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FORMAL NEIGHBOURHOODS AND FORMAL MODELS OF NON-DEGENERATE ARCS

Peter Petrov

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Abstract

An important result about the geometry of the arc space of an algebraic variety is the theorem of Drinfeld–Grinberg–Kazhdan, representing the formal neighbourhood of a non-degenerate arc. We start with a brief review of some important results and notions. Then the complete proof of the theorem with examples is given. A generalization to the relative case is discussed for smooth and étale morphisms.

Key words: non-degenerate arc, formal neighbourhood, formal model, embedding codimension

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1. Introduction. Let X be a variety over a field k. Locally it is defined by a system of polynomial equations

$$f_j(x_1, \ldots, x_n) = 0, \quad j = 1, \ldots, r.$$

An (closed) arc is a k-point in the space of arcs X_∞ parameterizing the solutions of that system in $k[[t]]$. That is, the arc space parameterizes all morphisms $\operatorname{Spec}(K[[t]]) \to X$, for any field extension K/k, and it comes with natural projection $\pi\colon X_\infty \to X$ defined by truncation. When $k = \mathbb{C}$ an arc could be viewed as a germ of analytic curve on X. If $\dim(X) > 0$, the space of arcs is non Noetherian scheme of infinite dimension, encoding information about the geometry of the singular locus X_{sing}.

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Remark 1.1. Some notations. As topological space, $\operatorname{Spec} K[[t]] = \{0, \eta\}$, with 0 being the closed point corresponding to the ideal (t), and η being the generic point. It could be viewed as infinitesimally small neighbourhood of $0 \in \mathbb{A}^1_k$.

Denote by FN_{γ_0} the formal neighbourhood of an arc $\gamma_0 \in X_\infty$, i.e. $\operatorname{Spf}(\widehat{\mathcal{O}_{X_\infty,\gamma_0}})$, the completion being taken with respect to the maximal ideal m_{γ_0}. The formal disc $\operatorname{Spf}(k[[t]])$ is denoted by $\mathbf{D}$, and the formal scheme $\operatorname{Spf}(k[[x_i, i \in \mathbb{N}]])$ is denoted by $\mathbf{D}^\infty$. Each element in its ring could be represented as $\sum_{i=0}^{\infty} f_i$, with f_i homogeneous polynomial of degree i, $i \in \mathbb{N}$. The ring itself is the completion of $k[x_i, i \in \mathbb{N}]$ with respect to the $(x_i, i \in \mathbb{N})$-adic topology, viewed as an object in the category of topological rings and continuous homomorphisms. Denote the ideal $(x_i, i \in \mathbb{N})$ as $(\underline{x})$. That is, we get complete local ring with maximal ideal $(\underline{x})$ and the topology of projective limit, having a neighbourhood basis at 0 consisting of ideals $I_n = \{f = \Sigma f_n \mid f_i = 0, i \leq n\}$. This linear topology is weaker than the $(\underline{x})$-adic one, and $k[[\underline{x}]]$ is non complete in the latter topology because $k[\underline{x}]$ is non Noetherian ring [9].

An arc γ whose generic point $\gamma(\eta)$ is not contained in X_{sing} is called *non degenerate arc.*

2. Few theorems and definitions. Some deep results given below describe how the geometry of X and the geometry of its arc and jet spaces are related.

Theorem 2.1 (Kolchin [11]). *If X is a variety over a field of characteristic 0, then X_∞ is irreducible.*

This may not hold if the characteristic is positive.

Theorem 2.2 (Nash [14]). *If $\operatorname{char} k = 0$, then there is an injective map: (irreducible components of $\{\pi^{-1}(X_{\text{sing}})\}$) $\hookrightarrow$ (essential divisors over X). It is called the Nash map for X.*

The space of arcs is associated with the variety by reflecting intrinsically its geometry, especially the geometry of X_{sing}. The Nash problem is asking for which classes of varieties the Nash map is bijective. It has positive answer in dimension 2 ([4]), but in higher dimensions there are counterexamples ([5,10]).

