SOME NOTES ON THE FOUR-PARAMETER KIES DISTRIBUTION

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Abstract

This short note is inspired by the work of [1] which is devoted on the four-parameter Kies distribution. Some of the results presented in this seminal paper need to be clarified. We present and prove the correct results following the original work.

Key words: Kies distribution, mean residual life function, moments, Whittaker function

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1. Introduction The original Kies distribution, first introduced in KIES [2] as a derivative of the Weibull distribution, has many further modifications. We refer to KUMAR and DHARMAJA [1,3,4], SANKU et al. [5], AL-BABTAIN et al. [6], and AL SOBHI [7]. This distribution has several nice features – finite domain, large flexibility, different shape of the hazard function, etc. – which destine its wide use in many real life fields. The main distributional and practical properties like moments, the mean residual life function, the failure rate, the quantile function, some conditional expectations, the maximum likelihood estimator, for the four-parameter distribution are obtained in Kumar and Dharmaja [1]. Some of them,

more precisely the results for the moments and the mean residual life function, need some clarification. We present the correct statements in this paper proving them in the same manner as in the original work. The results obtained will be applied further in the reaction network analysis – this research will appear elsewhere.

2. Some facts for the hypergeometric functions First we need to remind some results on the hypergeometric functions. The confluent hypergeometric function (also known as degenerate hypergeometric function), denoted by $_1F_1(c, d; z)$, may be defined as the following series expansion

$$(2.1) \qquad _1F_1(c, d; z) = \sum_{n=0}^{\infty} \frac{c_{(n)}}{n! d_{(n)}} z^n,$$

where $c_{(n)}$ is the Pochhammer symbol

$$(2.2) \qquad c_{(n)} = \frac{\Gamma(c+n)}{\Gamma(c)} = \prod_{j=0}^{n-1} (c+j).$$

Above we use $\Gamma(\cdot)$ for the gamma function. The function $_1F_1(c, d; z)$ is a special solution of the following differential equation

$$(2.3) \qquad zu''(z) + (d - z) u'(z) - cu(z) = 0.$$

A related to the confluent hypergeometric function is the Whittaker function, $M_{k,m}(z)$, which is a solution of the corresponding Whittaker differential equation

$$(2.4) \qquad u''(z) + \left(-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2} \right) u(z) = 0.$$

For more details see GRADSHTEYN and RYZHIK [8], p. 1033. It can be presented as

$$
\begin{aligned}
(2.5) \qquad M_{k,m}(z) &= e^{-\frac{z}{2}} z^{\frac{1}{2}+m} {}_1F_1\left(\frac{1}{2} + m - k, 2m + 1; z \right) \\
&\equiv e^{-\frac{z}{2}} z^{\frac{1}{2}+m} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2} + m - k \right)_{(n)} z^n}{n!(2m+1)_{(n)}}.
\end{aligned}
$$

We need also the following linear combination

$$(2.6) \qquad W_{k,m}(z) = \frac{\Gamma(-2m)}{\Gamma\left(\frac{1}{2} - k - m\right)} M_{k,m}(z) + \frac{\Gamma(2m)}{\Gamma\left(\frac{1}{2} - k + m\right)} M_{k,-m}(z).$$

Note that function (2.6) satisfies Whittaker equation (2.4) too. Also, we have $W_{k,m}(z) = W_{k,-m}(z)$ and the following presentations holds (Gradshteyn and Ryzhik [8], statements 9.220.4 and 9.222, page 1034)

$$(2.7) \qquad W_{k,m}(z) = e^{-\frac{z}{2}} z^{m+\frac{1}{2}} U\left(\frac{1}{2} + m - k, 2m + 1; z\right),$$

where $U(c, d; z)$ is the Tricomi confluent hypergeometric function. It has the following integral presentation for $\Re c > 0$ and $\Re z > 0$ (Gradshteyn and Ryzhik [8], equation 9.211.4, page 1032)

$$(2.8) \qquad U(c, d; z) = \frac{1}{\Gamma(c)} \int_0^{+\infty} e^{-zt} t^{c-1} (1 + t)^{d-c-1} \, dt.$$

Particularly, if $c = 1$, then

$$(2.9) \qquad U(1, d; z) = \int_0^{+\infty} e^{-zt} (1 + t)^{d-2} \, dt.$$

Let us remind also the integral presentation of the gamma functions

$$
\begin{aligned}
\Gamma(z) &= \int_0^{+\infty} e^{-t} t^{z-1} \, dt, \\
(2.10) \qquad \Gamma(z, x) &= \int_x^{+\infty} e^{-t} t^{z-1} \, dt, \\
\gamma(z, x) &= \Gamma(z) - \Gamma(z, x) = \int_0^x e^{-t} t^{z-1} \, dt
\end{aligned}
$$

for $\Re z > 0$.

3. Clarification of the existing results We shall denote the probability density function (PDF) of a random variable by $f(x)$, the cumulative distribution function (CDF) by $F(x)$, and the complementary cumulative distribution function (CCDF, also known as the survival function) by $\overline{F}(x)$.

