the singularities when $\mu > \left(\dfrac{n-2}{2}\right)^2$. For this purpose, following the idea of optimal control, see [14], we consider for any $u_0 \in L^2(\Omega)$ the functional $J_{u_0}(u, f) = \dfrac{1}{2} \displaystyle\int_{\Omega \times (0,T)} u^2(t, x) \, dx \, dt + \dfrac{1}{2} \displaystyle\int_0^T \|f\|_{H^{-1}(\Omega)} \, dt$ defined in the set

$$D(u_0) = \left\{ (u, f) \in L^2\left((0,T); H_0^1(\Omega)\right) \times L^2\left((0,T); H^{-1}(\Omega)\right) \right\},$$

where $u(t, x)$ satisfies (2). Here $f(t, x)$ is the control which is null in $(0, T) \times (\Omega \backslash \overline{\omega})$, $\omega \Subset \Omega \setminus \{0\}$.

We say that (2) can be stabilized if there exists a constant C_0 such that

$$\inf_{(u,f) \in D(u_0)} J_{u_0}(u, f) \leq C_0 \|u_0\|_{L^2(\Omega)}^2 \text{ for every } u_0 \in L^2(\Omega).$$

Let us consider the regularized problem

$$(9) \qquad \begin{cases} u_t - \Delta u - \mu \Psi_\varepsilon u = f(t, x) & \text{for } (t, x) \in (0, T) \times \Omega, \\ u(t, x) = 0 & \text{for } (t, x) \in (0, T) \times \partial\Omega, \\ u(0, x) = u_0(x) & \text{for } x \in \Omega, \end{cases}$$

where

$$(10) \qquad \Psi_\varepsilon(x) = (|x| + \varepsilon)^{-2} \left(1 + \varepsilon - |x|^{n-2} \varphi^{2-n}(x)\right)^{-2}.$$

For every $\varepsilon > 0$ problem (9) is well-posed. For the functional

$$J_{u_0}^\varepsilon(f) = \dfrac{1}{2} \int_{(0,T) \times \Omega} u^2(t, x) \, dt \, dx + \dfrac{1}{2} \int_0^T \|f\|_{H^{-1}(\Omega)} \, dt,$$

where f is localized in $(0, T) \times \omega$, $\omega \Subset \Omega \setminus \{0\}$ and u is a solution of (9), we have the following result.

Theorem 3. *Suppose* $\mu > \left(\dfrac{n-2}{2}\right)^2$, $\omega \Subset \Omega \setminus \{0\}$, $n \geq 3$ *and* f *is localized in* $(0, T) \times \omega$. *Then there is no constant* C_0 *such that for all* $\varepsilon > 0$ *and* $u_0 \in L^2(\Omega)$

$$\inf_{f \in D_1(f)} J_{u_0}^\varepsilon(f) \leq C_0 \|u_0\|_{L^2(\Omega)}^2,$$

where $D_1(f) = \left\{ f \in L^2\left((0,T); H^{-1}(\Omega)\right) \right\}$.

In order to prove Theorem 3 we need the following spectral estimates for the operator

(11)
$$\begin{cases} L^\varepsilon(u) \equiv -\Delta u - \mu \Psi_\varepsilon u \ \text{ in } \ \Omega, \\ u = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

Let λ_1^ε be the first eigenvalue of (11), $\phi_1^\varepsilon(x)$ be the corresponding first eigenfunction, $\|\phi_1^\varepsilon(x)\|_{L^2(\Omega)} = 1$, i.e.,

(12)
$$\begin{cases} -\Delta\phi_1^\varepsilon - \mu\Psi_\varepsilon(x)\phi_1^\varepsilon = \lambda_1^\varepsilon\phi_1^\varepsilon, & x \in \Omega, \\ \phi_1^\varepsilon = 0, & x \in \partial\Omega \end{cases}$$

and $\Psi_\varepsilon(x)$ is defined in (10).