The following result characterizes how "bad" are the singularities of the variety in terms of its jet spaces.

Theorem 2.3 (Mustata [13]). *If X is a variety over $\mathbb{C}$, then the jet scheme X_n is irreducible for any n iff X has at most rational singularities.*

The main topic in this article is the following theorem.

Theorem 2.4 (Drinfeld–Grinberg–Kazhdan). *If $\gamma_0 \in X_\infty \setminus (X_{\text{sing}})_\infty$ is a k-arc, then there exists a scheme of finite type Y over k and a point $y \in Y(k)$ with formal neighbourhood FN_y, such that $FN_{\gamma_0} \simeq FN_y \hat{\times} \mathbf{D}^\infty$.*

The theorem of Drinfeld–Grinberg–Kazhdan ([7,8])(or briefly, DGK theorem) claims that the singularity of $\gamma_0(0)$ is encoded in a finite dimensional scheme Y. The corresponding formal neighbourhood FN_y is called formal model for γ_0.

Definition 2.5. A minimal formal model of an arc γ is a formal model $\widehat{Y_y}$ which is indecomposable, i.e. cannot be represented as $FN_y = FN_z \hat{\times} \mathbf{D}$.

As follows from a theorem in [1], for any non-degenerate arc the minimal formal model exists and is unique up to isomorphism. If the arc is degenerate though, its image is contained in X_{sing} and the claim does not hold, as the following theorem shows.

Theorem 2.6 (Chiu, Hauser [3])**.** *Suppose* $\operatorname{char}(K) = 0$*. For* $\alpha = \alpha_x$*, the constant arc centred at* $x \in X$*, there is a decomposition* $\widehat{\mathcal{O}_{X_\infty,\alpha}} = \widehat{\mathcal{O}_{Y,y}} \hat{\otimes} K[[\underline{x}]]$ *iff there is such a decomposition for* $\widehat{\mathcal{O}_{X,x}}$*. In particular, if* $x \in X_{\text{sing}}$ *there is no such a decomposition for* α_x*.*

3. Proof of Drinfeld–Grinberg–Kazhdan theorem. In this section is given a complete proof over arbitrary field, which follows the original proof ([7,8]) with some additional explanations.

Definition 3.1. *Test ring* A is a local k-algebra having residue field k, with nilpotent maximal ideal m (i.e. $m^n = 0$ for some $n \in \mathbb{N}$). Let $Testrings_k$ be the category of test rings with local homomorphisms as morphisms.

We can make a few observations.

If $\gamma_0\colon \operatorname{Spec} k[[t]] \to X$, and X' is the closure of the irreducible component of X_{reg} containing $\gamma(\eta)$, then $X_{\infty,\gamma} = X'_{\infty,\gamma}$, so we can assume X to be reduced and irreducible. Because the claim is local we could take X to be affine as well. The scheme Y does not need to be neither reduced nor irreducible in general (see the example below).

The reason that the category $Testrings_k$ is enough to define the functor of points is that every $\widehat{\mathcal{O}}_{S,P}$ is a projective limit of test rings.

To start the proof we take X to be reduced irreducible scheme of dimension n, embedded in $\mathbf{A}^N$.

Claim 1. When working with local properties of $\pi^{-1}(X_{\text{sing}})$, we could suppose that X is locally complete intersection, which may be reducible. Indeed, let $r = \operatorname{codim}(X)$, and let the ideal defining X be $I_X = \{f_1, \ldots, f_s\}$. Put $F_i = \sum a_{ij} f_j$, $i = 1, \ldots, s$, with a_{ij} being generic elements in k, and let $M \subset \mathbf{A}^{\mathbf{N}}$ be the zero set of ideal $I_M = (F_1, \ldots, F_r)$, defined by the first r of F_i's. Then the following hold:

1) any irreducible component of M has dimension n, so M is a complete intersection scheme;

2) $X \hookrightarrow M$ is a closed subscheme, and X and M coincide at the generic point of X, that is, on an open nonempty subset;

3) there is some r-minor of the Jacobian matrix of M not vanishing at η_X;

4) $X_{\text{sing}} \subset M_{\text{sing}}$.

