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A GENERALIZATION OF DESCENT POLYNOMIALS

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The notion of a *descent polynomial*, a function in enumerative combinatorics that counts permutations with specific properties, enjoys a revived recent research interest due to its connection with other important notions in combinatorics, viz. *peak polynomials* and *symmetric functions*. We define the function $\mathfrak{d}^m(I,n)$ as a generalization of the descent polynomial and obtain an explicit formula for $\mathfrak{d}^m(I,n)$ when m is sufficiently large.

1. Introduction. The *descent polynomial* is a function in enumerative combinatorics that counts permutations with specific properties. It originates from 1915, when MacMahon [3] introduced it in his book *Combinatory Analysis*. However, in the following years, there was not much published about the descent polynomial. It was only in 2017 when the topic was revisited by Diaz-Lopez, Harris, Insko, Omar, and Sagan [2]. In their paper, Diaz-Lopez et al. gave two recurrence relations [2, Section 2] for the descent polynomial, looked at its coefficients, and investigated its roots. In 2018, Bencs [1] answered some questions about the coefficients of the descent polynomial, which were stated earlier by Diaz-Lopez et al. Since then, there were several other publications looking at some generalizations of the descent polynomial. In this paper, we look at an intuitive generalization about the multiplicity of the elements.

First, let us define the original descent polynomial that was studied in [2]. Let $I = \alpha_1, \alpha_2, \ldots, \alpha_t$ be a finite set of positive integers. We consider permutations of the set $\{1, 2, \ldots, n\}$. The *descent polynomial* is equal to the number of permutations for which the set of all positions in the permutation where the corresponding elements are bigger than the next ones, is exactly I. We study the following generalization, which has not been studied yet: instead of the set $\{1, 2, \ldots, n\}$, we are considering permutations of the multiset
$\{\underbrace{1,\ldots,1}_{m},\underbrace{2,\ldots,2}_{m},\ldots,\underbrace{n,\ldots,n}_{m}\}$, where each number is of multiplicity m. In this case we use the notation $\mathfrak{d}^m(I,n)$. Note that the descent polynomial is equal to $\mathfrak{d}^1(I,n)$.

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2. Preliminaries. For the rest of the paper, we assume that n and m are positive integers and I is a finite set of positive integers. If I is non-empty, we let $I = \{\alpha_1, \alpha_2, \ldots, \alpha_t\}$, where $\alpha_1 < \alpha_2 < \cdots < \alpha_t$. By I^- we denote the set I without its biggest element α_t, so $I^- = \{\alpha_1, \alpha_2, \ldots, \alpha_{t-1}\}$. Let $\mathrm{Comp}(t)$ be the set of all compositions of t. Recall that a composition of a positive integer t is an ordered sequence of positive integers with sum t.

Definition 2.1. *Let $v = (v_1, v_2, v_3, \ldots, v_\ell)$ be a finite sequence of positive integers. We define*

$$\mathrm{Des}(v) = \{i \in \{1, 2, 3, \ldots, \ell - 1\} : v_i > v_{i+1}\}$$

to be the descent set of the sequence v.

Example 2.2. *If $v = (1, 3, 2, 6, 1, 1, 9, 3)$, then $\mathrm{Des}(v) = \{2, 4, 7\}$.*

Next, we formally define the generalization of the descent polynomial, which we are considering in this paper.

Definition 2.3. *Let S_n be the set of all permutations of the set*

$$\{1, 2, 3, \ldots, n\}.$$

Then we define the descent polynomial $\mathfrak{d}(I, n)$ such that

$$\mathfrak{d}(I, n) = \#\{w \in S_n : \mathrm{Des}(w) = I\}.$$

Example 2.4. *If $n = 5$ and $I = \{2, 4\}$, then $\mathfrak{d}(\{1, 3\}, 4) = 5$, because there are 5 such permutations:*

$$(4, 1, 3, 2) \quad (4, 2, 3, 1) \quad (3, 1, 4, 2) \quad (3, 2, 4, 1) \quad (2, 1, 4, 3).$$

