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THE GENERAL DIFFERENTlAL OPERATORS GENERATED BY A QUASI-DIFFERENTIAL EXPRESSIONS WITH THEIR INTERIOR SINGULAR POINTS

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ABSTRACT. The general ordinary quasi-differential expression M of n-th order with complex coefficients and its formal adjoint M^+ are considered over a regoin (a, b) on the real line, $-\infty \leq a < b \leq \infty$, on which the operator may have a finite number of singular points. By considering M over various subintervals on which singularities occur only at the ends, restrictions of the maximal operator generated by M in $L^2_w(a, b)$ which are regularly solvable with respect to the minimal operators $T_0(M)$ and $T_0(M^+)$. In addition to direct sums of regularly solvable operators defined on the separate subintervals, there are other regularly solvable restrications of the maximal operator which involve linking the various intervals together in interface like style.

1. Introductions. In [8] Everitt and Zettl considered the problem of characterizing all self-adjoint differential operators generated by a countable number of quasi-differential expressions on the real line, and in [2, 10] Evans and

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Ibrahim gave a characterisation of all regularly solvable operators and their adjoints generated by a general ordinary quasi-differential expression M in $L^2_w(a,b)$. The domains of these operators are described in terms of boundary conditions involving the $L^2_w(a,b)-$solutions of the equation $M[u] = \lambda wu$ and its adjoint $M^+[v] = \overline{\lambda} wv$. The results include those of Sun Jiong [11].

Our objective in this paper is to extend the results in [2], [8], [9] and [10] for finitely many singular points or perhaps finitely many disjoint intervals on which singularities occur only at the ends by using the ideas and results from [7], [8], [10], [11], [12], [13] and [15]. Also, we discuss the possibility of the regularly solvable operators which are not expressible as the direct sums of regularly solvable operators defined in the separate intervals.

The minimal operators $T_0(M)$ and $T_0(M^+)$ are no longer symmetric, but direct sums of those over finitely many disjoint intervals and form an adjoint pair of closed densely-defined operators in the underlying L^2-space, that is $T_0(M) \subset [T_0(M^+)]^*$, and the operators which fulfil the role that the self-ajoint and maximal symmetric operators play in the symmetric case are those which are regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$: such an operator S satisfies $T_0(M) \subset S \subset [T_0(M^+)]^*$ and for some $\lambda \in \mathbb{C}$, $(S - \lambda I)$ is a Fredholm operator of zero index. In order to characterize all the operators which are regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ in L^2_w-solutions of $M[u] = \lambda wu$ over various subintervals, we need the results in [10] for the case when the end-points of the underlying interval are both singular. This is a result of special interest and extends one proved in [15] by Zai-Jiu Shang for formally symmetric and J-symmetric differential expressions.

2. Preliminaries. We begin with a brief survey of adjoint pairs of operators and their associated regularly solvable operators; a full treatment may be found in [1, Chapter III] and [3].

The domain and range of a linear operator T acting in a Hilbert space H will be denoted by $D(T)$, $R(T)$ respectively and $N(T)$ will denote its null space. The nullity of T, written $\operatorname{nul}(T)$, is the dimension of $N(T)$ and the deficiency of T, $\operatorname{def}(T)$, is the co-dimension of $R(T)$ in H; thus if T is densely-defined and $R(T)$ is closed, then $\operatorname{def}(T) = \operatorname{nul}(T^*)$. *The Fredholm domain* of T is (in the notation of [1]) the open subset $\Delta_3(T)$ of $\mathbb{C}$ consisting of those values $\lambda \in \mathbb{C}$ which are such that $(T - \lambda I)$ is a Fredholm operator, where I is the identity operator in H. Thus $\lambda \in \Delta_3(T)$ if and only if $(T - \lambda I)$ has closed range and finite nullity and deficiency. The index of $(T - \lambda I)$ is the number $\operatorname{ind}(T - \lambda I) = \operatorname{nul}(T - \lambda I) - \operatorname{def}(T - \lambda I)$, this being defined for $\lambda \in \Delta_3(T)$.

Two closed densely-defined operators A, B in H are said to form an *adjoint pair* if $A \subset B^*$ and consequently $B \subset A^*$; equivalently, $(Ax, y) = (x, By)$, for all $x \in D(A)$ and $y \in D(B)$, where $(\cdot, \cdot)$ denotes the inner product on H.

The joint field of reguiarity $\Pi(A, B)$ of A and B is the set of $\lambda \in \mathbb{C}$ which are such that $\lambda \in \Pi(A)$, the field of regularity of A, $\overline{\lambda} \in \Pi(B)$ and $\operatorname{def}(A - \lambda I)$ and $\operatorname{def}(B - \overline{\lambda} I)$ are finite. An adjoint pair A, B is said to be *compatible* if $\Pi(A, B) \neq \emptyset$. Recall that $\lambda \in \Pi(A)$ if and only if there exists a positive constant $K(\lambda)$ such that,

$$\|(A - \lambda I)x\| \geq K(\lambda)\|x\| \qquad \text{for all } x \in D(A),$$

or equivalently, on using the closed-Graph Theorem, $\operatorname{nul}(A - \lambda I) = 0$ and $R(A - \lambda I)$ is closed.

Definition 2.1. *A closed operator S in H is said to be* regularly solvable *with respect to the compatible adjoint pair A, B if $A \subset S \subset B^*$ and $\Pi(A, B) \cap \Delta_4(S) \neq \emptyset$, where*

$$\Delta_4(S) = \{\lambda : \lambda \in \Delta_3(S), \quad \operatorname{ind}(S - \lambda I) = 0\}.$$

The terminology "*regularly solvable*" comes from Visik's paper [16].

We now turn to the quasi-differential expressions defined in terms of a Shin-Zettl matrices A on an interval I, where I denotes an interval with left-end point a and the right-end point b $(-\infty \leq a < b \leq \infty)$. The set $Z_n(I)$ of Shin-Zettl matrices on I consists of $(n \times n)$-matrices $A = \{a_{rs}\}$ whose entries are complex-valued functions on I which satisfy the following conditions:

$$(2.1) \qquad \begin{cases} a_{rs} \in L^1_{\text{loc}}(I) & \text{a.e.} & (1 \leq r, s \leq n, \ n \geq 2), \\ a_{r,r+1} \neq 0 & \text{a.e. on } I & (1 \leq r \leq n-1), \\ a_{rs} = 0 & \text{a.e. on } I & (2 \leq r+1 < s \leq n). \end{cases}$$

For $A \in Z_n(I)$, the *quasi-derivatives* associated with A are defined by,

$$(2.2) \qquad \begin{cases} y^{[0]} := y \\ y^{[r]} := (a_{r,r+1})^{-1} \left\{ (y^{[r-1]})' - \sum\limits_{s=1}^{r} a_{rs} y^{[s-1]} \right\} \quad (1 \leq r \leq n-1), \\ y^{[n]} := (y^{[n-1]})' - \sum\limits_{s-1}^{n} a_{ns} y^{[s-1]} \end{cases}$$

where the prime $'$ denotes *differentiation.*

The quasi-differential expression M associated with A is given by:

$$M[y] := i^n y^{[n]}, \tag{2.3}$$

this being defined on the set

$$V(M) := \left\{ y : y^{[r-1]} \in AC_{\mathrm{loc}}(I),\ r = 1, \dots, n \right\}, \tag{2.4}$$

where $L^1_{\mathrm{loc}}(I)$ and $AC_{\mathrm{loc}}(I)$ denote respectively, the spaces of complex valued Lebesgue measurable functions on I which are locally integrable and locally absolutely continuous on all compact subintervals of I.

