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ON THE UNIFORM DECAY OF THE LOCAL ENERGY

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Abstract. It is proved in [1],[2] that in odd dimensional spaces any uniform decay of the local energy implies that it must decay exponentially. We extend this to even dimensional spaces and to more general perturbations (including the transmission problem) showing that any uniform decay of the local energy implies that it must decay like $O(t^{-2n})$, $t \gg 1$ being the time and n being the space dimension.

1. Introduction and statement of results. It is proved in [1] using the Lax-Phillips theory that in the case of obstacle scattering in odd dimensional spaces, if the local energy decays uniformly to zero, it must decay exponentially. This was extended in [2] for more general perturbations but still in odd dimensional space. The purpose of this note is to extend these results to the case of even dimensional spaces and to more general perturbations. Let $\Omega \subset \mathbb{R}^n, n \geq 2$ be a connected complement of a compact obstacle with smooth boundary. Let also $\Omega_{N_0} \subset \Omega_{N_0-1} \subset \cdots \subset \Omega_1 \subset \Omega_0 := \Omega$ be a finite number of open connected domains with smooth boundaries and bounded complements

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such that $\mathcal{O}_k = \Omega_{k-1} \backslash \Omega_k, k = 1, \dots, N_0$, are bounded connected domains. Define the Hilbert space $H = \oplus_{k=1}^{N_0} L^2(\mathcal{O}_k; c_k(x)dx) \oplus L^2(\Omega_{N_0})$. Let P_k, $k = 1, \dots, N_0$, be differential operators defined in $\mathcal{O}_k$, respectively, of the form

$$P_k = -c_k(x)^{-1} \sum_{i,j=1}^{n} \partial_{x_i}(g_{ij}^{(k)}(x)\partial_{x_j})$$

with smooth coefficients. Let P be a selfadjoint, positive operator on H with absolutely continuous spectrum only, such that

$$P|_{\mathcal{O}_k} = P_k, \quad P|_{\Omega_{N_0}} = -\Delta = -\sum_{j=1}^{n} \partial_{x_j}^2.$$

We also suppose that P is elliptic, i.e. the operator

$$(P+1)^{-m} : H \to \oplus_{k=1}^{N_0} H^{2m}(\mathcal{O}_k) \oplus H^{2m}(\Omega_{N_0})$$

is bounded for every $m \geq 0$.

Set $R(\lambda) = (P - \lambda^2)^{-1} : H \to H$ for $\operatorname{Im} \lambda < 0$, and let $\chi \in C_0^\infty(\mathbb{R}^n), \chi = 1$ on $B = \{x \in \mathbb{R}^n : |x| \leq \rho_0\}, \rho_0 \gg 1$. Then $R_\chi(\lambda) = \chi R(\lambda) \chi : H \to H$ extends to a meromorphic function on $\mathbb{C}$ if n is odd, and on the Riemann surface, Λ, of $\log \lambda$, if n is even (e.g. see [5]). Suppose that

$$\|\lambda R_\chi(-i\lambda)\| < \infty, \qquad \lambda \to 0, \lambda > 0, \tag{1.1}$$

where $\| \cdot \|$ denotes the norm in $\mathcal{L}(H, H)$.

Denote by $u(t)$ the solution of the Cauchy problem

$$\begin{cases} (\partial_t^2 + P)u = 0, \\ u(0) = f_1, \partial_t u(0) = f_2. \end{cases}$$

Let $a > \rho_0$ and set $B_a = \{x \in \mathbb{R}^n : |x| \leq a\}$. Given any $m \geq 0$, set

$$p_m(t) = \sup \left\{ \frac{\|\nabla_x u\|_{L^2(B_a \cap \Omega)} + \|\partial_t u\|_{L^2(B_a \cap \Omega)}}{\|\nabla_x f_1\|_{H^m(B_a \cap \Omega)} + \|f_2\|_{H^m(B_a \cap \Omega)}}, \right.$$

$$\left. (0,0) \neq (f_1, f_2) \in C^\infty(\Omega) \times C^\infty(\Omega), \operatorname{supp} f_j \subset B_a \right\}.$$

Our main result is the following

Theorem 1.1. *The following three statements are equivalent:*

i)

$$\lim_{t\to+\infty} p_0(t) = 0. \tag{1.2}$$

ii) There exist constants $C, C_1 > 0$ so that

$$\|\lambda R_\chi(\lambda)\| \le C, \quad \lambda \in \mathbb{R},\, |\lambda| \ge C_1. \tag{1.3}$$

iii) There exist constants $C, \gamma > 0$ so that

$$p_0(t) \le \begin{cases} Ce^{-\gamma t}, & n \quad odd, \\ Ct^{-n}, & n \quad even. \end{cases} \tag{1.4}$$

Note that a decay like (1.4) is known to hold for nontrapping perturbations (e.g. see [4]). Here *nontrapping* means that every generalized geodesic leaves any compact in a finite time. It is worth noticing however that the inverse statement is not true in general, i.e. there could be situations where the statements of the above theorem hold but the perturbation is not nontrapping in the sense of the above definition. In fact, to my best knowledge there is no interesting situation of scattering in which such an inverse statement to be shown to hold.