Definition 2.5. *Let $S_n^{(m)}$ be the set of all permutations of the multiset*

$$\{\underbrace{1, 1, \ldots, 1}_{m}, \underbrace{2, 2, \ldots, 2}_{m}, \ldots, \underbrace{n, n, \ldots, n}_{m}\}.$$

We define the function $\mathfrak{d}^m(I, n)$ such that

$$\mathfrak{d}^m(I, n) = \#\{w \in S_n^{(m)} : \mathrm{Des}(w) = I\}.$$

Example 2.6. *If $n = 3$, $m = 2$, and $I = \{2\}$, then $\mathfrak{d}^2(\{2\}, 3) = 5$, because there are 5 such permutations:*

$$(1, 2, 1, 2, 3, 3) \quad (1, 3, 1, 2, 2, 3) \quad (2, 2, 1, 1, 3, 3) \quad (2, 3, 1, 1, 2, 3) \quad (3, 3, 1, 1, 2, 2).$$

3. General properties of $\mathfrak{d}^m(I, n)$. In this section, we show that $\mathfrak{d}^m(I, n)$ is weakly increasing with m. We also prove that from some m onward, the function $\mathfrak{d}^m(I, n)$ stabilizes.

Proposition 3.1. *For a positive integer $n > \alpha_t$, we have the following inequality*

$$\mathfrak{d}^1(I, n) \leq \mathfrak{d}^2(I, n) \leq \mathfrak{d}^3(I, n) \leq \mathfrak{d}^4(I, n) \leq \cdots.$$

Proof. By determining the first α_t elements of the permutation, we determine the entire permutation, because the rest of the elements are in weakly increasing order. We juxtapose each permutation $w \in S_n^{(m)}$ with a permutation $w' \in S_n^{(m+1)}$, by using the first α_t elements of w in w'. Thus, we preserve the descent set I everywhere except possibly at position α_t. We want to compare w'_{α_t} and w'_{α_t+1} (α_t-th and $(\alpha_t + 1)$-th elements in w'). In the first α_t elements of w', the number 1 appears at most m times, so there is at least one 1 in the rest of the elements. The rest of the elements are in weakly increasing order, so $w'_{\alpha_t+1} = 1$. This leads us to:

$$w'_{\alpha_t} = w_{\alpha_t} > w_{\alpha_t+1} \geq 1 = w'_{\alpha_t+1}.$$

Because we can juxtapose each permutation in $S_n^{(m)}$ with a different permutation in $S_n^{(m+1)}$, it follows that $\mathfrak{d}^m(I,n) \leq \mathfrak{d}^{m+1}(I,n)$. $\square$

Proposition 3.2. *The function $\mathfrak{d}^m(I,n)$ stabilizes for $m \geq \alpha_t - t + 1$:*

$$\mathfrak{d}^{\alpha_t-t+1}(I,n) = \mathfrak{d}^{\alpha_t-t+2}(I,n) = \mathfrak{d}^{\alpha_t-t+3}(I,n) = \cdots.$$

Proof. To determine the entire permutation, it is enough to determine the first α_t elements of the permutation, because the rest of the elements are in weakly increasing order. Let $m \geq \alpha_t - t + 1$. The number 1 appears at most $\alpha_t - t$ times in the first α_t elements, because $w_{\alpha_i} \neq 1$ for $i: \ 1 \leq i \leq t$. So, the number 1 appears at least once in the rest of the elements, which means that $w_{\alpha_t+1} = 1$. Also, we can use each number from 1 to n as many times as we want in the first α_t elements, because the number 1 appears at most $\alpha_t - t$ times, and each number from 2 to n appears at most $\alpha_t - t + 1$ times there. Therefore, when $m \geq \alpha_t - t + 1$, the value of $\mathfrak{d}^m(I,n)$ is equal to the number of sequences $v = (v_1, v_2, \ldots, v_{\alpha_t})$, which satisfy the conditions:

- $\mathrm{Des}(v) = I^-$,
- $v_i \in \{1, 2, \ldots, n\}$,
- $v_{\alpha_t} \neq 1$.