The formal adjoint M^+ of M defined by the matrix $A^+ \in Z_n(I)$ and is given by

$$M^+[y] := i^n y^n_+ \quad \text{for all } y \text{ in} \tag{2.5}$$

$$V(M^+) := \left\{ y : y^{[r-1]}_+ \in AC_{\mathrm{loc}}(I),\ r = 1, \dots, n \right\}, \tag{2.6}$$

where $y^{[r-1]}_+$, the quasi-derivatives associated with the matrix A^+. Note that, $(A^+)^+ = A$ and so $(M^+)^+ = M$. We refer to [5], [9], [10] and [17] for a full account of the above and subsequent results on quasidifferential expressions.

For $u \in V(M)$, $v \in V(M^+)$ and $\alpha, \beta \in I$, we have *Green's formula*,

$$\int_\alpha^\beta \left\{ \overline{v} M[u] - -u\overline{M^+[v]} \right\} dx = [u, v](\beta) - [u, v](\alpha), \tag{2.7}$$

where,

$$\begin{aligned} [u, v](x) &= i^n \left(\sum_{r=0}^{n-1} (-1)^{n+r+1} u^{[r]}(x) \overline{v}^{[n-r-1]}_+(x) \right) \\ &= (-i)^n \left(u(x), \dots, u^{[n-1]}(x) \right) J_{n\times n} \begin{pmatrix} \overline{v}(x) \\ \vdots \\ \overline{v}^{[n-1]}_+(x) \end{pmatrix}; \end{aligned} \tag{2.8}$$

see [2] and [17, Corollary 1] for more details.

Let w be a function which satisfies,

$$w > 0 \text{ a.e. on } I, w \in L^1_{\mathrm{loc}}(I). \tag{2.9}$$

The equation

$$M[y] = \lambda w y \quad (\lambda \in \mathbb{C}) \tag{2.10}$$

on I is said to be *regular* at the left-end point a if it is finite and $x \in (a, b)$,

$$a \in \mathbb{R}, \quad w, a_{rs} \in L^1[a, X] \quad (r, s = 1, \dots, n). \tag{2.11}$$

Otherwise, (2.10) is said to be *singular* at a. Similarly we define the terms regular and singular at b. If (2.10) is regular at both end-points, then it is said to be regular, in this case we have,

$$a, b \in \mathbb{R}, \quad w, a_{rs} \in L^1[a, b] \quad (r, s = 1, \dots, n). \tag{2.12}$$

Note that: An end-point of the interval I is regular (See [9] and [10]) for the equation (2.10) if and only if it is regular for the equation

$$M^+[y] = \overline{\lambda} w y \qquad (\lambda \in \mathbb{C}). \tag{2.13}$$

Let $H = L^2_w(a, b)$ denote, the usual weighted L^2-space with inner-product

$$(f, g) := \int_a^b f(x)\overline{g}(x)w(x)dx, \tag{2.14}$$

and norm $\|f\| := (f, f)^{1/2}$; this is a Hilbert space on identifying functions which differ only on null sets.

We can without loss of generality assume that the interval (a, b) $-\infty \leq a < b \leq \infty$, in question is decomposed into four sets of subintervals:

(1) $\{I_i\}_{i=1}^m$. Considered on I_i, M is singular at both end-points.

(2) $\{J_i\}_{i=1}^n$. Considered on J_i, M is regular at the left end-point and singular at the right end-point.

(3) $\{K_i\}_{i=1}^p$. Considered on K_i, M is singular at the left end-point and regular at the right end-point.

(4) $\{L_i\}_{i=1}^q$. Considered on L_i, M is regular at both end-points.

Definition 2.2. *We denote by $D(M)$ the collection of those elements u satisfying the following:*

(1) $u \in V(M_i)$, $i = 1, \ldots, m$; $u \in V(M_i)$, $i = 1, \ldots, n$;
$u \in V(M_i)$, $i = 1, \ldots, p$; $u \in V(M_i)$, $i = 1, \ldots, q$.

(2) $u \in L^2_w(I_i)$, $i = 1, \ldots, m$; $u \in L^2_w(J_i)$, $i = 1, \ldots, n$;
$u \in L^2_w(K_i)$, $i = 1, \ldots, p$; $u \in L^2_w(L_i)$, $i = 1, \ldots, q$.

(3) $w^{-1}M_i[u] \in L^2_w(I_i)$, $i = 1, \ldots, m$; $w^{-1}M_i[u] \in L^2_w(J_i)$, $i = 1, \ldots, n$;
$w^{-1}M_i[u] \in L^2_w(K_i)$, $i = 1, \ldots, p$; $w^{-1}M_i[u] \in L^2_w(L_i)$, $i = 1, \ldots, q$,

and by $D(M^+)$ the collection of those elements v satisfying, the following:

(1) $v \in V(M_i^+)$, $i = 1, \ldots, m$; $v \in V(M_i^+)$, $i = 1, \ldots, n$;
$v \in V(M_i^+)$, $i = 1, \ldots, p$; $v \in V(M_i^+)$, $i = 1, \ldots, q$.

(2) $v \in L^2_w(I_i)$, $i = 1, \ldots, m$; $v \in L^2_w(J_i)$, $i = 1, \ldots, n$;
$v \in L^2_w(K_i)$, $i = 1, \ldots, p$; $v \in L^2_w(L_i)$, $i = 1, \ldots, q$.

(3) $w^{-1}M_i^+[v] \in L^2_w(I_i)$, $i = 1, \ldots, m$; $w^{-1}M_i^+[v] \in L^2_w(J_i)$, $i = 1, \ldots, n$;
$w^{-1}M_i^+[v] \in L^2_w(K_i)$, $i = 1, \ldots, p$; $w^{-1}M_i^+[v] \in L^2_w(L_i)$, $i = 1, \ldots, q$.

Definition 2.3. *We define the maximal operators $T(M)$ and $T(M^+)$ by setting: $T(M)u := w^{-1}M[u]$ and $T(M^+)v := w^{-1}M^+[v]$, for all $u \in D(M)$, $v \in D(M^+)$.*

The underlying Hilbert space is, of course,

$$H = \sum_{i=1}^{m} L^2_w(I_i) \oplus \sum_{i=1}^{n} L^2_w(J_i) \oplus \sum_{i=1}^{p} L^2_w(K_i) \oplus \sum_{i=1}^{q} L^2_w(L_i),$$

with the inner-product $\langle \cdot, \cdot \rangle$ over H,

$$\langle f, g \rangle = \sum_{i-1}^{m} \int_{I_i} f(x)\overline{g(x)}w(x)dx + \sum_{i-1}^{n} \int_{J_i} f(x)\overline{g(x)}w(x)dx +$$
$$+ \sum_{i-1}^{p} \int_{K_i} f(x)\overline{g(x)}w(x)dx + \sum_{i-1}^{q} \int_{L_i} f(x)\overline{g(x)}w(x)dx.$$

Green's formula over all of (a, b) is the sum of those over all subintervals of (a, b) such that: for $u \in V(M)$ and $v \in V(M^+)$, then

$$\langle w^{-1}M[u]v \rangle - \langle u, w^{-1}M^+[v] \rangle = \sum_{i=1}^{m} ([u, v]_i(\beta_i) - [u, v]_i(\alpha_i)) +$$

$$+\sum_{i=1}^{n}\left([u,v]_i(\beta_i)-[u,v]_i(\alpha_i)\right)+\sum_{i=1}^{p}\left([u,v]_i(\beta_i)-[u,v]_i(\alpha_i)\right)+$$
$$+\sum_{i=1}^{q}\left([u,v]_i(\beta_i)-[u,v]_i(\alpha_i)\right).$$