We will derive the above theorem from the following

Theorem 1.2. *Suppose that $R_\chi(\lambda)$ admits analytic continuations in $\Lambda_\pm := \{\lambda \in \mathbb{C} : 0 \le \operatorname{Im}\lambda \le C, \pm\operatorname{Re}\lambda > 0\}, C > 0$, such that*

$$\|\lambda R_\chi(\lambda)\| \le C_1|\lambda|^k, \quad \operatorname{Im}\lambda \le C,\, |\operatorname{Re}\lambda| \ge C_2, \tag{1.5}$$

for some $C_1, C_2 > 0$ and $k \ge 0$. Then there exist constants $C_3, \gamma > 0$ so that

$$p_k(t) \le \begin{cases} C_3e^{-\gamma t}, & n \quad odd, \\ C_3t^{-n}, & n \quad even. \end{cases} \tag{1.6}$$

The paper is organized as follows. In Section 2 we derive Theorem 1.1 from Theorem 1.2. In Section 3 we prove Theorem 1.2.

2. Proof of Theorem 1.1. The purpose of this section is to show how to derive Theorem 1.1 from Theorem 1.2. We begin with the following

Lemma 2.1. *The condition* (1.2) *implies* (1.5) *with $k = 0$.*

P r o o f. Recall the formula

$$\lambda R(\lambda) = i \int_0^\infty e^{-it\lambda} \cos(t\sqrt{P})dt, \qquad \operatorname{Im}\lambda < 0.$$

Let $\chi \in C_0^\infty(\mathbb{R}^n), \chi = 1$ on B. Assume (1.2) fulfilled. Then for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ so that

$$\|\lambda R_\chi(\lambda)\| \leq \int_0^\infty e^{-t|\operatorname{Im}\lambda|} \|\chi \cos(t\sqrt{P})\chi\| dt$$

$$\leq C_\varepsilon + \varepsilon \int_0^\infty e^{-t|\operatorname{Im}\lambda|} dt = C_\varepsilon + \varepsilon |\operatorname{Im}\lambda|^{-1}, \quad \operatorname{Im}\lambda < 0. \tag{2.1}$$

Choose functions $\chi_1, \chi_2, \chi_3, \eta \in C_0^\infty(\mathbb{R}^n), \chi_1 = 1$ on B, $\chi_2 = 1$ on $\operatorname{supp}\chi_1$, $\chi_3 = 1$ on $\operatorname{supp}\chi_2$, $\chi = 1$ on $\operatorname{supp}\chi_3$, and $\eta = 1$ on $\operatorname{supp}(1-\chi_2)\chi$, $\eta = 0$ on $\operatorname{supp}\chi_1$. As in [5], we have

$$R_\chi(z)(1 - K(z)) = K_1(z), \tag{2.2}$$

where

$$K(z) = ([\chi_1, \Delta]R_0(z)\eta - [\chi_1, \Delta]R_0(\lambda)\eta)\widetilde{K}(\lambda) + (z^2 - \lambda^2)\chi_2 R_\chi(\lambda),$$

$$K_1(z) = (1 - \chi_1)(\chi R_0(z)\eta - \chi R_0(\lambda)\eta)\widetilde{K}(\lambda) + R_\chi(\lambda),$$

$$\widetilde{K}(\lambda) = (1 - \chi_2)\chi + [\chi_2, \Delta]R_0(\lambda)\chi_3 + [\chi_2, \Delta]R_0(\lambda)[\chi_3, \Delta]R_\chi(\lambda),$$

where $\lambda \in \mathbb{C}$, $\operatorname{Im}\lambda < 0$, and $R_0(z)$ denotes the free outgoing resolvent of the Laplacian in $\mathbb{R}^n$. Clearly, $K(z)$ and $K_1(z)$ are analytic on $\mathbb{C}$ when n is odd and on Λ when n is even. Moreover, $K(z)$ takes values in the compact operators on H. Let $z \in \mathbb{C}$, $\operatorname{Re} z \geq 1, 0 \leq \operatorname{Im} z \leq \delta, 0 < \delta \leq 1$, and let $\lambda \in \mathbb{C}$ be such that $\operatorname{Re}\lambda = \operatorname{Re} z$ and $\operatorname{Im}\lambda = \delta$. In view of (2.1) we have

$$\|\widetilde{K}(\lambda)\| \leq C_\varepsilon + \varepsilon\delta^{-1}. \tag{2.3}$$

On the other hand, we have

$$\|[\chi_1, \Delta]R_0(z)\eta - [\chi_1, \Delta]R_0(\lambda)\eta\| \leq |z - \lambda| \|Q(\tau)\| \leq 2\delta \|Q(\tau)\| \tag{2.4}$$

for some $\tau = \beta\lambda + (1-\beta)z, \beta \in [0,1]$, where

$$Q(\sigma) = \frac{d}{d\sigma}[\chi_1, \Delta]R_0(\sigma)\eta.$$

We need the following

Lemma 2.2. *For $\sigma \in \mathbb{C}, |\operatorname{Im}\sigma| \leq 1$, we have*

$$\|Q(\sigma)\| \leq C \tag{2.5}$$

with a constant $C > 0$ independent of σ.