Claim 2. There exists closed affine complete intersection scheme of finite type $X' \supset X$ of the same dimension such that $\operatorname{Im}(\gamma_0)$ is not contained in $\overline{X' \setminus X}$.

Indeed, take L to be the index set of all the r-tuples $(i_1, \ldots, i_r)$ of distinct integers with $i_j \in \{1, \ldots, s\}$, and let M_l, for $l \in L$, be the corresponding complete intersection scheme. If there is no such an X' as claimed, for all M_l we would have

$\operatorname{Im}(\gamma_0) \subset \overline{M_l \setminus X}$, thus $\operatorname{Im}(\gamma_0)$ is contained in their intersection. But $\overline{M_l \setminus X} \cap X \subset \operatorname{Sing}(M_l)$, and $\bigcap_l \operatorname{Sing}(M_l) = X_{\text{sing}}$, contradicting the choice of γ_0.

For such an X' we have $FN^X_{\gamma_0} = FN^{X'}_{\gamma_0}$, because $\gamma_0(\eta) \in X_{\text{reg}}$, so without loss of generality we can replace X by X'.

Now take X to be complete intersection affine variety, contained in $\operatorname{Spec} k[x_1, \ldots, x_n, y_1, \ldots, y_r]$, and defined by equations $p_i = 0$, $i = 1, \ldots, r$. Also, $\gamma_0(t) = (x^o(t), y^o(t)) = (x^o_1(t), \ldots, x^o_n(t), y^o_1(t), \ldots, y^o_r(t))$ is not contained for any t in $X_{\text{sing}} = Z(\det(\partial p_i/\partial y_j))$.

For a test ring A let $\gamma = (x(t), y(t))$ with $x(t) \in A[[t]]^n$, $y(t) \in A[[t]]^r$ be an A-deformation of $\gamma_)$, i.e. its reduction modulo $m \subset A$ is equal to γ_0. Because $\operatorname{Im}(\gamma_0) \not\subset Z(\det(\partial p_i/\partial y_j))$, not all coefficients of the power series $\det(\partial p_i/\partial y_j))(\gamma_0)$ are 0. So we apply:

Lemma 3.2 (Weierstrass preparation theorem)**.** *Let (R, m) be complete local separated ring with respect to a linear topology, which is weaker than the m-adic topology, $f = \sum_i c_i t^i \in R[[t]]$, with not all $c_i \in m$. If d is the first index such that $c_i \notin m$, then we have unique representation $f = q.u$, for some monic polynomial $q = t^d + \sum_{0 \le l < d} a_l t^l \in R[t]$ of degree d, with $a_l \in m$ for all l, and $u \in R[[t]]$ invertible.*

Thus, $\det(\partial p_i/\partial y_j))(x(t), y(t)) = q(t)u(t)$, for some $u(t) \in A[[t]]^*$, and $q(t) \in A[t]$ a monic polynomial of degree d whose reduction modulo m is t^d. The degree d depends on γ_0 only, not on the choice of its deformation γ. We may assume $d \ge 1$ because, if $d = 0$ we can eliminate y, and the claim holds.

The idea of the proof is to consider q as a new variable. Then all A-deformations of γ_0 are in one-to-one correspondence with the solutions of the following system of equations with unknowns $q \in A[t]$, $x \in A[[t]]^n$, $y \in A[[t]]^r$:

$$\text{(I)} \qquad \det\left(\frac{\partial p_i}{\partial y_j}\right)(x, y) = 0 \mod q, \qquad p(x, y) = (p_1, \ldots, p_r) = 0.$$

Here, if the first equation holds, $q^{-1}\det(\partial p_i/\partial y_j)$ is invertible because it is invertible modulo m which is nilpotent.