So, for $m \geq \alpha_t - t + 1$, the value of $\mathfrak{d}^m(I,n)$ does not depend on m, which means that for $m \geq \alpha_t - t + 1$, the function $\mathfrak{d}^m(I,n)$ stabilizes. $\square$

4. Formula for $d^\infty(I,n)$. In the previous section, we proved that $\mathfrak{d}^m(I,n)$ stabilizes for $m \geq \alpha_t - t + 1$, so it is reasonable to introduce a new notation for the stabilized function.

Definition 4.1. *Let $d^\infty(I,n)$ equals the function $\mathfrak{d}^m(I,n)$ for $m \geq \alpha_t - t + 1$.*

In this section, we derive an explicit formula for $d^\infty(I,n)$. However, we first need to introduce some crucial notations.

Definition 4.2. *For the set $I = \{\alpha_1, \alpha_2, \ldots, \alpha_t\}$, we let $\beta = (\beta_1, \beta_2, \ldots, \beta_t)$ denote the sequence of the first differences of the sequence $(0, \alpha_1, \alpha_2, \ldots, \alpha_t)$. So, $\beta_1 = \alpha_1 - 0$, $\beta_2 = \alpha_2 - \alpha_1, \ldots, \ \beta_t = \alpha_t - \alpha_{t-1}$.*

When we talk about compositions we imagine putting separators between balls in a line. For example, the division of the balls below corresponds to the composition $(3, 1, 2, 2)$.

$$. \ . \ . \mid . \mid . \ . \mid . \ .$$

As we can see the numbers of balls between the separators give us the elements of the composition. Let us put a weight on each ball and instead of taking the number of balls, we take the total weight of the balls between the separators. Let us look at the previous example, but this time let us put weights on the balls. We take the weights to be $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8$. Now the composition $(3, 1, 2, 2)$ corresponds to

$$\beta_1 \ \beta_2 \ \beta_3 \mid \beta_4 \mid \beta_5 \ \beta_6 \mid \beta_7 \ \beta_8.$$

We can write this in short as $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, where $\sigma_1 = \beta_1 + \beta_2 + \beta_3$, $\sigma_2 = \beta_4$, $\sigma_3 = \beta_5 + \beta_6$, and $\sigma_4 = \beta_7 + \beta_8$. In the next definition, we define this more generally.

Definition 4.3. *Let $A \in \mathrm{Comp}(t)$ be a composition of t and $A = (a_1, a_2, \dots, a_s)$. For the sequence $\beta = (\beta_1, \beta_2, \dots, \beta_t)$, we define the function f_β, such that $f_\beta(A) = (\sigma_1, \sigma_2, \dots, \sigma_s)$, where*

$$\sigma_1 = \beta_1 + \beta_2 + \dots + \beta_{a_1}$$

$$\sigma_i = \beta_{a_1+a_2+\dots+a_{i-1}+1} + \beta_{a_1+a_2+\dots+a_{i-1}+2} + \dots + \beta_{a_1+a_2+\dots+a_i} \quad \text{for all } i:\ 2 \le i \le s.$$

To derive an explicit formula for $d^\infty(I, n)$ we first need to prove the following lemma.

Lemma 4.4. *Let j and n be positive integers such that $j \le n$. For the non-empty set $I = \{\alpha_1, \alpha_2, \dots, \alpha_t\}$, we define $\beta = (\beta_1, \beta_2, \dots, \beta_t)$ as in Definition 4.2. Then, the number of sequences $v = (v_1, v_2, \dots, v_{\alpha_t})$, which satisfy the conditions:*

- $v_i \in \{1, 2, \dots, n\}$
- $v_{\alpha_t} = j$
- $\mathrm{Des}(v) = I^-$

is

$$(1) \qquad \sum_{A \in \mathrm{Comp}(t)} (-1)^{t-s} \binom{n-1+\sigma_1}{\sigma_1} \dots \binom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \binom{j-1+\sigma_s-1}{\sigma_s-1},$$

where the summation is over all compositions $A \in \mathrm{Comp}(t)$. Recall that from Definition 4.3 the variables $\sigma_1, \sigma_2, \dots, \sigma_s$ depend on A by the relation $f_\beta(A) = (\sigma_1, \sigma_2, \dots, \sigma_s)$.