Proof. It is easy to see that $\|Q(\tau)\|$ is polynomially bounded in $|\operatorname{Im}\sigma| \leq 1$. Hence, by Phragmèn-Lindelöf principle it suffices to prove (2.5) on the lines $\operatorname{Im}\sigma = \pm 1$. Let $\operatorname{Im}\sigma = -1, |\operatorname{Re}\sigma| \geq 1$. We have

$$\|Q(\sigma)\| \leq C_1 \|\frac{d}{d\sigma}R_0(\sigma)\|_{L^2(\mathbb{R}^n)\to H^1(\mathbb{R}^n)}$$

$$= 2C_1|\sigma| \|(\Delta + \sigma^2)^{-2}\|_{L^2(\mathbb{R}^n)\to H^1(\mathbb{R}^n)}$$

$$\leq 2C_1|\sigma| \|(\Delta + \sigma^2)^{-1}\|_{L^2(\mathbb{R}^n)\to L^2(\mathbb{R}^n)} \|(\Delta + \sigma^2)^{-1}\|_{L^2(\mathbb{R}^n)\to H^1(\mathbb{R}^n)} \leq C_2, \tag{2.6}$$

with a constant $C_2 > 0$ independent of σ.

Let now $\operatorname{Im}\sigma = 1, \operatorname{Re}\sigma \geq 1$. In view of (2.6) it is clear that it suffices to prove (2.5) on $\operatorname{Im}\sigma = 1$ with $Q(\sigma)$ replaced by

$$L(\sigma) = \frac{d}{d\sigma}[\chi_1, \Delta](R_0(\sigma) - R_0(-\sigma))\eta.$$

It is well known that the kernel of $R_0(\sigma) - R_0(-\sigma)$ is given by

$$(i/2)(2\pi)^{-n+1}\sigma^{n-2} \int_{S^{n-1}} e^{i\sigma\langle x-y,w\rangle} dw,$$

where S^{n-1} is the unit sphere in $\mathbb{R}^n$. It is easy to see that the kernel of $L(\sigma)$ is a finite sum of integrals of the form

$$K(x,y) = \sigma^p a(x)b(y) \int_{S^{n-1}} \varphi(w)e^{i\sigma\langle x-y,w\rangle} dw,$$

where $p \leq n-1$, $a, b \in C_0^\infty(\mathbb{R}^n), \varphi \in C^\infty(S^{n-1})$, independent of σ. Denote by $\mathcal{K}(\sigma)$ the operator with kernel $K(x,y)$. Clearly, (2.5) would follow from the estimate

$$\|\mathcal{K}(\sigma)\| \leq C \qquad \text{for} \quad |\mathrm{Im}\,\sigma| \leq 1 \tag{2.7}$$

with a constant $C > 0$ independent of σ. Set $\sigma_1 = \mathrm{Re}\,\sigma$ and $S_{\sigma_1}^{n-1} = \{x \in \mathbb{R}^n : |x| = \sigma_1\}$. We can write

$$K(x,y) = \sigma^p \sigma_1^{-n+1} a(x) b(y) \int_{S_{\sigma_1}^{n-1}} \varphi(w/\sigma_1) e^{i\sigma\langle x-y,w\rangle/\sigma_1} dw.$$

Hence $\mathcal{K}(\sigma) = \mathcal{K}_1(\sigma)\mathcal{K}_2(\sigma)$ with $\mathcal{K}_1 \in \mathcal{L}(L^2(S_{\sigma_1}^{n-1}), L^2(\mathbb{R}^n)), \mathcal{K}_2 \in \mathcal{L}(L^2(\mathbb{R}^n), L^2(S_{\sigma_1}^{n-1}))$ with kernels

$$K_1(x,w) = \sigma^p \sigma_1^{-n+1} a(x) \varphi(w/\sigma_1) e^{i\sigma\langle x,w\rangle/\sigma_1}, \qquad K_2(w,y) = b(y) e^{-i\sigma\langle y,w\rangle/\sigma_1}.$$

Clearly,

$$|K_1(x,w)| + |K_2(w,y)| \leq C_1, \qquad \forall (x,y,w) \in \mathbb{R}^n \times \mathbb{R}^n \times S_{\sigma_1}^{n-1},$$

with a constant $C_1 > 0$ independent of σ. Hence,

$$\|\mathcal{K}_1\|_{\mathcal{L}(L^2(S_{\sigma_1}^{n-1}), L^2(\mathbb{R}^n))} + \|\mathcal{K}_2\|_{\mathcal{L}(L^2(\mathbb{R}^n), L^2(S_{\sigma_1}^{n-1}))} \leq C_2$$

with a constant $C_2 > 0$ independent of σ, which clearly implies (2.7). $\Box$

By (2.3)–(2.5) we have

$$\|K(z)\| \leq C_\varepsilon C\delta + C\varepsilon$$

with a constant $C > 0$ independent of ε and δ. Choosing $\varepsilon > 0$ so that $C\varepsilon \leq 1/4$ and $\delta \in (0,1)$ so that $C_\varepsilon C\delta \leq 1/4$, we conclude that $\|K(z)\| \leq 1/2$ for $\mathrm{Re}\,z \geq 1, 0 \leq \mathrm{Im}\,z \leq \delta$, and in this region

$$\|R_\chi(z)\| \leq 2\|K_1(z)\| \leq C'|\mathrm{Re}\,z|^{-1}.$$

Clearly, the same is true in the region $\mathrm{Re}\,z \leq -1, 0 \leq \mathrm{Im}\,z \leq \delta$, as well. $\Box$

Lemma 2.3. *The condition* (1.3) *implies* (1.5) *with* $k = 0$.