For the regular problem, the *minimal operators* $T_0(M_i)$ and $T_0(M_i^+)$ are the restrications of $w^{-1}M_i[\cdot]$ and $w^{-1}M_i^+[\cdot]$ to the subspaces,

$$(2.15)\quad \begin{cases} D_0(M_i) := \left\{u \in D(M_i) : u^{[r-1]}(a_i) = u^{[r-1]}(b_i) = 0, \quad i = 1,\dots,q\right\} \\ D_0(M_i^+) := \left\{v \in D(M_i^+) : v_+^{[r-1]}(a_i) = v_+^{[r-1]}(b_i) = 0, \quad i = 1,\dots,q\right\} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (r = 1,\dots,n) \end{cases}$$

respectively. The subspaces $D_0(M_i)$ and $D_0(M_i^+)$ are dense in $L_w^2(L_i)$ and $T(M_i)$, $T(M_i^+)$ are closed operators (see [17, Section 3]). In the singular problem we first introduce the operators $T_0'(M_i)$ and $T_0'(M_i^+)$ being the restriction of $w^{-1}M_i[\cdot]$ to

$$(2.16)\qquad D_0'(M_i) := \{u : u \in D(M_i),\ \operatorname{supp} u \subset (a_i, b_i) \text{ on } J_i \text{ and } K_i\},$$

and with $T_0'(M_i^+)$ defined similarly. These operators are densely-defined and closable in $L_w^2(J_i)$, $i = 1,\dots,n$ and $L_w^2(K_i)$, $i = 1,\dots,p$.

We define the minimal operators $T_0(M_i)$, $T_0(M_i^+)$ to be their respective closures (cf. [17, Section 5]). We denote the domains of $T_0(M_i)$ and $T_0(M_i^+)$, by $D_0(M_i)$ and $D_0(M_i^+)$ respectively. It can be shown that, (2.10) is regular at a_i, then

$$(2.17)\qquad \begin{cases} u \in D_0(M_i) \quad u^{[r-1]}(a_i) = 0 \quad \text{on } J_i,\ i = 1,\dots,n \\ u \in D_0(M_i) \quad u^{[r-1]}(a_i) = 0 \quad \text{on } K_i,\ i = 1,\dots,p \end{cases}$$

and similarly,

$$(2.18)\qquad \begin{cases} v \in D_0(M_i^+) \quad \Rightarrow \quad v_+^{[r-1]}(a_i) = 0 \quad \text{on } J_i,\ i = 1,\dots,n \\ v \in D_0(M_i^+) \quad \Rightarrow \quad v_+^{[r-1]}(a_i) = 0 \quad \text{on } K_i,\ i = 1,\dots,p \end{cases}$$

$r = 1,\dots,n$. Moreover, in both regular and singular problems we have,

$$(2.19)\qquad [T_0(M_i)]^* = T(M_i^+) \text{ and } [T(M_i)]^* = T_0(M_i^+) \text{ on } J_i \text{ and } K_i;$$

see [17, Section 5] in the case when $M_i = M_i^+$ and compare with the treatment in [1, Section III.10.3] in the general case.

In the case of two singular end-points, the problem on (a_i, b_i), $i = 1, \ldots, m$ is effectively reduced to the problems with one singular end-point on the intervals $(a_i, c_i]$ and $[c_i, b_i)$, where $c_i \in (a_i, b_i)$. We denote by $T(M_i; a_i)$ and $T_0(M_i; b_i)$ the maximal operators with domains $D(M_i; a_i)$ and $D(M_i; b_i)$ and denote by $T_0(M_i; a_i)$ and $T_0(M_i; b_i)$ the closures of the operators $T_0'(M_i; a_i)$ and $T_0'(M_i; b_i)$ defined in (2.16) on the intervals $(a_i, c_i]$ and $[c_i, b_i)$ respectively.

Let $\tilde{T}_0'(M_i)$ be the orthogonal sum

$$\tilde{T}_0'(M_i) = T_0'(M_i; a_i) \oplus T_0'(M_i; b_i) \text{ in}$$

$$L^2_w(a_i, b_i) = L^2_w(a_i, c_i) \oplus L^2_w(c_i, b_i),\ i = 1, 2, \ldots, m;$$

$\tilde{T}_0'(M_i)$ is densely-defined and closable in $L^2_w(a_i, b_i)$ and its closure is given by

$$\tilde{T}_0(M_i) = T_0(M_i; a_i) \oplus T_0(M_i; b_i),\ i = 1, \ldots, m.$$

Also,

$$\operatorname{nul}[\tilde{T}_0(M_i) - \lambda I] = \operatorname{nul}[T_0(M_i; a_i) - \lambda I] + \operatorname{nul}[T_0(M_i; b_i) - \lambda I],$$

$$\operatorname{def}[\tilde{T}_0(M_i) - \lambda I] = \operatorname{def}[T_0(M_i; a_i) - \lambda I] + \operatorname{def}[T_0(M_i; b_i) - \lambda I],$$

and $R[\tilde{T}_0(M_i) - \lambda I]$ is closed if, and only if, $R[T_0(M_i; a_i) - \lambda I]$ and $R[T_0(M_i; b_i) - \lambda I]$ are both closed. These results imply in particular that,

$$\Pi[\tilde{T}_0(M_i)] = \Pi[T_0(M_i; a_i)] \cap \Pi[T_0(M_i; b_i)], \quad i = 1, \ldots, m.$$

We refer to [1, Section III.10.4], [3], [6] and [14] for more details.

Next, we state the following results; the proof is similar to that in [1, Section III. 10.4].

Theorem 2.4. *$\tilde{T}_0(M_i) \subset T_0(M_i)$, $T(M_i) \subset T(M_i; a_i) \oplus T(M_i; b_i)$ and $\dim\{D[T_0(M_i)]/D[\tilde{T}_0(M_i)]\} = n$. If $\lambda \in \Pi[\tilde{T}_0(M_i) \cap \Delta_3[T_0(M_i) - \lambda I]$, then*

$$\operatorname{ind}[T_0(M_i) - \lambda I] = n - \operatorname{def}[T_0(M; a_i) - \lambda I] - \operatorname{def}[T_0(M_i; b_i) - \lambda I],$$

$i = 1, \ldots, m$, and in particular, if $\lambda \in \Pi[T_0(M_i)]$,

$$\operatorname{def}[T_0(M_i) - \lambda I] = \operatorname{def}[T_0(M_i; a_i) - \lambda I] + \operatorname{def}[T_i(M_i; b_i) - \lambda I] - n, \tag{2.20}$$

Remark. It can be shown for $i = 1, \ldots, m$ that,

$$(2.21)\quad \begin{cases} D[\tilde{T}_0(M_i)] = \{u : u \in D[T_0(M_i)] \text{ and } u^{[r-1]}(c_i) = 0,\ r = 1, \ldots, n\} \\ D[\tilde{T}_0(M_i^+)] = \left\{v : v \in D[T_0(M_i^+)] \text{ and } v_+^{[r-1]}(c_i) = 0,\ r = 1, \ldots, n\right\} \end{cases}$$

see [1, Section III.10.4].