Proposition 1. *Suppose* $\mu > \left(\dfrac{n-2}{2}\right)^2$, $n \geq 3$. *Then we have*

(13)
$$\lim_{\varepsilon \to 0} \lambda_1^\varepsilon = -\infty$$

and for all $\rho > 0$, $\delta > 0$, $\rho < (1-\delta)\varphi(x)$, $U_{\rho,\delta} = \{x \colon \rho < |x| < (1-\delta)\varphi(x)\}$,

(14)
$$\lim_{\varepsilon \to 0} \|\phi_1^\varepsilon\|_{H^1(U_{\rho,\delta})} = 0.$$

Proof. We assume by contradiction that λ_1^ε is bounded from below with a constant C_1. Then from the Rayleigh identity it follows that

(15)
$$\mu \int_\Omega \Psi_\varepsilon(x)u^2\,dx \leq \int_\Omega |\nabla u|^2\,dx - C_1\int_\Omega u^2\,dx$$

for every $u \in H_0^1(\Omega)$. For every $a \geq 1$ we define $u_a = a^n u(ax)$ so that (15) becomes

(16) $\quad \mu a^{2n}\displaystyle\int_\Omega u^2(ax)\Psi_\varepsilon(x)\,dx \leq a^{2n+2}\int_\Omega |\nabla u(ax)|^2\,dx - C_1 a^{2n}\int_\Omega u^2(ax)\,dx.$

After the limit $\varepsilon \to 0$ and then the change of variables $ax = y$ we get from (16)

(17) $\quad \mu a^2 \displaystyle\int_{|y|<a\varphi(y)} u^2(y)|y|^{-2}\left(1 - |y|^{n-2}(\varphi(y)a)^{2-n}\right)^{-2} dy$

$$\leq a^2 \int_{|y|<a\varphi(y)} |\nabla u(y)|^2\,dy - C_1\int_{|y|<a\varphi(y)} u^2(y)\,dy.$$

For every $u \in C_0^\infty(\Omega)$ and for every fixed y we have

$$\lim_{a\to\infty} u^2(y)|y|^{-2}\left(1 - |y|^{n-2}(\varphi(y)a)^{2-n}\right)^{-2} = u^2(y)|y|^{-2}.$$

Hence supp $u \subset \{|y| < \delta\varphi(y)\}$ for some $\delta \in (0,1)$. Then for every $a \geq 1$ and every y it follows

$$u^2(y)|y|^{-2}\left(1 - |y|^{n-2}(\varphi(y)a)^{2-n}\right)^{-2} \leq u^2(y)|y|^{-2}\left(1 - \delta^{n-2}\right)^{-2} < \infty.$$

Thus after the limit $a \to \infty$ in (17) we get

$$(18) \qquad \int_\Omega |\nabla u(y)|^2\, dy \geq \mu \int_\Omega \frac{u^2(y)\, dy}{|y|^2}$$

for every $u \in C_0^\infty(\Omega)$ and by the continuity (18) holds for every $u \in H_0^1(\Omega)$.

Since $\mu > \left(\dfrac{n-2}{2}\right)^2$, inequality (18) contradicts the Hardy inequality (8) with optimal constant $\left(\dfrac{n-2}{2}\right)^2$ and (13) is proved.

In order to prove (14) let us consider a non-negative smooth function $\eta(x)$ such that $\|\eta\|_{L^\infty(R^n)} \leq 1$ and

$$\eta(x) = \begin{cases} 1, & \text{for } x \in \{\rho < |x| < (1-\delta)\varphi(x)\}, \\ 0, & \text{in } \left\{|x| < \dfrac{\rho}{2}\right\} \cup \left\{\left(1 - \dfrac{\delta}{2}\right)\varphi(x) < |x| < \varphi(x)\right\}. \end{cases}$$