P r o o f. We are going to take advantage of (2.2). Let $z \in \mathbb{C}, \operatorname{Re} z \geq 1, 0 \leq \operatorname{Im} z \leq 1$. Clearly, (2.2) extends for $\lambda \in \mathbb{R}$. Choose $\lambda = \operatorname{Re} z$. In the same way as in the proof of Lemma 2.1, using (1.2) instead of (2.1), we get

$$\|K(z)\| \leq C_1 \operatorname{Im} z$$

with a constant $C_1 > 0$ independent of z. Hence $\|K(z)\| \leq 1/2$ for $\operatorname{Im} z \leq (2C_1)^{-1}$, $\operatorname{Re} z \geq 1$, and by (2.2), $R_\chi(z)$ extends analytically to this region and satisfies there the desired estimate. $\square$

3. Proof of Theorem 1.2. We begin this section with the following

Proposition 3.1. *If $n \geq 3$ is odd, we have*

$$R_\chi(\lambda) = \lambda^{-1}\mathcal{P}_n + \mathcal{E}_n(\lambda), \tag{3.1}$$

where $\mathcal{E}_n(\lambda)$ is analytic at $\lambda = 0$, and $\mathcal{P}_n = 0$ if $n \geq 5$, $\operatorname{rank} \mathcal{P}_3 \leq 1$.

If $n \geq 2$ is even, we have

$$R_\chi(\lambda) = \mathcal{M}_n \lambda^{n-2}\log\lambda + \mathcal{F}_n(\lambda) + O(|\lambda|^{n-2}), \quad \lambda \to 0, |\arg\lambda + \pi/2| \leq \pi, \tag{3.2}$$

where $\operatorname{rank}\mathcal{M}_n = 1$ *and $\mathcal{F}_n(\lambda)$ is a polynomial of degree $\leq n-3$ if $n \geq 4$, $\mathcal{F}_2(\lambda) \equiv 0$.*

P r o o f. Recall first that for the free cutoff resolvent we have that $\chi R_0(z)\chi$ is analytic when $n \geq 3$ is odd, while for n even it is of the form

$$\chi R_0(z)\chi = E_n(z) z^{n-2}\log z + F_n(z), \tag{3.3}$$

where $E_n(z)$ and $F_n(z)$ are entire operator-valued functions. We are going to take advantage of (2.2). We can write

$$K(z) = A(z) + B(z) + C(\lambda)$$

with

$$A(z) = ([\chi_1, \Delta]\widetilde{R}_0(z)\eta - [\chi_1, \Delta]\widetilde{R}_0(0)\eta)\widetilde{K}(\lambda) + z^2\chi_2 R_\chi(\lambda),$$

$$B(z) = T(\lambda)(\log z - \log\lambda),$$

$$C(\lambda) = ([\chi_1, \Delta]\widetilde{R}_0(0)\eta - [\chi_1, \Delta]\widetilde{R}_0(\lambda)\eta)\widetilde{K}(\lambda) - \lambda^2\chi_2 R_\chi(\lambda),$$

where

$$[\chi_1, \Delta]\widetilde{R}_0(z)\eta = [\chi_1, \Delta]R_0(z)\eta, \quad T(\lambda) = 0,$$

if $n \geq 3$, while for $n = 2$,

$$[\chi_1, \Delta]\widetilde{R}_0(z)\eta = [\chi_1, \Delta]R_0(z)\eta - (([\chi_1, \Delta]E_2(z)\eta)|_{z=0}) \log z,$$

$$T(\lambda) = (([\chi_1, \Delta]E_2(z)\eta)|_{z=0})\widetilde{K}(\lambda).$$

Let us now compute $E_n(z)$ when n is even. To this end, we will make use of the following well known formula:

$$[R_0(e^{i\pi m}z) - R_0(z)](x, y) = m(i/2)(2\pi)^{-n+1}z^{n-2} \int\limits_{S^{n-1}} e^{iz\langle x-y,w\rangle}dw \tag{3.4}$$

for any integer m. Using (3.4) with $m = 2$ and combining with (3.3) gives

$$[E_n(z)](x, y) = (2\pi)^{-n+2}\chi(x)\chi(y) \int\limits_{S^{n-1}} e^{iz\langle x-y,w\rangle}dw$$

$$= (2\pi)^{-n+2}\chi(x)\chi(y) \sum_{k=0}^{\infty} \frac{z^{2k}}{(2k)!} \int\limits_{S^{n-1}} (i\langle x - y, w\rangle)^{2k}dw.$$

In particular, this gives $\operatorname{rank} E_n(0) = 1$ and

$$[(([\chi_1, \Delta]E_2(z)\eta)|_{z=0}](x, y) = (\Delta\chi_1)(x)\eta(y),$$

and hence $\operatorname{rank} T(\lambda) \leq 1$. Let us now compute

$$Q_n = (\frac{d}{dz}[\chi_1, \Delta]\widetilde{R}_0(z)\eta)|_{z=0}.$$

Clearly,

$$2Q_n = (\frac{d}{dz}([\chi_1, \Delta]\widetilde{R}_0(z)\eta - [\chi_1, \Delta]\widetilde{R}_0(-z)\eta))|_{z=0}$$

$$= (\frac{d}{dz}([\chi_1, \Delta]R_0(z)\eta - [\chi_1, \Delta]R_0(-z)\eta))|_{z=0}.$$