Let us remind the definition of the mean residual life function. Also, we present an alternative form of this function. For more details see GUPTA and BRADLEY [9].

Proposition 3.1. *Suppose that a random variable ξ has a positive support. The mean residual life function is defined as the following conditional expectation*

$$(3.1) \qquad m(x) = E[\xi - x \mid \xi > x].$$

This function can be rewritten also as

$$(3.2) \qquad m(x) = \frac{1}{\overline{F}(x)} \int_x^{+\infty} \overline{F}(t) \, dt.$$

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We shall follow hereafter the parametrization of the original work of Kumar and Dharmaja [1]. The four-parameter Kies distribution is defined by its PDF:

$$(3.3) \qquad f(z) = \frac{\lambda\beta(b-a)(x-a)^{\beta-1}\exp\left(-\lambda\left(\frac{x-a}{b-x}\right)^{\beta}\right)}{(b-x)^{\beta+1}} I_{x\in(a,b)}.$$

The constants a, b, λ, and β are positive, $a < b$, and the domain of the distribution is the interval (a,b). Equivalently, the distribution can be defined by its CDF or CCDF

$$(3.4) \qquad \begin{aligned} F(x) &= 1 - \exp\left(-\lambda\left(\frac{x-a}{b-x}\right)^{\beta}\right) \\ \overline{F}(x) &= \exp\left(-\lambda\left(\frac{x-a}{b-x}\right)^{\beta}\right). \end{aligned}$$

We need now the following series expansion, formulated in the lemma below.

Lemma 3.1. *If $i > 0$, then the following binomial series holds*

$$(3.5) \qquad \begin{aligned} (1+x)^{-i} &= \sum_{j=0}^{\infty} \frac{(-1)^j(i)_j x^j}{j!}, \quad |x| < 1 \\ (1+x)^{-i} &= \sum_{j=0}^{\infty} \frac{(-1)^j(i)_j x^{-(i+j)}}{j!}, \quad |x| > 1. \end{aligned}$$

We shall examine the moments and the mean residual life function. They are derived as Results 7 and 8 in Kumar and Dharmaja [1], but there are some little mistakes. We present and prove below the correct statements.

Proposition 3.2. *The n-th moment of a Kies distributed random variable ξ, $\mu_n := E[\xi^n]$, can be presented as the following sum*

$$(3.6) \qquad \begin{aligned} \mu_n &= \sum_{i=0}^{n}\left[\binom{n}{i}(b-a)^i b^{n-i}\sum_{j=0}^{\infty}\left(\frac{(-1)^{i+j}(i)_j\gamma\left(\frac{j}{\beta}+1,\lambda\right)}{j!\lambda^{\frac{j}{\beta}}}\right)\right] \\ &+ \sum_{i=0}^{n}\left[\binom{n}{i}(b-a)^i b^{n-i}\sum_{j=0}^{\infty}\left(\frac{(-1)^{i+j}(i)_j\lambda^{\frac{i+j}{2\beta}}e^{-\frac{\lambda}{2}}W_{-\frac{i+j}{2\beta},\frac{\beta-i-j}{2\beta}}(\lambda)}{j!}\right)\right]. \end{aligned}$$

We shall use below the notations

$$(3.7) \qquad \eta(x) = \left(\frac{x-a}{b-x}\right)^{\beta}, \quad \Delta_j(x) = \frac{(-1)^j(2)_j W_{-\frac{j+1+\beta}{2\beta},-\frac{j+1}{2\beta}}(x)}{j!}.$$

The mean residual life function has different forms in the first and second halves of the distribution domain. If $x \leq (a+b)/2$, then

$$(3.8) \quad m(x) = \frac{b-a}{\beta} e^{\lambda \eta(x)} \left[\sum_{j=0}^{\infty} \left\{ \frac{(-1)^j (2)_j \left[\Gamma\left(\frac{j+1}{\beta}\right) - \gamma\left(\frac{j+1}{\beta}, \lambda\eta(x)\right) - \Gamma\left(\frac{j+1}{\beta}, \lambda\right) \right]}{j! \lambda^{\frac{j+1}{\beta}}} \right\} + e^{-\frac{\lambda}{2}} \sum_{j=0}^{\infty} \lambda^{\frac{j+1-\beta}{2\beta}} \Delta_j(\lambda) \right].$$

Otherwise, if $x > (a+b)/2$, then

$$(3.9) \qquad m(x) = \frac{b-a}{\beta} e^{\frac{\lambda \eta(x)}{2}} \sum_{j=0}^{\infty} \lambda^{\frac{j+1-\beta}{2\beta}} \eta(x)^{-\frac{j+1+\beta}{2\beta}} \Delta_j\left(\lambda\eta(x)\right).$$

Proof. Following Kumar and Dharmaja [1], using form (3.3) of the PDF, changing the variables $u = \eta(x)$ (the function $\eta(\cdot)$ is given in formulas (3.7)), and applying formulas (3.5) – the first one when expanding $\left(1 + u^{\frac{1}{\beta}}\right)^{-i}$ for $u < 1$ and the second one when $u > 1$ – we obtain for the n-th moment μ_n