Next, for any fixed $e \in \mathbf{N}$, consider the following system with unknowns $q \in A[t]$, $x \in A[[t]]^n$, $\overline{y} \in A[[t]]^r/(q^e)$, such that q is monic polynomial of degree d, $q = t^d \mod m$, $x = x^o \mod m$, $\overline{y} \mod m = y^o \mod t^{ed}$:

$$\text{(II)} \qquad \begin{gathered} \det\left(\frac{\partial p_i}{\partial y_j}\right)(x, \overline{y}) = 0 \mod q, \\ p(x, \overline{y}) \in \operatorname{Im}\left(q^e \left(\frac{\partial p_i}{\partial y_j}\right)(x, \overline{y}) \colon \frac{A[[t]]^r}{qA[[t]]^r} \to \frac{q^e A[[t]]^r}{q^{e+1}A[[t]]^r}\right). \end{gathered}$$

The second condition makes sense, if one takes the Taylor expansion of $p(x, y)$ and noting that $p(x, \overline{y})$ is well defined modulo $\operatorname{Im}\left(q^e \left(\frac{\partial p_i}{\partial y_j}\right)(x, \overline{y})\right)$. Moreover, it is equivalent to the equation $\hat{C}p(x, y) = 0 \mod q^{e+1}$, where $\hat{C}$ is the adjoint matrix to $C = \left(\frac{\partial p_i}{\partial y_j}(x,y)\right)$ (i.e. $C\hat{C} = \det(C).I_r$), with $y = y(t) \in A[[t]]^r$ in the pre-image of $\overline{y}$. Indeed, if $p(x,\overline{y}) = q^e C(x,\overline{y}).v$ for some $v \in A[[t]]^r/(q)$, then $\hat{C}p(x,\overline{y}) = q^{e+1}v$. Conversely, if $\hat{C}p(x,\overline{y}) = 0 \mod q^{e+1}$, there is some w such that $\hat{C}p(x,\overline{y}) = q^{e+1}w$, so $C\hat{C}p(x,\overline{y}) = Cq^{e+1}w$. Then for some invertible u, $p(x,\overline{y}) = Cq^e uw$, that is $p(x,\overline{y}) \in \operatorname{Im} q^e C$.

Furthermore, for any fixed $e \in \mathbf{N}$ the last condition in (II) is equivalent to the following equations:

$$
\begin{aligned}
&p(x,\overline{y}) = 0 \mod q^e;\\
&\hat{B}p(x,\overline{y}) = 0 \mod q^{e+1}, \text{ where } B = \left(\frac{\partial p_i}{\partial y_j}(x,\overline{y})\right).
\end{aligned}
$$

Both come from the second condition in (II), and the second equation makes sense once the first one holds. So (II) is equivalent to the following system which does not need any choice of $y \in A[[t]]^r$ such that $y \mod q^e = \overline{y}$, and $x(t)$ is relevant up to $\overline{x} = x \mod q^{e+1}$:

$$
\text{(III)}\qquad
\begin{aligned}
&\det\left(\frac{\partial p_i}{\partial y_j}\right)(x,\overline{y}) = 0 \mod q,\\
&p(x,\overline{y}) = 0 \mod q^e;\\
&\hat{B}p(x,\overline{y}) = 0 \mod q^{e+1}, \text{ where } B = \left(\frac{\partial p_i}{\partial y_j}(x,\overline{y})\right).
\end{aligned}
$$

Lemma 3.3. *For any e, the natural map from the set of solutions over A of system (I) to the set of solutions over A of system (II) is bijective.*