On the other hand, using (3.4) with $m = 1$, one gets

$$[\frac{d}{dz}([\chi_1, \Delta]R_0(z)\eta - [\chi_1, \Delta]R_0(-z)\eta)](x, y)$$

$$= (i/2)(2\pi)^{-n+1}(n-2)z^{n-3}(\Delta\chi_1)(x)\eta(y) + O(z^{n-1}), \quad z \to 0.$$

Hence, $Q_n = 0$ if $n \neq 3$, $\operatorname{rank} Q_3 = 1$. Using (1.1) we get

$$C(\lambda) = Q_n[\chi_2, \Delta]R_0(\lambda)[\chi_3, \Delta]\lambda R_\chi(\lambda) + O(|\lambda|)$$

as $\lambda \to 0$. If $n = 3$, in view of (1.1), $\lambda R_\chi(\lambda)$ is analytic at $\lambda = 0$, so $\lim_{\lambda\to 0} \lambda R_\chi(\lambda)$ exists, and we can write, as $\lambda \to 0$,

$$C(\lambda) = \begin{cases} O(|\lambda|), & n \neq 3, \\ \widetilde{Q}_3 + O(|\lambda|), & n = 3, \end{cases} \tag{3.5}$$

where $\widetilde{Q}_3$ is of the form $Q_3\widetilde{Q}$, and hence $\operatorname{rank} \widetilde{Q}_3 \leq 1$. Fix λ so that $\|C(\lambda) - C(0)\| \leq 1/2$. Clearly,

$$A(z) = O(|z|), \quad z \to 0.$$

We can write, for $|z| \ll 1$,

$$1 - K(z) = (1 - B_1(z) - L_1(z))(1 - A(z)(1 - C(\lambda) + C(0))^{-1})(1 - C(\lambda) + C(0)), \tag{3.6}$$

where

$$B_1(z) = B(z)D(z), \quad L_1(z) = C(0)D(z),$$

$$D(z) = (1 - C(\lambda) + C(0))^{-1}(1 - A(z)(1 - C(\lambda) + C(0))^{-1})^{-1}.$$

If $n \geq 4$, clearly $B_1(z) \equiv 0, L_1(z) \equiv 0$, and in view of (3.6), $1 - K(z)$ is invertible near $z = 0$. Therefore, by (2.2) we can conclude that the leading singularities of $R_\chi(z)$ at $z = 0$ are of the same type as those of $\chi R_0(z)\chi$, so the proposition in this case follows from (3.3).

To study the other cases, observe that if π is an operator of rank 1 such that $\pi w = aw$ for some $w \in H, w \neq 0$, and a scalar a, then

$$(1 - \pi)^{-1} = 1 - (a\|w\|^2 - 1)^{-1}\pi \qquad \text{if} \quad a\|w\|^2 \neq 1. \tag{3.7}$$

Let $n = 3$. Then $B_1(z) \equiv 0$, $\operatorname{rank} L_1(z) \leq 1$. Moreover, we have $L_1(z)w = a(z)w$ with $w = \Delta\chi_1$ and $a(z) = Const\langle\widetilde{Q}D(z)w, w\rangle$. Hence $a(z)$ is analytic at $z = 0$, and $a(z)\|w\|^2 - 1 = a_p z^p(1 + O(|z|)), z \to 0, a_p \neq 0$, for some integer $p \geq 0$. Thus, by (2.2), (3.3), (3.6) and (3.7), we conclude that the leading singularity of $R_\chi(z)$ at $z = 0$ is of the form $z^{-p}NL_1(0)$. In view of (3.5), however, $p \leq 1$, which establishes (3.1) with $\mathcal{P}_3 = NL_1(0)$ which is of rank ≤ 1 as so is $L_1(0)$.

Let $n = 2$. Then $L_1(z) \equiv 0$, $\operatorname{rank} B_1(z) \le 1$. Moreover, we have $B_1(z)w = b(z)w$ with $w = \Delta\chi_1$ and $b(z) = (\log z - \log \lambda) b_1(z)$, $b_1(z) = Const\langle \widetilde{K}(\lambda) D(z) w, w\rangle$. If $b_1(0) = 0$, then clearly $b(z)\|w\|^2 - 1$ is invertible near $z = 0$, and by (2.2) and (3.7) we conclude that $R_\chi(z)$ has the same type of leading singularity at $z = 0$ as does $\chi R_0(z)\chi$. Suppose $b_1(0) \neq 0$. Then

$$(b(z)\|w\|^2 - 1)^{-1} = Const(\log z)^{-1}(1 + O((\log z)^{-1})),$$

and by (2.2), (3.3), (3.6) and (3.7) we obtain

$$R_\chi(z) = N_1 E_2(0) \log z + O(1), \quad z \to 0,$$

for some bounded, invertible operator N_1. This completes the proof of Proposition 3.1. $\square$

Remark 3.2. Clearly, Proposition 3.1 gives that $\lambda R_\chi(\lambda)$ is analytic at $\lambda = 0$ if n is odd, while for n even, modulo a function analytic at $\lambda = 0$, it is of the form

$$\lambda R_\chi(\lambda) = \mathcal{M}_n \lambda^{n-1} \log \lambda + O(|\lambda|^{n-1}), \quad \lambda \to 0.$$

Moreover, it is easy to see from the proof that the same conclusion holds for $\chi \partial_{x_j} R(\lambda) \chi$, $j = 1, \dots, n$.