Multiplying (12) by $\eta\phi_1^\varepsilon(x)$ and integrating in Ω we get

$$(19) \quad \int_\Omega \eta|\nabla\phi_1^\varepsilon|^2\, dx - \lambda_1^\varepsilon \int_\Omega \eta(\phi_1^\varepsilon)^2\, dx = \mu \int_\Omega \eta\Psi_\varepsilon(x)(\phi_1^\varepsilon)^2\, dx + \frac{1}{2}\int_\Omega (\phi_1^\varepsilon)^2\Delta\eta\, dx.$$

From (19), the choice of η and the unit L^2 norm of ϕ_1^ε it follows that

$$-\lambda_1^\varepsilon \int_\Omega \eta(\phi_1^\varepsilon)^2\, dx \leq 4\mu\rho^{-2}\left[1 - \left(1 - \frac{\delta}{2}\right)^{n-2}\right]^{-2} + \frac{1}{2}\|\Delta\eta\|_{L^\infty(\Omega)}.$$

By means of (13) we get $\displaystyle\lim_{\varepsilon\to 0}\int_\Omega \eta(\phi_1^\varepsilon)^2\, dx = 0$ and hence

$$(20) \qquad \lim_{\varepsilon\to 0}\int_{U_{\rho,\delta}} (\phi_1^\varepsilon)^2\, dx = 0 \quad \text{for every} \ \ U_{\rho,\delta} = \{\rho < |x| < (1-\delta)\varphi(x)\}.$$

Now using (19), (20) for $U_{\frac{\rho}{2},\frac{\delta}{2}} = \left\{\dfrac{\rho}{2} < |x| < \left(1 - \dfrac{\delta}{2}\right)\varphi(x)\right\}$ it follows that

$$\int_{U_{\rho,\delta}} |\nabla\phi_1^\varepsilon|^2\, dx \leq \int_\Omega \eta|\nabla\phi_1^\varepsilon|^2\, dx$$

$$\leq \left[4\mu\rho^{-2}\left(1 - \left(1 - \frac{\delta}{2}\right)^{n-2}\right)^{-2} + \frac{1}{2}\|\Delta\eta\|_{L^\infty(\Omega)}\right]\int_{U_{\frac{\rho}{2},\frac{\delta}{2}}} (\phi_1^\varepsilon)^2\, dx \xrightarrow[\varepsilon\to 0]{} 0,$$

which proves (14). $\qquad\qquad\square$

Proof of Theorem 3. Due to (13) we fix $\varepsilon > 0$ sufficiently small such that $\lambda_1^\varepsilon < 0$ and choose $u_0 = \phi_1^\varepsilon$, $\|\phi_1^\varepsilon\|_{L^2(\Omega)} = 1$, where ϕ_1^ε is the first eigenfunction of (11). Let us consider the functions $a(t) = \int_\Omega u(t,x)\phi_1^\varepsilon \, dx$, $b(t) = \langle f, \phi_1^\varepsilon \rangle_{L^2(\Omega)}$. Simple computations give us

$$
\begin{aligned}
a'(t) &= \int_\Omega \phi_1^\varepsilon \left(\Delta u + \mu \Psi_\varepsilon(x)u + f \right) dx \\
&= \int_\Omega \left(\Delta \phi_1^\varepsilon + \mu \Psi_\varepsilon(x)\phi_1^\varepsilon(x) \right) u \, dx + \int_\Omega f \phi_1^\varepsilon \, dx \\
&= -\lambda_1^\varepsilon \int_\Omega \phi_1^\varepsilon u \, dx + \int_\Omega f \phi_1^\varepsilon \, dx = -\lambda_1^\varepsilon a(t) + b(t).
\end{aligned}
$$