Proof. Let $c \in \mathbf{N}$ be the minimal number such that $m^c = 0$; we will prove the lemma by induction on c. If $c = 1$, i.e. $m = 0$, $A = k$, this holds because both systems have one solution. Let $c \geq 2$, and suppose the claim holds for $c-1$, that is, there exists $\tilde{y} \in A[[t]]^r$ with $\tilde{y} \mod q^e = \overline{y}$ and $p(x(t,\tilde{y}(t))) \in m^{c-1}[[t]]^r$. This is the second equation of system (I) taken over A/m^{c-1}. Such a $\tilde{y}$ is unique modulo $q^e A[[t]]^r \cap m^{c-1}[[t]]^r$. To prove that the map between the set of solutions of system (I) and the set of solutions of system (II) is bijective we have to prove it has an inverse. That is, we have to find $z(t) \in q^e A[[t]]^r \cap m^{c-1}[[t]]^r$ s.t. $p(x, \tilde{y} - z) = 0$. As before, letting $C := \left(\frac{\partial p}{\partial y}\right)(x(t), \tilde{y}(t))$, we would have $p(x,\tilde{y}) - C.z = 0$, because $m^{2c-2} = 0$ by the assumption $c \geq 2$. By the first equation in (II) and the argument following system (I), $\det(C) = q.u$ for some invertible $u(t) \in A[[t]]$. This means that $z(t)$ will be unique if it exists, because $C.z = p(x,\tilde{y})$, and $\det(C) \neq 0$. By the second equation in (II), $p(x,\tilde{y}) \in q^e C.A[t]^r + q^{e+1}A[t]^r$. But $C.A[t]^r \supset C.\hat{C}.A[t]^r = \det(C)I_s.A[t]^r = q.u.A[t]^r$, so $q^{e+1}A[t]^r = q.q^e A[t]^r \subset q^e C.A[t]^r$. Thus $p(x,\tilde{y}) \in q^e CA[t]^r$, that is, there is $z \in q^e A[t]^r$ such that $p(x,\tilde{y}) = C.z$. It remains to prove that $z \in m^{c-1}[t]^r$. By the induction we have $C.z = p(x,\tilde{y}) = 0 \mod m^{c-1}$, and multiplying both sides by $\hat{C}$ we have $\det(C)I_s.z = q.u.I_s.z = 0$

mod m^{c-1}. Thus $q.z = 0 \mod m^{c-1}$, and as q is monic, $z = 0 \mod m^{c-1}$ as expected. $\square$

We are continuing the proof of Theorem 2.4. From Lemma 3.3, the A-deformations of γ_0 are in one-to-one correspondence with the set of solutions of system (II), and thus, with the set of solutions of system (III) (for any fixed $e \in \mathbf{N}$). The latter is defined by finite number of equations in finitely many variables, because $x(t)$ could be replaced with $\overline{x} = x(t) \mod q^{e+1}$. Take $e = 1$, for example, so that $x(t) = q^2.\xi + \overline{x}$, where $\xi \in A[[t]]^n$, $\overline{x} \in A[t]^n$, $\deg(\overline{x}) < 2d$. We can consider $\overline{x}$, $\overline{y}$, ξ, q as a new system of unknowns, replacing x, $\overline{y}$, q. Then (II) becomes a finite system of equations over k for $\overline{x}$, $\overline{y}$, q, without involving ξ. By the remark about the restricted functor of points above, this proves that the formal neighbourhood of γ_0 is $FN_{\gamma 0} \simeq \mathrm{Spf}(R[[z_i, i \in \mathbf{N}]])$, where R is a complete local Noetherian ring which defines the formal neighbourhood of a point on a scheme of finite type $y \in Y(k)$. FN_y is defined by equations including the variables $\overline{x}$, $\overline{y}$, q in terms of its functor of points. Thus for any k-algebra S, if $D := \left(\frac{\partial p}{\partial y}\right)(\overline{x}, \overline{y})$, let $Y(S) := \{(q, \overline{x}, \overline{y})\colon q \in S[t], \overline{x} \in S[t]^n/(q^2), \overline{y} \in S[t]^s/(q)\colon \det(D) = 0 \mod q, p(\overline{x}, \overline{y}) = 0 \mod q, \hat{D}.p(\overline{x}, \overline{y}) = 0 \mod q^2\}$. The point $y \in Y(k)$ corresponds to $(q = t^d, \overline{x} = x^0(t) \mod (t^{2d}), \overline{y} = y^0(t) \mod (t^d))$. Then the second factor is $\mathbf{D}^\infty = \mathrm{Spf}(k[[z_i, i \in \mathbf{N}]])$, because for $\xi \in A[[t]]^n$ there is no restriction. This completes the proof. $\square$

Remark 3.4. The number $d = \deg(q)$ in the proof of Theorem 2.2 is equal to $\mathrm{ord}_{\gamma_0}(Jac_X)$, if the variables y_j, $j = 1, \ldots, r$, are chosen so that the $r \times r$ minor defined by them in the Jacobian matrix of X has the minimal possible order defining locally the ideal of X_{sing}.