We now establish by [1], [7], [8] and [13] some further notation:

$$(2.22)\quad \begin{cases} D_0(M) = \sum_{i=1}^{m} D_0(M_i) \oplus \sum_{i=1}^{n} D_0(M_i) + \sum_{i=1}^{p} D_0(M_i) \oplus \sum_{i=1}^{q} D_0(M_i) \\ D_0(M^+) = \sum_{i=1}^{m} D_0(M_i^+) \oplus \sum_{i=1}^{n} D_0(M_i^+) + \sum_{i=1}^{p} D_0(M_i^+) \oplus \sum_{i=1}^{q} D_0(M_i^+) \end{cases},$$

$$(2.23)\quad \begin{cases} T_0(M)f = \Big\{T_0(M_1)f, \ldots, T_0(M_m)f; T_0(M_1)f, \ldots, T_0(M_n)f; \\ \qquad T_0(M_1)f, \ldots, T_0(M_p)f; T_0(M_1)f, \ldots, T_0(M_q)f\Big\}, \\ T_0(M^+)g = \Big\{T_0(M_1^+)g, \ldots, T_0(M_m^+)g; T_0(M_1^+)g, \ldots, T_0(M_n^+)g; \\ \qquad T_0(M_1^+)g, \ldots, T_0(M_p^+)g; T_0(M_1^+)g, \ldots, T_0(M_q^+)g\Big\}, \\ f \in D_0(M_i), \quad g \in D_0(M_i^+); \qquad i = 1, \ldots, m; \\ f \in D_0(M_i), \quad g \in D_0(M_i^+); \qquad i = 1, \ldots, n; \\ f \in D_0(M_i), \quad g \in D_0(M_i^+); \qquad i = 1, \ldots, p; \\ f \in D_0(M_i), \quad g \in D_0(M_i^+); \qquad i = 1, \ldots, q. \end{cases}$$

Also,

$$(2.24)\quad \begin{cases} T(M)f = \Big\{T(M_1)f, \ldots, T(M_m)f; T(M_1)f, \ldots, T(M_n)f; \\ \qquad T(M_1)f, \ldots, T(M_p)f; T(M_1)f, \ldots, T(M_q)f\Big\}, \\ T(M^+)g = \Big\{T(M_1^+)g, \ldots, T(M_m^+)g; T(M_1^+)g, \ldots, T(M_n^+)g; \\ \qquad T(M_1^+)g, \ldots, T(M_p^+)g; T(M_1^+)g, \ldots, T(M_q^+)g\Big\}, \\ f \in D(M_i), \quad g \in D(M_i^+); \qquad i = 1, \ldots, m; \\ f \in D(M_i), \quad g \in D(M_i^+); \qquad i = 1, \ldots, n; \\ f \in D(M_i), \quad g \in D(M_i^+); \qquad i = 1, \ldots, p; \\ f \in D(M_i), \quad g \in D(M_i^+); \qquad i = 1, \ldots, q \end{cases}$$

and

$$[f,g]=\sum_{i=1}^{m}\Big\{[f,g]_i(b_i)-[f,g]_i(a_i)\Big\}+\sum_{i=1}^{n}\Big\{[f,g]_i(b_i)-[f,g]_i(a_i)\Big\}+$$
$$+\sum_{i=1}^{p}\Big\{[f,g]_i(b_i)-[f,g]_i(a_i)\Big\}+\sum_{i=1}^{q}\Big\{[f,g]_i(b_i)-[f,g]_i(a_i)\Big\},$$

where $[\cdot,\cdot]$ is the bilinear form defined in (2.12).

Note that: $T_0(M)$ is a closed densely-defined operator in H.

We summarize a few additional properties of $T_0(M)$ in the form of a lemma:

Lemma 2.5. *We have,*

(*a*) $$[T_0(M)]^*=\sum_{i=1}^{m}[T(M_i)]^*\oplus\sum_{i=1}^{n}[T(M_i)]^*\oplus\sum_{i=1}^{p}[T(M_i)]^*\oplus\sum_{i=1}^{q}[T(M_i)]^*,$$

$$[T_0(M^+)]^*=\sum_{i=1}^{m}[T(M_i^+)]^*\oplus\sum_{i=1}^{n}[T(M_i^+)]^*\oplus\sum_{i=1}^{p}[T(M_i^+)]^*\oplus\sum_{i=1}^{q}[T(M_i^+)]^*,$$

In particular

$$\begin{aligned}D[T_0(M)]^* &= D[T(M^+)]\\ &=\sum_{i=1}^{m}D[T(M_i^+)]\oplus\sum_{i=1}^{n}D[T(M_i^+)]\oplus\sum_{i=1}^{p}D[T(M_i^+)]\oplus\sum_{i=1}^{q}D[T(M_i^+)]\\ D[T_0(M^+)]^* &= D[T(M)]\\ &=\sum_{i=1}^{m}D[T(M_i)]\oplus\sum_{i=1}^{n}D[T(M_i)]\oplus\sum_{i=1}^{p}D[T(M_i)]\oplus\sum_{i=1}^{q}D[T(M_i)]\end{aligned}$$

(*b*) $$\begin{aligned}\mathrm{nul}[T_0(M)-\lambda I] &=\sum_{i=1}^{m}\mathrm{nul}[T_0(M_i)-\lambda I]+\sum_{i=1}^{n}\mathrm{nul}[T_0(M_i)-\lambda I]\\ &\quad+\sum_{i=1}^{p}\mathrm{nul}[T_0(M_i)-\lambda I]+\sum_{i=1}^{q}\mathrm{nul}[T_0(M_i)-\lambda I],\end{aligned}$$

$$\begin{aligned}\mathrm{nul}[T_0(M^+)-\overline{\lambda}I] &=\sum_{i=1}^{m}\mathrm{nul}[T_0(M_i^+)-\overline{\lambda}I]+\sum_{i=1}^{n}\mathrm{nul}[T_0(M_i^+)-\overline{\lambda}I]\\ &\quad+\sum_{i=1}^{p}\mathrm{nul}[T_0(M_i^+)-\overline{\lambda}I]+\sum_{i=1}^{q}\mathrm{nul}[T_0(M_i^+)-\overline{\lambda}I]\end{aligned}$$

$$(c)\quad \operatorname{def}[T_0(M)-\lambda I] = \sum_{i=1}^{m}\operatorname{def}[T_0(M_i)-\lambda I]+\sum_{i=1}^{n}\operatorname{def}[T_0(M_i)-\lambda I] + \sum_{i=1}^{p}\operatorname{def}[T_0(M_i)-\lambda I]+\sum_{i=1}^{q}\operatorname{def}[T_0(M_i)-\lambda I],$$

$$\operatorname{def}[T_0(M^+)-\overline{\lambda} I] = \sum_{i=1}^{m}\operatorname{def}[T_0(M_i^+)-\overline{\lambda} I]+\sum_{i=1}^{n}\operatorname{def}[T_0(M_i^+)-\overline{\lambda} I] + \sum_{i=1}^{p}\operatorname{def}[T_0(M_i^+)-\overline{\lambda} I]+\sum_{i=1}^{q}\operatorname{def}[T_0(M_i^+)-\overline{\lambda} I]$$

for all $\lambda \in \Pi[T_0(M)]$ *and* $\overline{\lambda} \in \Pi[T_0(M^+)]$.