So $a(t)$ satisfies the problem

$$
a'(t) + \lambda_1^\varepsilon a(t) = b(t), \quad a(0) = 1.
$$

Hence $a(t) = e^{-\lambda_1^\varepsilon t} + \int_0^t e^{-\lambda_1^\varepsilon(t-s)} b(s) \, ds$ and

$$
\begin{aligned}
\int_{(0,T)\times\Omega} u^2(t,x) \, dx \, dt &\geq \int_0^T a^2(t) \, dt \\
&\geq \frac{1}{2} \int_0^T e^{-2\lambda_1^\varepsilon t} \, dt - \int_0^T \left(\int_0^t e^{-\lambda_1^\varepsilon(t-s)} b(s) \, ds \right)^2 dt \\
&\geq -\frac{1}{4\lambda_1^\varepsilon} \left(e^{-2\lambda_1^\varepsilon T} - 1 \right) + \frac{1}{2\lambda_1^\varepsilon} \int_0^T \left(e^{-2\lambda_1^\varepsilon t} - 1 \right) \int_0^t b^2(s) \, ds \, dt \\
&\geq -\frac{1}{4\lambda_1^\varepsilon} \left(e^{-2\lambda_1^\varepsilon T} - 1 \right) + \frac{1}{2\lambda_1^\varepsilon} \int_0^T e^{-2\lambda_1^\varepsilon t} \, dt \int_0^T b^2(s) \, ds \\
&= -\frac{1}{4\lambda_1^\varepsilon} \left(e^{-2\lambda_1^\varepsilon T} - 1 \right) - \frac{e^{-2\lambda_1^\varepsilon T} - 1}{4(\lambda_1^\varepsilon)^2} \int_0^T b^2(s) \, ds.
\end{aligned}
$$
(21)

Since $b^2(t) \leq \|f\|_{H^{-1}(\omega)} \|\phi_1^\varepsilon\|_{H^1(\omega)}$, $\omega \Subset \Omega \setminus \{0\}$ we get from (21)

$$
-\frac{e^{-2\lambda_1^\varepsilon T} - 1}{4\lambda_1^\varepsilon} \leq \int_{(0,T)\times\Omega} u^2(t,x) \, dx \, dt + \frac{e^{-2\lambda_1^\varepsilon T} - 1}{4(\lambda_1^\varepsilon)^2} \|\phi_1^\varepsilon\|_{H^1(\omega)} \int_0^T \|f(t,\cdot)\|_{H^{-1}(\Omega)} \, dt.
$$

Therefore either

$$
-\frac{e^{-2\lambda_1^\varepsilon T} - 1}{8\lambda_1^\varepsilon} \leq \frac{e^{-2\lambda_1^\varepsilon T} - 1}{4(\lambda_1^\varepsilon)^2} \|\phi_1^\varepsilon\|_{H^1(\omega)} \int_0^T \|f(t,\cdot)\|_{H^{-1}(\Omega)} \, dt,
$$

or

$$
-\frac{e^{-2\lambda_1^\varepsilon T} - 1}{8\lambda_1^\varepsilon} \leq \int_{(0,T)\times\Omega} u^2(t,x) \, dx \, dt.
$$

N. Kutev, Ts. Rangelov

In any case we have for every f localized in $(0, T) \times \omega$, $\omega \Subset \Omega \setminus \{0\}$ the estimate

$$J_{u_0}^\varepsilon(f) \geq \inf \left\{ -\frac{e^{-2\lambda_1^\varepsilon T} - 1}{16\lambda_1^\varepsilon}, \frac{-\lambda_1^\varepsilon}{4\|\phi_1^\varepsilon\|_{H^1(\omega)}} \right\}$$

holds.

From Proposition 1, if $\omega \subset U_{\rho,\delta} = \{\rho < |x| < (1-\delta)\varphi(x)\}$ for some positive constants ρ, δ, it follows that

$$\lim_{\varepsilon \to 0} \|\phi_1^\varepsilon\|_{U_{\rho,\delta}} = 0,$$

and hence $\lim_{\varepsilon \to 0} J_{u_0}^\varepsilon(f) = \infty$ which proves Theorem 3. $\qquad\square$

REFERENCES

[1] BARAS P., J. A. GOLDSTEIN (1984) The heat equation with singular potential, Trans. Am. Math. Soc., **284**, 121–139.