Proof. To prove this statement we make an induction on t — the number of elements in I.

When $t = 1$, we have that $I = \alpha_1$ and $I^- = \varnothing$. Therefore, the sequence v should look like

$$1 \le v_1 \le v_2 \le v_3 \le \dots \le v_{\alpha_1} = j.$$

There are $\binom{j-1+\alpha_1-1}{\alpha_1-1}$ such sequences. Let us see what result we get by applying formula (1). When $t = 1$, we get that $\beta = (\alpha_1)$ and the set $\mathrm{Comp}(1) = \{(1)\}$. Therefore, $f_\beta((1)) = \alpha_1$ and $\sigma_1 = \alpha_1$. Substituting these values in formula (1), we get

$$\sum_{A \in \mathrm{Comp}(t)} (-1)^{t-s} \binom{n-1+\sigma_1}{\sigma_1} \dots \binom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \binom{j-1+\sigma_s-1}{\sigma_s-1} =$$

$$= (-1)^{1-1} \binom{j-1+\sigma_1-1}{\sigma_1-1} = \binom{j-1+\alpha_1-1}{\alpha_1-1},$$

so formula (1) is true for $t = 1$. Let us carry out the induction step from $t-1$ to t. We fix $v_{\alpha_{t-1}} = i$. From the induction hypothesis, we know that there are

$$\sum_{A \in \mathrm{Comp}(t-1)} (-1)^{t-s-1} \binom{n-1+\sigma_1}{\sigma_1} \dots \binom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \binom{i-1+\sigma_s-1}{\sigma_s-1}$$

ways to choose the first α_{t-1} elements of the sequence. Let us look at the last $\alpha_t - \alpha_{t-1}$

elements of the sequence:

$$i > v_{\alpha_{t-1}+1} \leq v_{\alpha_{t-1}+2} \leq \cdots \leq v_{\alpha_t - 1} \leq v_{\alpha_t} = j.$$

If $i+1 \geq j$, there are

$$\binom{j-1+\alpha_t-\alpha_{t-1}-1}{\alpha_t-\alpha_{t-1}-1} = \binom{j-1+\beta_t-1}{\beta_t-1}$$

ways to choose the last $\alpha_t - \alpha_{t-1}$ elements of the sequence.

If $i \leq j$, there are

$$\binom{j-1+\beta_t-1}{\beta_t-1} - \binom{j-i+\beta_t-1}{\beta_t-1}$$

ways to choose the last $\alpha_t - \alpha_{t-1}$ elements of the sequence.

Therefore, the number of sequences v with a descent set $I^- = \{\alpha_1, \alpha_2, \ldots, \alpha_{t-1}\}$, last element $v_{\alpha_t} = j$ and elements from the set $\{1, 2, \ldots, n\}$ is