(3.10)

$$\mu_n \equiv E\left[\xi^n\right] = \int_a^b x^n f(x)\, dx$$

$$= \lambda \sum_{i=0}^{n} \left[\binom{n}{i} (b-a)^i b^{n-i} \sum_{j=0}^{\infty} \left[\frac{(-1)^{i+j} (i)_j}{j!} \left(\int_0^1 u^{\frac{j}{\beta}} e^{-\lambda u}\, du + \int_1^{+\infty} u^{-\frac{i+j}{\beta}} e^{-\lambda u}\, du \right) \right] \right].$$

For the first integral we change the variables as $t = \lambda u$ and apply the third formula from (2.10); for the second one we use $t = u - 1$ and equations (2.7) and (2.9):

$$\int_0^1 u^{\frac{j}{\beta}} e^{-\lambda u}\, du = \int_0^\lambda \left(\frac{t}{\lambda}\right)^{\frac{j}{\beta}} e^{-t}\, d\frac{t}{\lambda} = \lambda^{-\frac{j}{\beta}-1} \gamma\left(\frac{j}{\beta}+1, \lambda\right)$$

$$\int_1^{+\infty} u^{-\frac{i+j}{\beta}} e^{-\lambda u}\, du = \int_0^{+\infty} (t+1)^{-\frac{i+j}{\beta}} e^{-\lambda(t+1)}\, dt$$

$$(3.11) \qquad\qquad = e^{-\lambda} \int_0^{+\infty} (t+1)^{-\frac{i+j}{\beta}} e^{-\lambda t}\, dt$$

$$= e^{-\lambda} U\left(1, 2 - \frac{i+j}{\beta}, \lambda\right)$$

$$= \lambda^{\frac{i+j}{2\beta}-1} e^{-\frac{\lambda}{2}} W_{-\frac{i+j}{2\beta}, \frac{\beta-i-j}{2\beta}}(\lambda).$$

We derive result (3.6) estimating integrals (3.11) in equation (3.10). Note that the sum in formula (3.10) is multiplied by λ.

Let us turn to the mean residual life function – we use equation (3.2). If x is in the first half of the distribution domain, the equation (33) from Kumar and Dharmaja [1] gives

(3.12)

$$m(x) = \frac{b-a}{\beta} e^{\lambda\eta(x)} \sum_{j=0}^{\infty} \frac{(-1)^j (2)_j}{j!} \left[\begin{array}{c} \int_0^{+\infty} u^{\frac{j+1}{\beta}-1} e^{-\lambda u}\, du - \int_0^{\eta(x)} u^{\frac{j+1}{\beta}-1} e^{-\lambda u}\, du \\ - \int_1^{+\infty} u^{\frac{j+1}{\beta}-1} e^{-\lambda u}\, du + \int_1^{+\infty} u^{-\frac{j+1+\beta}{\beta}} e^{-\lambda u}\, du \end{array} \right].$$

For the first three integrals we change the variables as $t = \lambda u$ and use the formulas (2.10). For the fourth integral we use again $t = u - 1$ and equations (2.7) and (2.9). The integrals are derived through equations (3.11).

If $x > (b+a)/2$, then equation (34) from Kumar and Dharmaja [1] gives

(3.13) $$m(x) = \frac{b-a}{\beta} e^{\lambda\eta(x)} \sum_{j=0}^{\infty} \frac{(-1)^j (2)_j}{j!} \int_{\eta(x)}^{+\infty} u^{-\frac{j+1+\beta}{\beta}} e^{-\lambda u}\, du.$$

Analogously, changing the variables as $t = \dfrac{u}{\eta(x)} - 1$, we derive formula (3.9) by the use of

$$\int_{\eta(x)}^{+\infty} u^{-\frac{j+1+\beta}{\beta}} e^{-\lambda u}\, du = \eta(x)^{-\frac{j+1}{\beta}} e^{-\lambda\eta(x)} \int_0^{+\infty} (t+1)^{-\frac{j+1+\beta}{\beta}} e^{-\lambda\eta(x)t}\, dt$$

(3.14) $$= \eta(x)^{-\frac{j+1}{\beta}} e^{-\lambda\eta(x)} U\left(1, 1 - \frac{j+1}{\beta}, \lambda\eta(x)\right)$$

$$= e^{-\frac{\lambda\eta(x)}{2}} \lambda^{\frac{j+1-\beta}{2\beta}} \eta(x)^{-\frac{j+1+\beta}{2\beta}} W_{-\frac{j++1+\beta}{2\beta}, -\frac{j+1}{2\beta}} (\lambda\eta(x)). \quad \square$$

Using the moments μ_n presented in formula (3.6), we can recover the moment generating function:

Proposition 3.3. *The moment generating function of a four-parameter Kies distributed random variable is*

(3.15) $$\psi(t) = \sum_0^{\infty} \frac{\mu_n}{n!} t^n.$$

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