[2] CABRÉ X., Y. MARTEL (1999) Existence versus instantaneous blow-up for linear heat equations with singular potentials, C. R. Acad. Sci. Paris Sér. I Math., **329**, 973–978.

[3] DE CASTRO A. S. (2004) Bound states of the Dirac equation for a class of effective quadratic plus inversely quadratic potentials, Ann. Physics, **311**, 170–181.

[4] PERAL I., J. VAZQUEZ (1995) On the stability or instability of the singular solution of the semilinear heat equation with exponential reaction term, Arch. Ration. Mech. Anal., **129**, 201–224.

[5] LEVY-LEBLOND J. M. (1967) Electron capture by polar molecules, Phys. Rev., **153**(1), 1–4.

[6] BERESTYCKI H., M. J. ESTEBAN (1997) Existence and bifurcation of solutions for an elliptic degenerate problem, J. Diff. Eq., **134**(1), 1–25.

[7] GIRI P. R., K. S. GUPTA, S. MELJANAC, A. SAMSAROV (2008) Electron capture and scaling anomaly in polar molecules, Phys. Lett. A, **372**(17), 2967–2970.

[8] ANSINI L., L. GIACOMELLI (2004) Doubly nonlinear thin-film equation in one space dimension, Arch. Ration. Mech. Anal., **173**, 89–131.

[9] AZORERO J. P. G., I. P. ALONSO (1998) Hardy inequalities and some critical elliptic and parabolic problems, J. Diff. Eq., **144**, 441–476.

[10] VÁZQUEZ J., E. ZUAZUA (2000) The Hardy inequality and the asymptotic behaviour of the heat equation with an inverse-square potential, J. Funct. Anal., **173**, 103–153.

[11] GOLDSTEIN G. R., J. A. GOLDSTEIN, A. RHANDI (2012) Weighted Hardy's inequality and the Kolmogorov equation perturbed by an inverse-square potential, Appl. Anal., **91**(11), 2057–2071.

[12] JUNQIANG H., W. YONGDA, N. PENGCHENG (2012) Existence of solutions to the parabolic equation with a singular potential of the Sobolev–Hardy type, Acta Math. Sci., **32B**(5), 1901–1918.

C. R. Acad. Bulg. Sci., **75**, No 2, 2022

195

[13] VANCOSTENOBLE J. (2011) Lipschitz stability in inverse source problems for singular parabolic equations, Comm. Part. Diff. Eq., **36**(8), 1287–1317.

[14] ERVEDOZA S. (2008) Control and stabilization properties for a singular heat equation with an inverse-square potential, Comm. Part. Diff. Eq., **33**(11), 1996–2019.

[15] VANCOSTENOBLE J., E. ZUAZUA (2008) Null controllability for the heat equation with singular inverse-square potentials, J. Funct. Anal., **254**(7), 1864–1902.

[16] CAZACU C. (2012) Schrödinger operators with boundary singularities: Hardy inequality, Pohozaev identity and controllability results, J. Funct. Anal., **263**, 3741–3783.

[17] BICCARI U., E. ZUAZUA (2016) Null controllability for a heat equation with a singular inverse-square potential involving the distance to the boundary function, J. Diff. Eq., **261**, 2809–2853.

[18] FABRICANT A., N. KUTEV, T. RANGELOV (2013) Hardy-type inequality with double singular kernels, Centr. Eur. J. Math., **11**(9), 1689–1697.

[19] BOCCARDO L., F. MURAT (1992) Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations, Nonlinear Anal., **19**(6), 437–477.

[20] HARDY G., G. POLYA, J. LITTLEWOOD (1952) Inequalities, Cambridge, Cambridge University Press.

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Akad. G. Bonchev St, Bl. 8
1113 Sofia, Bulgaria
e-mails: `kutev@math.bas.bg`
`rangelov@math.bas.bg`