Denote by $\widetilde{H}_k, k = 1, \dots, N_0$, the closure of $C_{(0)}^\infty(\mathcal{O}_k)$ with respect to the norm

$$\left(\int_{\mathcal{O}_k} \sum_{i,j=1}^{n} g_{ij}^{(k)}(x) \partial_{x_i} f \partial_{x_j} \overline{f} dx \right)^{1/2},$$

and by $\widetilde{H}_{N_0+1}$ the closure of $C_{(0)}^\infty(\Omega_{N_0})$ with respect to the norm $(\int_{\Omega_{N_0}} |\nabla_x f|^2 dx)^{1/2}$. Set $\widetilde{H} = \oplus_{k=1}^{N_0+1} \widetilde{H}_k$, and $\mathcal{H} = \widetilde{H} \oplus H$. Given $a > \rho_0$, define the Hilbert space $\mathcal{H}_a = \widetilde{H}_a \oplus H_a$, where $\widetilde{H}_a = \oplus_{k=1}^{N_0} \widetilde{H}_k \oplus \widetilde{H}_{N_0+1}^a$, $H_a = \oplus_{k=1}^{N_0} L^2(\mathcal{O}_k; c_k(x)dx) \oplus L^2(\Omega_{N_0} \cap B_a)$. Here $\widetilde{H}_{N_0+1}^a$ is the closure of $C_{(0)}^\infty(\Omega_{N_0} \cap B_a)$ with respect to the norm $(\int_{\Omega_{N_0} \cap B_a} |\nabla_x f|^2 dx)^{1/2}$. In what follows $\|\cdot\|$ will denote the norm on $\mathcal{H}$, while $\|\cdot\|_a$ will denote the norm on $\mathcal{H}_a$. Consider the operator

$$G = -i \begin{pmatrix} 0 & Id \\ P & 0 \end{pmatrix}$$

on the Hilbert space $\mathcal{H}$ with domain of definition

$$D(G) = \{(u_1, u_2) \in \mathcal{H} : u_1 \in D(P), Pu_1 \in H, u_2 \in \widetilde{H}\}.$$

It is easy to see that the operator G is selfadjoint and, for $\operatorname{Im}\lambda < 0$,

$$(G-\lambda)^{-1} = -i \begin{pmatrix} i\lambda R(\lambda) & R(\lambda) \\ \lambda^2 R(\lambda) - Id & i\lambda R(\lambda) \end{pmatrix}.$$

Hence, $(G-\lambda)^{-1} : \mathcal{H}_{comp} \to \mathcal{H}_{loc}$ extends to a meromorphic function on $\mathbb{C}$ if n is odd and on Λ if n is even. Moreover, in view of Remark 3.2, $(G-\lambda)^{-1}$ is analytic at $\lambda = 0$ if n is odd and it has the form, modulo a function analytic at $\lambda = 0$,

$$(G-\lambda)^{-1} = \mathcal{M}'_n \lambda^{n-1} \log\lambda + O(|\lambda|^{n-1}), \quad \lambda \to 0, \tag{3.8}$$

if n is even. Furthermore, it is easy to see that under the assumptions of Theorem 1.2, $(G-\lambda)^{-1} : \mathcal{H}_{comp} \to \mathcal{H}_{loc}$ extends analytically to $\Lambda_\pm = \{\lambda \in \mathbb{C} : 0 \le \operatorname{Im}\lambda \le C, \pm\operatorname{Re}\lambda > 0\}$ and satisfies the estimate

$$\|(G-\lambda)^{-1} f\|_a \le C_1 |\lambda|^k \|f\| \quad \text{for} \quad |\operatorname{Im}\lambda| \le C, |\operatorname{Re}\lambda| \ge C_2, \tag{3.9}$$

for every compactly supported $f \in \mathcal{H}$. Clearly, Theorem 1.2 follows from the following

Proposition 3.3. *Under the same assumptions as in Theorem* 1.2*, we have, for* $t \gg 1$,

$$\|(1+G^2)^{-k/2} e^{itG} f\|_a \le \begin{cases} Ce^{-\gamma t}\|f\|, & n \quad \textit{odd}, \\ Ct^{-n}\|f\|, & n \quad \textit{even}, \end{cases} \tag{3.10}$$

for every compactly supported $f \in \mathcal{H}$.