P r o o f. Part (a) follows immediately from the definition of $T_0(M)$ and from the general definition of an adjoint operator. The other parts are either direct consequences of part (a) or follows immediately from the definitions. □

Lemma 2.6. *Let*

$$T_0(M)=\sum_{i=1}^{m}T_0(M_i)\oplus\sum_{i=1}^{n}T_0(M_i)\oplus\sum_{i=1}^{p}T_0(M_i)\oplus\sum_{i=1}^{q}T_0(M_i)$$

be a closed densely-defined operator on H. *Then,*

$$\Pi[T_0(M)]=\bigcap_{i=1}^{m}\Pi[T_0(M_i)]\bigcap_{i=1}^{n}\Pi[T_0(M_i)]\bigcap_{i=1}^{p}\Pi[T_0(M_i)]\bigcap_{i=1}^{q}\Pi[T_0(M_i)]. \tag{2.25}$$

P r o o f. The proof follows from Lemma 2.5 and since $R[T_0(M)-\lambda I]$ is closed if, and only if R$[T_0(M)-\lambda I]$, $i=1,\dots,m$; $R[T_0(M_i)-\lambda I)$, $i=1,\dots,n$; $R[T_0(M_i)-\lambda I]$, $i=1,\dots,p$ and $R[T_0(M_i)-\lambda I]$, $i=1,\dots,q$ are closed. □

Lemma 2.7. *If* S_i *are regularly solvable operators with respect to* $T_0(M_i)$ *and* $T_0(M_i^+)$ *on all subintervals* I_i, J_i, K_i *and* L_i *respectively then,*

$$S=\sum_{i=1}^{m}S_i\oplus\sum_{i=1}^{n}S_i\oplus\sum_{i=1}^{p}S_i\oplus\sum_{i=1}^{q}S_i$$

is regularly solvable with respect to $T_0(M)$ *and* $T_0(M^+)$.

P r o o f. The proof follows from Lemmas 2.5 and 2.6.

3. The regularly solvable operators with two singular end-points. We see from (2.19) that $T_0(M_i)\subset T(M_i)=[T_0(M_i^+)]^*$ and hence $T_0(M_i)$ and $T_0(M_i^+)$ form an adjoint pair of closed densely-defined operators in $L_w^2(I_i)$, $i=1,\dots,m$.

Lemma 3.1. *For* $\lambda \in \Pi[T_0(M_i), T_0(M_i^+)]$, $\operatorname{def}[T_0(M_i) - \lambda I] + \operatorname{def}[T_0(M_i^+) - \overline{\lambda}]$ *is constant and*

$$0 \leq \operatorname{def}[T_0(M_i) - \lambda I] + \operatorname{def}[T_0(M_i^+) - \overline{\lambda} I] \leq 2n, \quad i = 1, \dots, m. \tag{3.1}$$

In the problem with one singular end points, i.e., on J_i, *and* K_i,

$$n \leq \operatorname{def}[T_0(M_i) - \lambda I] + \operatorname{def}[T_0(M_i^+) - \overline{\lambda} I] \leq 2n.$$

In the regular problem, i.e., on L_i, $i = 1, \dots, q$,

$$\operatorname{def}[T_0(M_i) - \lambda I] + \operatorname{def}[T_0(M_i^+) - \overline{\lambda} I] = 2n.$$

Proof. The proof is similar to that in [10] and therefore omitted. □

For $\lambda \in \Pi[T_0(M_i), T_0(M_i^+)]$, $i = 1, \dots, m$, we define r_i, s_i and m_i as follows,

$$\begin{cases} r_i = r_i(\lambda) : = \operatorname{def}[T_0(M_i) - \lambda I] \\ \qquad\qquad\quad = \operatorname{def}[T_0(M_i; a_i) - \lambda I] + \operatorname{def}[T_0(M_i; b_i) - \lambda I] - n \\ \qquad\qquad\quad = r_i^1 + r_i^2 - n, \\ s_i = s_i(\lambda) : = \operatorname{def}[T_0(M_i^+) - \overline{\lambda} I] \\ \qquad\qquad\quad = \operatorname{def}[T_0(M_i^+; a_i) - \overline{\lambda} I] + \operatorname{def}[T_0(M_i^+; b_i) - \overline{\lambda} I] - n \\ \qquad\qquad\quad = s_i^1 + s_i^2 - n \\ \text{and} \\ \qquad m_i : = r_i + s_i. \end{cases} \tag{3.2}$$

Since,

$$r_i = r_i^1 + r_i^2 - n, \quad s_i = s_i^1 + s_i^2 - n, \qquad i = 1, \dots, m,$$

then,

$$\begin{cases} m_i = r_i + s_i \\ \quad = (r_i^1 + r_i^2 - n) + (s_i^1 + s_i^2 - n) \\ \quad = (r_i^1 + s_i^1) + (r_i^2 + s_i^2) - 2n \\ \quad = m_i^1 + m_i^2 - 2n, \quad i = 1, \dots, m. \end{cases} \tag{3.3}$$

Also, since,

$$n \le m_i^j \le 2n \qquad (j = 1, 2;\ i = 1, \ldots, m),$$

then by Lemma 3.1, we have that,

$$0 \le m_i \le 2n, \quad i = 1, \ldots, m. \tag{3.4}$$

For $\Pi[T_0(M_i), T_0(M_i^+)] \neq \emptyset$ the operators which are regularly solvable with respect to $T_0(M_i)$ and $T_0(M_i^+)$ are characterized by the following theorem.

Theorem 3.2. *For $\lambda \in \Pi[T_0(M_i), T_0(M_i^+)]$, let r_i and m_i be defined by* (3.2) *and let $\psi_{j,i}$, ($j = 1, \ldots, r_i$) and $\phi_{k,i}$ ($k = r_i + 1, \ldots, m_i$), $i = 1, \ldots, m$ be arbitrary functions satisfying:*

- *(i)* *$\{\psi_{j,i} : j = 1, \ldots, r_i\} \subset D(M_i)$ is linearly independed modulo $D_0(M_i)$ and $\{\phi_{k,i} : k = r_i + 1, \ldots, m_i\} \subset D_0(M_i^+)$ is linearly independed modulo $D_0(M_i^+)$, $i = 1, \ldots, m$.*
- *(ii)* *$[\psi_{j,i}, \phi_{k,i}]_i(b_i) - [\psi_{j,i}, \phi_{k,i}]_i(a_i) = 0$ ($j = 1, \ldots, r_i$, $k = r_i + 1, \ldots, m$; $i = 1, \ldots, m$).*

Then the set,

$$\begin{aligned} \{u \ : \ u \in D(M_i), \quad & [u, \phi_{k,i}]_i(b_i) - [u, \phi_{k,i}]_i(a_i) = 0, \\ & (k = r_i + 1, \ldots, m_i; \quad i = 1, \ldots, m)\} \end{aligned} \tag{3.5}$$

is the domain of an operator S_i which is regularly solvable with respect to $T_0(M_i)$ and $T_0(M_i^+)$ and the set,

$$\begin{aligned} \{v \ : \ v \in D(M_i^+), \quad & [\psi_{j,i}, v]_i(b_i) - [\psi_{j,i}, v]_i(a_i) = 0, \\ & (j = 1, \ldots, r_i; \quad i = 1, \ldots, m)\} \end{aligned} \tag{3.6}$$

is the domain of S_i^*; moreover $\lambda \in \Delta_4(S_i)$, $i = 1, \ldots, m$.

Conversely, if S_i is regularly solvable with respect to $T_0(M_i)$ and $T_0(M_i^+)$ and $\lambda \in \Pi[T_0(M_i), T_0(M_i^+)] \cap \Delta_4(S_i)$, then with r_i and m_i defined by (3.2) *there exists functions $\psi_{j,i}$ ($j = 1, \ldots, r_i$), $\phi_{k,i}$ ($k = r_i + 1, \ldots, m_i$), $i = 1, \ldots, m$ which satisfy (i) and (ii) and are such that* (3.5) *and* (3.6) *are the domains of S_i and S_i^* respectively.*

S_i is self-adjoint if and only if $M_i^+ = M_i$, $r_i = s_i$ and $\phi_{k,i} = \psi_{k-r_i,i}$ ($k = r_i + 1, \ldots, m_i$, $i = 1, \ldots, m$), S_i is J-self-adjoint if and only if $M_i = JM_i^+J$ (J is a complex conjugate), $r_i = s_i$ and $\phi_{k,i} = \overline{\psi}_{k-r_i,i}$, ($k = r_i + 1 \ldots, m_i$, $i = 1, \ldots, m$).