$$\begin{aligned}
&\sum_{i=1}^{j} \left(\tbinom{j-1+\beta_t-1}{\beta_t-1} - \tbinom{j-i+\beta_t-1}{\beta_t-1}\right) \sum_{A\in\mathrm{Comp}(t-1)} (-1)^{t-s-1} \tbinom{n-1+\sigma_1}{\sigma_1} \cdots \tbinom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \tbinom{i-1+\sigma_s-1}{\sigma_s-1} \\
&\quad + \sum_{i=j+1}^{n} \tbinom{j-1+\beta_t-1}{\beta_t-1} \sum_{A\in\mathrm{Comp}(t-1)} (-1)^{t-s-1} \tbinom{n-1+\sigma_1}{\sigma_1} \cdots \tbinom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \tbinom{i-1+\sigma_s-1}{\sigma_s-1} \\
&= \sum_{i=1}^{n} \sum_{A\in\mathrm{Comp}(t-1)} (-1)^{t-s-1} \tbinom{n-1+\sigma_1}{\sigma_1} \cdots \tbinom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \tbinom{i-1+\sigma_s-1}{\sigma_s-1} \tbinom{j-1+\beta_t-1}{\beta_t-1} \\
&\quad + \sum_{i=1}^{j} \sum_{A\in\mathrm{Comp}(t-1)} (-1)^{t-s} \tbinom{n-1+\sigma_1}{\sigma_1} \cdots \tbinom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \tbinom{i-1+\sigma_s-1}{\sigma_s-1} \tbinom{j-i+\beta_t-1}{\beta_t-1} \\
&= \sum_{A\in\mathrm{Comp}(t-1)} (-1)^{t-(s+1)} \tbinom{n-1+\sigma_1}{\sigma_1} \cdots \tbinom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \tbinom{n-1+\sigma_s}{\sigma_s} \tbinom{j-1+\beta_t-1}{\beta_t-1} \\
&\quad + \sum_{A\in\mathrm{Comp}(t-1)} (-1)^{t-s} \tbinom{n-1+\sigma_1}{\sigma_1} \cdots \tbinom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \tbinom{j-1+\sigma_s+\beta_t-1}{\sigma_s+\beta_t-1} \\
&= \sum_{A\in\mathrm{Comp}(t)} (-1)^{t-s} \tbinom{n-1+\sigma_1}{\sigma_1} \cdots \tbinom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \tbinom{j-1+\sigma_s-1}{\sigma_s-1},
\end{aligned}$$

which finishes the induction. □

In the next theorem, we derive an explicit formula for $d^\infty(I,n)$. To do this we use the result from the previous lemma.

Theorem 4.5. *For the non-empty set $I = \{\alpha_1, \alpha_2, \ldots, \alpha_t\}$, we define $\beta = (\beta_1, \beta_2, \ldots, \beta_t)$ as in Definition 4.2. Then:*

$$d^\infty(I,n) = \sum_{A\in\mathrm{Comp}(t)} (-1)^{t-s} \binom{n-1+\sigma_1}{\sigma_1} \cdots \binom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \left(\binom{n-1+\sigma_s}{\sigma_s} - 1\right),$$

where, in the summation above, we sum over all compositions $A \in \mathrm{Comp}(t)$. Recall that from Definition 4.3 the variables $\sigma_1, \sigma_2, \ldots, \sigma_s$ depend on A by the relation $f_\beta(A) = (\sigma_1, \sigma_2, \ldots, \sigma_s)$.

Proof. From the proof of Proposition 3.2, we know that $d^\infty(I,n)$ is equal to the number of sequences $v = (v_1, v_2, v_3, \ldots, v_{\alpha_t})$, which satisfy the conditions:

- $\mathrm{Des}(v) = I^-$,
- $v_i \in \{1, 2, \ldots, n\}$,
- $v_{\alpha_t} \neq 1$.

We can compute the number of sequences v by summing over all j from 2 to n in Lemma 4.4. Therefore, we obtain that

$$d^{\infty}(I,n) = \sum_{j=2}^{n} \sum_{A\in \mathrm{Comp}(t)} (-1)^{t-s} \binom{n-1+\sigma_1}{\sigma_1} \cdots \binom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \binom{j-1+\sigma_s-1}{\sigma_s-1}$$

$$= \sum_{A\in \mathrm{Comp}(t)} (-1)^{t-s} \binom{n-1+\sigma_1}{\sigma_1} \cdots \binom{n-1+\sigma_{s-1}}{\sigma_{s-1}} \left(\binom{n-1+\sigma_s}{\sigma_s} - 1 \right).$$

□

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ОБОБЩЕНИЕ НА СПУСКАЩИТЕ СЕ ПОЛИНОМИ

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В комбинаториката *спускащият се полином* е функция, с която се изброяват пермутации със специфични свойства. В последните години интересът към него е подновен поради връзката му с други важни понятия в комбинаториката като *върхови полиноми* и *симетрични функции.* Дефинираме функцията $\mathfrak{d}^m(I,n)$ като обобщение на спускащия се полином и извеждаме явна формула за $\mathfrak{d}^m(I,n)$, когато m е достатъчно голямо.