Proof. We will proceed in a way similar to that one in [3]. Let $\varphi(t) \in C^\infty(\mathbb{R}^n), \varphi = 0$ for $t \le 1$, $\varphi = 1$ for $t \ge 2$. Set $U(t) = e^{itG}$ and $V(t) = \varphi(t)U(t)$. The Fourier transform

$$\widehat{V}(\lambda) = \int_{-\infty}^{+\infty} e^{-it\lambda} V(t)\,dt$$

is well defined for $\operatorname{Im}\lambda < 0$ as a bounded operator on $\mathcal{H}$. We have

$$V(t) = (2\pi)^{-1} \int_{\operatorname{Im}\lambda = -\varepsilon} e^{it\lambda} \widehat{V}(\lambda)\,d\lambda, \tag{3.11}$$

$\forall \varepsilon > 0$, and

$$\widehat{V}(\lambda) = i(G-\lambda)^{-1}\widehat{\varphi' U}(\lambda), \quad \operatorname{Im}\lambda < 0.$$

By the finite speed of the wave propagation, we have that for every compactly supported $f \in \mathcal{H}, \forall t \in \mathbb{R}$, $\varphi'(t)U(t)f$ is supported in some compact independent of t. Therefore, $\widehat{\varphi' U}(\lambda) : \mathcal{H}_{comp} \to \mathcal{H}_{comp}$ extends to an entire function on $\mathbb{C}$. Set $S(\lambda) = (1+G^2)^{-k/2}(G-\lambda)^{-1}$.

Lemma 3.4. *$S(\lambda) : \mathcal{H}_{comp} \to \mathcal{H}_{loc}$ admits analytic continuations in $\Lambda_\pm = \{\lambda \in \mathbb{C} : 0 \le \operatorname{Im}\lambda \le C, \pm\operatorname{Re}\lambda > 0\}, C > 0$, such that*

$$\|S(\lambda)f\|_a \le C_1\|f\| \quad \text{for} \quad |\operatorname{Im}\lambda| \le C, \tag{3.12}$$

for every compactly supported $f \in \mathcal{H}$. Moreover, $S(\lambda)$ is analytic at $\lambda = 0$ when n is odd and it has the form, modulo a function analytic at $\lambda = 0$,

$$S(\lambda) = S_0\lambda^{n-1}\log\lambda + O(|\lambda|^{n-1}), \quad \lambda \to 0, \tag{3.13}$$

if n is even.

Proof. The lemma is clearly true for $k = 0$. If $k > 0$, by the spectral theorem we have

$$S(\lambda) = \int_{-\infty}^{+\infty} (1+x^2)^{-k/2}(x-\lambda)^{-1}E(x)dx, \quad \operatorname{Im}\lambda < 0,$$

where

$$E(x) = \lim_{\varepsilon\to 0^+}(2\pi i)^{-1}((G-x-i\varepsilon)^{-1} - (G-x+i\varepsilon)^{-1}).$$

For any $\alpha > \beta > 0$ we can write

$$S(\lambda) = \int_{\alpha-\beta}^{\alpha+\beta} (1+x^2)^{-k/2}(x-\lambda)^{-1}E(x)dx$$

$$+ \left(\int_{-\infty}^{\alpha-\beta} + \int_{\alpha+\beta}^{+\infty}\right)(1+x^2)^{-k/2}(x-\lambda)^{-1}E(x)dx = J_1(\lambda) + J_2(\lambda).$$

Clearly, $J_2(\lambda) : \mathcal{H} \to \mathcal{H}$ extends analytically to $\mathbb{C}_\alpha = \{\lambda \in \mathbb{C} : \alpha - \beta/2 \leq \operatorname{Re}\lambda \leq \alpha + \beta/2, \operatorname{Im}\lambda \leq 1\}$ and

$$\|J_2(\lambda)\|_{\mathcal{L}(\mathcal{H},\mathcal{H})} \leq 2\beta^{-1}, \quad \lambda \in \mathbb{C}_\alpha. \tag{3.14}$$

On the other hand, $(G-z)^{-1} : \mathcal{H}_{comp} \to \mathcal{H}_{loc}$ extends analytically from $\pm\operatorname{Im} z < 0$ to $\{z \in \mathbb{C} : \pm\operatorname{Im} z \leq \gamma, \operatorname{Re} z > 0\}$. Denote this continuation by $F^\pm(z)$. Without loss of generality we can suppose $\gamma < 1$. Now by the Cauchy theorem we have

$$J_1(\lambda) = -(2\pi i)^{-1} \int\limits_{\substack{\operatorname{Im} z=\gamma \\ \alpha-\beta\leq\operatorname{Re} z\leq\alpha+\beta}} (1+z^2)^{-k/2}(z-\lambda)^{-1}(F^+(z) - F^-(z))dz$$

$$-(2\pi i)^{-1} \int\limits_{\substack{0\leq\operatorname{Im} z\leq\gamma \\ \operatorname{Re} z=\alpha-\beta}} (1+z^2)^{-k/2}(z-\lambda)^{-1}(F^+(z) - F^-(z))dz$$

$$+(2\pi i)^{-1} \int\limits_{\substack{0\leq\operatorname{Im} z\leq\gamma \\ \operatorname{Re} z=\alpha+\beta}} (1+z^2)^{-k/2}(z-\lambda)^{-1}(F^+(z) - F^-(z))dz$$

considered as an operator from $\mathcal{H}_{comp}$ to $\mathcal{H}_{loc}$. It follows from the above representation that $J_1(\lambda) : \mathcal{H}_{comp} \to \mathcal{H}_{loc}$ extends analytically to $\mathbb{C}'_\alpha = \{\lambda \in \mathbb{C} : \alpha - \beta/2 \leq \operatorname{Re}\lambda \leq \alpha + \beta/2, 0 \leq \operatorname{Im}\lambda \leq \gamma/2\}$, and in view of (3.9) it satisfies the estimate, for $\alpha \gg 1$,