P r o o f. The proof is entirely similar to that in [2, 9] and therefore omitted. □

For $\lambda \in \Pi[T_0(M_i), T_0(M_i^+)]$, define r_i, s_i and m_i, $i = 1, \ldots, m$ as in (3.2) and (3.3). Let $\{\psi_{j,i}^{a_i}, j = 1, \ldots, s_i^1\}$, $\{\phi_{k,i}^{a_i}, k = s_i^1 + 1, \ldots, m_i^1\}$ be bases for $N[T(M_i; a_i) - \lambda I]$ and $N[T(M_i^+; a_i) - \overline{\lambda} I]$ respectively, thus $\psi_{j,i}^{a_i}, \phi_{k,i}^{a_i} \in L_w^2(a_i, c_i)$ $(j = 1, \ldots, s_i^1;\ k = s_i^1 + 1 \ldots, m_i^1)$ and

$$M_i[\psi_{j,i}^{a_i}] = \lambda w \psi_{j,i}^{a_i}, \quad M_i^+[\phi_{k,i}^{a_i}] = \overline{\lambda} w \phi_{k,i}^{a_i} \quad \text{on } [a_i, c_i),\ i = 1, \ldots, m. \tag{3.7}$$

Similarly, let $\{\psi_{j,i}^{b_i} : j = 1, \ldots, s_i^2\}$, $\{\phi_{k,i}^{b_i} : k = s_i^2 + 1, \ldots, m_i^2\}$ be bases for $N[T(M_i; b_i) - \lambda I]$ and $N[T(M_i^+; b_i.) - \overline{\lambda} I]$ respectively; thus $\psi_{j,i}^{b_i}$, $\phi_{k,i}^{b_i} \in L_w^2(c_i, b_i)$ and

$$M_i[\psi_{j,i}^{b_i}] = \lambda w \psi_{j,i}^{b_i}, \quad M_i^+[\phi_{k,i}^{b_i}] = \overline{\lambda} w \phi_{k,i}^{b_i} \quad \text{on } [c_i, b_i), i = 1, \ldots, m. \tag{3.8}$$

Since, $[T_0(M_i^+; a_i) - \overline{\lambda} I]$ and $[T_0(M_i^+; b_i) - \overline{\lambda} I]$ have closed ranges, so do their adjoints $[T(M_i; a_i) - \lambda I]$ and $[T(M_i; b_i) - \lambda I]$ and moreover $R[T(M_i; a_i) - \lambda I]^{\perp} = N[T_0(M_i; a_i) - \lambda I) = \{0\}$ and $R[T(M_i^+; b_i) - \overline{\lambda} I]^{\perp} = N[T_0(M_i^+; b_i) - \overline{\lambda} I] = \{0\}$. Hence $R[T(M_i; a_i) - \lambda I] = L_w^2(a_i, c_i)$ and $R[T(M_i^+; b_i) - \overline{\lambda} I] = L_w(c_i, b_i)$, $i = 1, \ldots, m$. Similarly, $R[T(M_i; a_i) - \lambda I] = L_w^2(a_i, c_i)$ and $R[T(M_i^+; b_i) - \overline{\lambda} I] = L_w^2(c_i; b_i)$, $i = 1, \ldots, m$. We can therefore define the following:

$$\begin{cases} x_{j,i}^{a_i} := \psi_{j,i}^{a_i} & (j = 1, \ldots, s_i^1;\ i = 1, \ldots, m), \\ [T(M_i; a_i) - \lambda I] x_{j,i}^{a_i} := \phi_{j,i}^{a_i} & (j = s_i^1 + 1, \ldots, m_i^1), \\ [T(M_i^+; a_i) - \overline{\lambda} I] y_{j,i}^{a_i} := \psi_{j,i}^{a_i} & (j = 1, \ldots, s_i^1), \\ y_{j,i}^{a_i} := \phi_{j,i}^{a_i} & (j = s_i^1 + 1, \ldots, m_i^1); \end{cases} \tag{3.9}$$

$$\begin{cases} x_{j,i}^{b_i} := \psi_{j,i}^{b_i} & (j = 1, \ldots, s_i^2;\ i = 1, \ldots, m), \\ [T(M_i; b_i) - \lambda I] x_{j,i}^{b_i} := \phi_{j,i}^{b_i} & (j = s_i^2 + 1, \ldots, m_i^2), \\ [T(M_i^+; b_i) - \overline{\lambda} I] y_{j,i}^{b_i} := \psi_{j,i}^{b_i} & (j = 1, \ldots, s_i^2), \\ y_{j,i}^{b_i} := \phi_{j,i}^{b_i} & (j = s_i^2 + 1, \ldots, m_i^2). \end{cases} \tag{3.10}$$

Next we state the following results; the proofs are similar to those in [2, Section 4], [9] and [10].

Lemma 3.3. *The sets* $\{x_{j,i}^{a_i} : j = 1, \ldots, m_i^1, i = 1, \ldots, m\}$, $\{x_{k,i}^{b_i} : k = 1, \ldots, m_i^2,\ i = 1, \ldots, m\}$ *are bases of* $N([T(M_i^+; a_i) - \overline{\lambda} I][T(M_i; a_i) -$

$\lambda I]$) and $R([T(M_i^+;b_i)-\overline{\lambda}I][T(M_i;b_i)-\lambda I])$ respectively, $\{y_{j,i}^{a_i}: j=1,\dots,m_i^1; i=1,\dots,m\}$ and $\{y_{k,i}^{b_i}: k=1,\dots,m_i^2; i=1,\dots,m\}$ are bases of $N([T(M_i;a_i)-\lambda I][T(M_i^+;a_i)-\overline{\lambda}I])$ and $N([T(M_i;b_i)-\lambda I][T(M_i^+;b_i)-\overline{\lambda}I])$ respectively.

On applying [1, Theorem III.3.1] we obtain,

Corollary 3.4. *Any $z_i^{a_i}\in D(M_i;a_i)$ and $(z_i^{a_i})^+\in D(M_i^+;a_i)$ have the unique representations,*

$$z_i^{a_i}=z_{0i}^{a_i}+\sum_{j=1}^{m_i^1}\alpha_{ji}^i x_{j,i}^{a_i}\quad (z_{0i}^{a_i}\in D_0(M_i;a_i),\alpha_{ji}^i\in\mathbb{C}),\tag{3.11}$$

$$(z_i^{a_i})^+=(z_{0i}^{a_i})^++\sum_{j=1}^{m_i^1}\beta_{ji}^i y_{j,i}^{a_i}\quad ((z_{0i}^{a_i})^+\in D_0(M_i^+;a_i),\beta_{ji}^i\in\mathbb{C}).\tag{3.12}$$

Also, any $z_i^{b_i}\in D(M_i;b_i)$ and $(z_i^{b_i})^+\in D(M_i;b_i)$ have the unique representations,

$$z_i^{b_i}=z_{0i}^{b_i}+\sum_{k=1}^{m_i^2}\gamma_{ki}^i x_{k,i}^{b_i}\quad (z_{0i}^{b_i}\in D_0(M_i;b_i),\gamma_{ki}^i\in\mathbb{C}),\tag{3.13}$$

$$(z_i^{b_i})^+=(z_{0i}^{b_i})^++\sum_{k=1}^{m_i^2}\delta_{ki}^i y_{k,i}^{b_i}\quad ((z_{0i}^{b_i})^+\in D_0(M_i^+;b_i),\delta_{ki}^i\in\mathbb{C}),\tag{3.14}$$

$(i=1,\dots,m)$.