Example 3.5. 1) Let $\mathrm{char}(k) = 0$ and $X\colon f(x_1, \ldots, x_n) + x_{n+1}^s y = 0$ be a hypersurface in $\mathbf{A}^{n+2}$, for $f \neq 0$ a polynomial, $f(0, \ldots, 0) = 0$ and $s \geq 1$ an integer. Take $\gamma_0 = (0, \ldots, 0, t, 0) \in X_\infty$, so that an A-deformation γ is an $(n+2)$-tuple of power series $(x_1(t), \ldots, x_{n+1}(t), y(t))$ satisfying

(i) $$x_i(t) \in m[[t]], \quad i = 1, \ldots, n, \quad y(t) \in m[[t]],$$

where $m \subset A$ is the maximal ideal of the test ring A. By Weierstrass division theorem, any A-deformation of $x_{n+1}^0(t) = t$ will be of the form $x_{n+1}(t) = (t - \alpha).u(t)$ for some $\alpha \in m$ and $u(t) \in A[[t]]$ invertible. Now, given $\alpha, u(t), x_1(t), \ldots, x_n(t)$, there will be at most one $y(t)$, satisfying (i), and it exists iff

(ii)
$$f(x_1(\alpha), \ldots, x_n(\alpha)) = f'(x_1(\alpha), \ldots, x_n(\alpha)) = \cdots = f^{(s-1)}(x_1(\alpha), \ldots, x_n(\alpha)) = 0,$$

the derivation taken with respect to t.

Indeed, if a solution $y(t)$ of $f(x_1(t), \ldots, x_n(t)) + x_{n+1}^s(t)y(t) = 0$ exists, then it is unique, and $(t - \alpha)^s$ divides $f(x_1(t), \ldots, x_n(t))$. Conversely, if for an $\alpha \in m$, $f(\alpha) = f'(\alpha) = \cdots = f^{(s-1)}(\alpha) = 0$, then $y(t)$ exists because $(t - \alpha)^s$ divides $f(x_1(t), \ldots, x_n(t))$.

That is, (ii) defines a scheme of finite type Y with k-point $y = (0, \ldots, 0)$.

(2) Let again $\mathrm{char}(k) = 0$ and take $X\colon \Sigma_1^n x_i^2 = 0 \subset \mathbb{A}^n$ with $\gamma_0 = (a_1 t, \ldots, a_n t)$ such that $\Sigma_1^n a_i^2 = 0$ and $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$. Then $\deg q = 1$ and we take $s = 1$, so a deformation γ is given by $x_i(t) = u_i(t - \alpha) + v_i$, with u_i invertible and $\alpha \in A$, the testring. We may take $Y = \{(\alpha, u_1, \ldots, u_n, v_1, \ldots, v_n) \in \mathbb{A}^{2n+1}\colon v_n = 0, \Sigma_1^{n-1} u_i v_i = 0, \Sigma_1^{n-1} v_i^1 = 0\}$ and the point $(a_1, \ldots, a_n) \in Y(k)$. For $n = 3$, $Y \simeq \mathrm{Spec}\ k[t]/(t^2)$ is the minimal formal model for γ_0.

4. The relative case of Drinfeld–Grinberg–Kazhdan theorem. One may ask how the relative case of the theorem would look like. In this section we discuss this in the case of smooth and étale morphisms.