$$\|J_1(\lambda)f\|_a \leq C_\beta\|f\|, \quad \lambda \in \mathbb{C}'_\alpha, \tag{3.15}$$

for every compactly supported $f \in \mathcal{H}$, with a constant C_β independent of α. Hence, $S(\lambda) : \mathcal{H}_{comp} \to \mathcal{H}_{loc}$ extends analytically to $\{\operatorname{Im}\lambda \leq \gamma/2, \operatorname{Re}\lambda > 0\}$ and (3.12) follows from (3.14) and (3.15). In a similar way one can extend $S(\lambda)$ to $\{\operatorname{Im}\lambda \leq \gamma/2, \operatorname{Re}\lambda < 0\}$.

To study the singularity of $S(\lambda)$ at $\lambda = 0$, observe that the difference

$$S(\lambda) - (1+\lambda^2)^{-k/2}(G-\lambda)^{-1}$$

$$= \int\limits_{|x|\geq 1} ((1+x^2)^{-k/2} - (1+\lambda^2)^{-k/2})(x-\lambda)^{-1}E(x)dx + \sum_{\mu,\nu=0}^{\infty} C_{\mu\nu}\lambda^\mu \int_{-1}^{1} x^\nu E(x)dx$$

is analytic at $\lambda = 0$. Thus (3.13) follows from (3.8). $\square$

By (3.11) and Lemma 3.4, for every compactly supported $f \in \mathcal{H}$, we have

$$W(t)f := (1+G^2)^{-k/2}V(t)f = (2\pi)^{-1} \int\limits_{\mathrm{Im}\,\lambda=-\varepsilon} e^{it\lambda} S(\lambda)\widehat{\varphi' U}(\lambda) f d\lambda$$

$$= e^{-Ct}(2\pi)^{-1} \int\limits_{-\infty}^{+\infty} e^{itz} S(z+iC)\widehat{\varphi' U}(z+iC) f dz$$

$$+(2\pi)^{-1} \lim_{\varepsilon\to 0^+} \left(\int\limits_{\substack{\mathrm{Re}\,\lambda=-\varepsilon \\ 0\le \mathrm{Im}\,\lambda\le C}} e^{it\lambda} S(\lambda)\widehat{\varphi' U}(\lambda) f d\lambda - \int\limits_{\substack{\mathrm{Re}\,\lambda=\varepsilon \\ 0\le \mathrm{Im}\,\lambda\le C}} e^{it\lambda} S(\lambda)\widehat{\varphi' U}(\lambda) f d\lambda \right)$$

$$= e^{-Ct} W_1(t)f + W_2(t)f. \tag{3.16}$$

Clearly, $W_2(t)f \equiv 0$ if n is odd, while for n even, we have in view of (3.13),

$$W_2(t) = S_0' \int\limits_0^C e^{-ty} y^{n-1} dy + O(t^{-n}) = O(t^{-n}).$$

In other words,

$$\|W_2(t)f\|_a \le \widetilde{C} t^{-n} \|f\| \tag{3.17}$$

for every compactly supported $f \in \mathcal{H}$.

To estimate $\|W_1(t)f\|_a$ we will use Plancherel identity together with (3.12). We have

$$\int\limits_{-\infty}^{+\infty} \|W_1(t)f\|_a^2 dt = \int\limits_{-\infty}^{+\infty} \|S(z+iC)\widehat{\varphi' U}(z+iC) f\|_a^2 dz$$

$$\le C_1 \int\limits_{-\infty}^{+\infty} \|\widehat{\varphi' U}(z+iC) f\|^2 dz = C_1 \int\limits_{-\infty}^{+\infty} e^{2Ct} \|\varphi'(t)U(t)f\|^2 dt \le C_2 \|f\|^2. \tag{3.18}$$

Let $\chi \in C_0^\infty(\mathbb{R}^n), \chi = 1$ for $|x| \leq a$. An easy computation gives

$$(\partial_t - iG)\chi W_1(t)f = -i[G, \chi]W_1(t)f + C\chi W_1(t)f$$

$$-i\chi(1+G^2)^{-k/2}(2\pi)^{-1} \int\limits_{-\infty}^{+\infty} e^{itz}\widehat{\varphi' U}(z+iC)f dz = \widetilde{W}_1(t)f,$$

and hence

$$\chi W_1(t)f = U(t)\chi W_1(0)f + \int\limits_0^t U(t-s)\widetilde{W}_1(s)f ds.$$

This implies

$$\|W_1(t)f\|_a \leq \|\chi W_1(t)f\| \leq C_3\|f\| + \int\limits_0^t \|\widetilde{W}_1(s)f\|ds$$

$$\leq C_3\|f\| + t^{1/2}(\int\limits_{-\infty}^{+\infty} \|\widetilde{W}_1(s)f\|^2 ds)^{1/2}.$$

It is easy to see that (3.18) holds with $\|W_1(t)f\|_a$ replaced by $\|\widetilde{W}_1(t)f\|$. Hence, for $t \geq 1$,

$$\|W_1(t)f\|_a \leq C_4 t^{1/2}\|f\|. \tag{3.19}$$

Thus, (3.10) follows from (3.16), (3.17) and (3.19). □

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