A central role in the algorithm is played by the matrices

Lemma 3.5. *Let*

$$E_{m_i^1\times m_i^1}:=\left([x_{j,i}^{a_i},y_{k,i}^{a_i}]_i(a_i)\right)_{\substack{1\le j\le m_i^1\\ 1\le k\le m_i^1}},\quad i=1,\dots,m,\tag{3.15}$$

$$E_{m_i^2\times m_i^2}:=\left([x_{j,i}^{b_i},y_{k,i}^{b_i}]_i(b_i)\right)_{\substack{1\le j\le m_i^2\\ 1\le k\le m_i^2}},\quad i=1,\dots,m,\tag{3.16}$$

and

$$E_{s_i^1\times s_i^1}^{1,2}:=\left([x_{j,i}^{a_i},y_{k,i}^{a_i}]_i(a_i)\right)_{\substack{1\le j\le s_i^1\\ s_i^1+1\le k\le m_i^1}},\quad i=1,\dots,m,\tag{3.17}$$

$$E^{1,2}_{s_i^2\times s_i^2} := \Big([x^{b_i}_{j,i}, y^{b_i}_{k,i}]_i(b_i)\Big)_{\substack{1\le j\le s_i^2\\ s_i^2+1\le k\le m_i^2}}, \quad i=1,\dots,m. \tag{3.18}$$

Then,

$$\text{Rank}E^{1,2}_{s_i^j\times r_i^j} = \text{Rank}E_{m_i^j\times m_i^j} = m_i^j - n, \quad (j=1,2;\ i=1,\dots,m). \tag{3.19}$$

In view of Lemma 3.5 and since $r_i^j,\ s_i^j \ge m_i^j - n$ $(j=1,2;\ i=1,\dots,m)$, we may suppose, without loss of generality, that the matrices,

$$E^{1,2}_{(m_i^1-n)\times(m_i^1-n)} := \Big([x^{a_i}_{j,i}, y^{a_i}_{k,i}]_i(a_i)\Big)_{\substack{1\le j\le m_i^1-n\\ n+1\le k\le m_i^1}}, \tag{3.20}$$

and

$$E^{1,2}_{(m_i^2-n)\times(m_i^2-n)} := \Big([x^{b_i}_{j,i}, y^{b_i}_{k,i}]_i(b_i)\Big)_{\substack{1\le j\le m_i^2-n\\ n+1\le k\le m_i^2}}, \tag{3.21}$$

satisfy

$$\text{Rank}E^{1,2}_{(m_i^j-n)\times(m_i^j-n)} = m_i^j - n, \quad (j=1,2;\ i=1,\dots,m). \tag{3.22}$$

If we partation $E_{m_i^j\times m_i^j}$ $(j=1,2;\ i=1,\dots,m)$ as

$$E_{m_i^j\times m_i^j} = \begin{bmatrix} E^{1,1}_{(m_i^j-n)\times n} & \vdots & E^{1,2}_{(m_i^j-n)\times(m_i^j-n)} \\ \cdots\cdots & \cdots & \cdots\cdots \\ E^{2,1}_{n\times n} & \vdots & E^{2,2}_{n\times(m_i^j-n)} \end{bmatrix}, \tag{3.23}$$

and set

$$\begin{cases} E^{1}_{(m_i^j-n)\times(m_i^j-n)} = E^{1,1}_{(m_i^j-n)\times n} \oplus E^{1,2}_{(m_i^j-n)\times(m_i^j-n)} \\ E^{2}_{n\times m_i^j} = E^{2,1}_{n\times n} \oplus E^{2,2}_{n\times(m_i^j-n)}, \end{cases} \tag{3.24}$$

$$\begin{cases} F^{1}_{m_i^j\times n} = E^{1,1}_{(m_i^j-n)\times n} \oplus^{\top} E^{2,1}_{n\times n} \\ F^{2}_{m_i^j\times(m_i^j-n)} = E^{1,2}_{(m_i^j-n)\times(m_i^j-n)} \oplus^{\top} E^{2,2}_{n\times(m_i^j-n)}, \end{cases} \tag{3.25}$$

then (3.22) yields the results,

$$\mathrm{Rank}E^1_{(m_i^j-n)\times m_i^j} = \mathrm{Rank}F^2_{m_i^j\times(m_i^j-n)} = m_i^j - n, \quad (j = 1, 2;\ i = 1, \ldots, m). \tag{3.26}$$

Lemma 3.6. *Let $D_i(M_i; a_i)$, $i = 1, \ldots, m$ be the linear span of $\{z_{j,i}^{a_i} : j = 1, \ldots, n;\ i = 1, \ldots, m\}$ where $z_{j,i}^{a_i} \in D(M_i; a_i)$ satisfy the following conditions for $k = 1, \ldots, n$ and some $c_i \in (a_i; b_i)$, $i = 1, \ldots, m$.*

$$\begin{cases} (z_{j,i}^{a_i})^{[k-1]}(a_i) = \delta_{jk}^i & (z_{j,i}^{a_i})^{[k-1]}(c_i) = 0 \\ z_{j,i}^{a_i}(t) = 0, & \text{for } t \geq c_i,\ i = 1, \ldots, m, \end{cases} \tag{3.27}$$

and let $D_2(M_i; a_i)$, $i = 1, \ldots, m$ be the linear span of $\{x_{j,i}^{a_i} : j = 1, \ldots, m_i^1 - n,\ i = 1, \ldots, m\}$ with (3.22) satisfied. Then,

$$D(M_i; a_i) = D_0(M_i; a_i) \dot{+} D_1(M_i; a_i) \dot{+} D_2(M_i; a_i), \quad i = 1, \ldots, m.$$

Similarly,

$$D(M_i; b_i) = D_0(M_i; b_i) \dot{+} D_1(M_i; b_i) \dot{+} D_2(M_i; b_i), \quad i = 1, \ldots, m.$$

If $D_1(M_i^+; a_i)$ and $D_2(M_i^+; a_i)$ are the linear spaces of $\{(z_{j,i}^{a_i})^+ : j = 1, \ldots, n;\ i = 1, \ldots, m\}$ and $\{y_{k,i}^{a_i}, k = n+1, \ldots, m_i^1;\ i = 1, \ldots, m\}$ respectively, then

$$D(M_i^+; a_i) = D_0(M_i^+; a_i) \dot{+} D_1(M_i^+; a_i) \dot{+} D_2(M_i^+; a_i), \quad i = 1, \ldots, m. \tag{3.28}$$

Similarly,

$$D(M_i^+; b_i) = D_0(M_i^+; b_i) \dot{+} D_1(M_i^+; b_i) \dot{+} D_2(M_i^+; b_i), \quad i = 1, \ldots, m. \tag{3.29}$$

We shall now characterize all the operators which are regularly solvable with respect to $T_0(M_i)$ and $T_0(M_i^+)$ in terms of boundary conditions featuring $L_w^2(I_i)$-solutions of the equations $M_i[u] = \lambda wu$ and $M_i^+[v] = \overline{\lambda} wv$ ($\lambda \in \mathbb{C}$), $i = 1, \ldots, m$ by the following two theorems with a brief sketch of the proof.