Suppose $f\colon X \to Y$ is a morphism between algebraic varieties, inducing $f_\infty\colon X_\infty \to Y_\infty$. Given non degenerate k-arcs $\gamma \in X_\infty$, $\delta \in Y_\infty$ such that $f_\infty(\gamma) = \delta$, one has the induced morphism $\hat{f}\colon FN_\gamma \to FN_\delta$. By DGK theorem the domain is $FN_u \hat{\times} \mathbf{D}^\infty$ with $u \in U(k)$ for a scheme of finite type U, and the target is $FN_v \hat{\times} \mathbf{D}^\infty$ with $v \in V(k)$ for some scheme of finite type V. The question above is for which classes of morphisms f holds that f_∞ induces a well defined morphism between the first factors? If this is the case for any choice of γ, δ as above, we say that the relative case of DGK theorem holds for f.

Take k to be algebraically closed and let f be an étale morphism of finite type. As the fibres of f are finite sets of reduced points ([12]) and we have $X \times Y_\infty = X_\infty$, for any $\gamma \in Y_\infty$ non-degenerated arc, with $\gamma(0) = P$ and $f^{-1}(P) = \{Q_1, \ldots, Q_s\}$, we get uniquely determined arcs $\alpha_i \in X_\infty$ centred at Q_i for all i. As $\pi_Y^{-1}(\gamma) \simeq \pi_X^{-1}(\alpha_i)$ for all i, the formal neighbourhoods are isomorphic. So, taking the minimal formal models of $\gamma, \alpha_1, \ldots, \alpha_s$ in the representation given by Theorem 2.4, from this isomorphism the first factor of FN_γ becomes isomorphic to the first factor of FN_{α_i} for all i.

Theorem 4.1. *Let $f\colon X \to Y$ be a smooth morphism of finite type of relative dimension n, $f_\infty(\gamma) = \delta$ for non-degenerate k-arcs γ, δ. Then the relative case of DGK theorem holds.*

Proof. In an open neighbourhood $P \in U$, f could be represented as a composition of étale $g\colon U \to Y \times \mathbf{A}^n$ followed by the projection $pr_1\colon Y \times \mathbf{A}^n \to Y$. Now we need the following:

Lemma 4.2. *For any smooth V, and $pr_1\colon X \times V \to X$, if γ, δ are non-degenerated closed arcs such that $pr_{1,\infty}(\gamma) = \delta$, then GKD for pr_1, γ, δ holds.*

Proof. As $(X \times_k V)_\infty = X_\infty \times V_\infty$, then $\gamma = (\delta, \eta)$, and $FN_\gamma = FN_\delta \hat{\times} FN_\eta$. By Theorem 2.4 $FN_\delta = \mathrm{Spf}\, R[[x_i, i \in \mathbf{N}]]$ for a complete local Noetherian ring R, thus $FN_\gamma = \mathrm{Spf}(R[[x_i, i \in \mathbf{N}]] \hat{\otimes} k[[z_j, j \in \mathbb{N}]])$. Define $\phi\colon R \to R \hat{\otimes} k \simeq R$, $\phi(q) = q \otimes 1$, take ψ to be the inclusion $k[[x_i, i \in \mathbf{N}]] \hookrightarrow k[[x_i, z_j, i, j \in \mathbf{N}]]$. Then $\phi \hat{\otimes} \psi$ defines the morphism $FN_\gamma \to FN_\delta$ induced by $pr_{1,\infty}$. □

Now the claim of the theorem holds by Lemma 4.2 and the observations prior to Theorem 4.1. □

Recently Chiu, de Fernex and Docampo [2] generalized the notion of embedding codimension for arbitrary local ring (A, m, k) as $\mathrm{ecodim}(A) := ht(\ker(\phi))$ for the natural homomorphism $\phi\colon Sym_k(m/m^2) \to gr(A)$. It coincides with the embedding codimension in the Noetherian setting, and could be viewed as a rough measure of singularities. Taking the local ring of a non degenerate arc, it permits to provide a converse to DGK theorem and moreover, it gives an optimal bound of the embedding codimension of the formal model. Given a morphism $f\colon X \to Y$ it would be interesting to understand how the embedding codimension of the minimal formal model changes for smooth f.

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Institute of Mathematics and Informatics, Bulgarian Academy of Sciences
Akad. G. Bonchev St, Bl. 8, 1113 Sofia, Bulgaria
e-mail: pk5rov@gmail.com