Theorem 3.7. *Let $\lambda \in \Pi[T_0(M_i); T_0(M_i^+)]$, let r_i, s_i and m_i, $i = 1, \ldots, m$ deffined by (3.2) and let $x_{j,i}^{a_i}$, $y_{j,i}^{a_i}$ ($j = 1, \ldots, m_i^1;\ i = 1, \ldots, m$) and $x_{k,i}^{b_i}$, $y_{k,i}^{b_i}$ ($k = 1, \ldots, m_i^2;\ i = 1, \ldots, m$) be defined by (3.9) and (3.10) respectively and arranged to satisfy (3.22). Let $M^i_{s_i\times(m_i^1-n)}$, $N^i_{s_i\times(m_i^2-n)}$, $K^i_{r_i\times(m_i^1-n)}$*

and $L^i_{r_i\times(m_i^2-n)}$*,* $i=1,\dots,m$ *be numerical matrices which satisfy the following conditions:*

$$(i)\qquad \text{Rank}\left\{M^i_{s_i\times(m_i^1-n)}\oplus N^i_{s_i\times(m_i^2-n)}\right\}=s_i\quad i=1,\dots,m,$$
$$\text{Rank}\left\{K^i_{r_i\times(m_i^1-n)}\oplus L^i_{r_i\times(m_i^2-n)}\right\}=r_i\quad i=1,\dots,m.$$

$$(ii)\qquad \left\{K^i_{r_i\times(m_i^1-n)}E^{1,2}_{(m_i^1-n)\times(m_i^1-n)}\left(M^i_{s_i\times(m_i^1-n)}\right)^{\top}\right\}_{r_i\times s_i}=$$
$$\left\{L^i_{r_i\times(m_i^2-n)}E^{1,2}_{(m_i^2-n)\times(m_i^2-n)}\left(N^i_{s_i\times(m_i^2-n)}\right)^{\top}\right\}_{r_i\times s_i},\quad i=1,\dots,m.$$

Then the set of all $u\in D[T(M_i)]$ *such that*

$$B_i(u,I_i)=M^i_{s_i\times(m_i^1-n)}\begin{bmatrix}[u,y^{a_i}_{n+1,i}]_i(a_i)\\ \vdots\\ [u,y^{a_i}_{m_i^1,i}]_i(a_i)\end{bmatrix}-N^i_{s_i\times(m_i^2-n)}\begin{bmatrix}[u,y^{b_i}_{n+1,i}]_i(b_i)\\ \vdots\\ [u,y^{b_i}_{m_i^2,i}]_i(b_i)\end{bmatrix}=$$
$$=O_{s_i\times 1},\qquad i=1,\dots,m,\tag{3.30}$$

is the domain of an operator S_i*,* $i=1,\dots,m$ *which is regularly solvable with respect to* $T_0(M_i)$ *and* $T_0(M_i^+)$ *and* $D(S_i^*)$ *is the set of all* $v\in D[T(M_i^+)]$ *which are such that*

$$B_i^*(v,I_i)=K^i_{r_i\times(m_i^1-n)}\begin{bmatrix}[x^{a_i}_{1,i},v]_i(a_i)\\ \vdots\\ [x^{a_i}_{(m_i^1-n),i},v]_i(a_i)\end{bmatrix}-L^i_{r_i\times(m_i^2-n)}\begin{bmatrix}[x^{b_i}_{1,i},v]_i(b_i)\\ \vdots\\ [x^{b_i}_{(m_i^2-n),i},v]_i(b_i)\end{bmatrix}=$$
$$=O_{r_i\times 1},\qquad i=1,\dots,m.\tag{3.31}$$

P r o o f. Let,

$$M^i_{s_i\times(m_i^1-n)}=\left(\alpha^i_{jk}\right)_{\substack{r_i+1\le j\le m_i\\ n+1\le k\le m_i^1}},\quad N^i_{s_i\times(m_i^2-n)}=\left(\beta^i_{jk}\right)_{\substack{r_i+1\le j\le m_i\\ n+1\le k\le m_i^2}},\tag{3.32}$$

and set

$$g^{a_i}_{j,i}=\sum_{k=n+1}^{m_i^1}\overline{\alpha}^i_{jk}y^{a_i}_{k,i}\qquad g^{b_i}_{j,i}=\sum_{k=n+1}^{m_i}\overline{\beta}^i_{jk}y^{a_i}_{k,i},\quad j=r_i+1,\dots,m_i.\tag{3.33}$$

Then $g_{j,i} \in D[T(M_i^+)]$ where

$$g_{j,i} = \begin{cases} g_{j,i}^{a_i} & \text{in} \quad (a_i, c_i], \\ g_{j,i}^{b_i} & \text{in} \quad [c_i, b_i), \qquad i = 1, \dots, m. \end{cases}$$

By [17, Theorem 8], we may choose $\phi_{j,i}$, $(j = r_i + 1, \dots, m_i) \in D[T(M_i^+)]$,

$$\phi_{j,i} = \begin{cases} \phi_{j,i}^{a_i} & \text{in} \quad (a_i, c_i], \\ \phi_{j,i}^{b_i} & \text{in} \quad [c_i, b_i), \qquad i = 1, \dots, m, \end{cases}$$

such that for $a_i' \in (a_i, c_i)$ and $k = 1, \dots, n$,

$$(3.34) \qquad (\phi_{j,i}^{a_i})_+^{[k-1]}(c_i) = 0, \quad (\phi_{j,i}^{a_i})_+^{[k-1]}(a_i') = (g_{j,i}^{a_i})_+^{[k-1]}(a_i'),$$

$$\phi_{j,i}^{a_i} = g_{j,i}^{a_i} \text{ on } (a_i, a_i'],\ j = r_i + 1, \dots, m;\ i = 1, \dots, m,$$

and for $b_i' \in (c_i, b_i)$,

$$(3.35) \qquad (\phi_{j,i}^{b_i})_+^{[k-1]}(c_i) = 0, \quad (\phi_{j,i}^{b_i})_+^{[k-1]}(b_i') = (g_{j,i}^{b_i})_+^{[k-1]}(b_i'),$$

$$\phi_{j,i}^{a_i} = g_{j,i}^{a_i} \text{ on } [b_i', b_i),\ j = r_i + 1, \dots, m;\ i = 1, \dots, m.$$

This gives,

$$M^i_{s_i \times (m_i^1 - n)} \begin{bmatrix} [u, y_{n+1,i}^{a_i}]_i(a_i) \\ \vdots \\ [u, y_{m_i^1,i}^{a_i}]_i(a_i) \end{bmatrix} = \begin{bmatrix} \Big[u, \sum_{k=n+1}^{m_i^1} \overline{\alpha}^i_{r_i+1,k} y_{k,i}^{a_i}\Big]_i(a_i) \\ \vdots \\ \Big[u, \sum_{k=n+1}^{m_i^1} \overline{\alpha}^i_{m_i,k} y_{k,i}^{a_i}\Big]_i(a_i) \end{bmatrix} = \begin{bmatrix} [u, \phi_{r_i+1,i}^{a_i}]_i(a_i) \\ \vdots \\ [u, \phi_{m_i,i}^{a_i}]_i(a_i) \end{bmatrix}.$$

Similarly,

$$N^i_{s_i \times (m_i^2 - n)} \begin{bmatrix} [u, y_{n+1,i}^{b_i}]_i(b_i) \\ \vdots \\ [u, y_{m_i^2,i}^{a_i}]_i(b_i) \end{bmatrix} = \begin{bmatrix} [u, \phi_{r_i+1,i}^{b_i}]_i(b_i) \\ \vdots \\ [u, \phi_{m_i,i}^{b_i}]_i(b_i) \end{bmatrix}.$$

The boundary conditions (3.30) therefore coincides with that in (3.5). Similarly, (3.31) coincides with (3.6) on making the following choices,

$$K^i_{r_i\times(m_i^1-n)} = \left(\tau^i_{jk}\right)_{\substack{1\le j\le r_i\\ 1\le k\le m_i^1-n}}, \quad L^i_{s_i\times(m_i^2-n)} = \left(\epsilon^i_{jk}\right)_{\substack{1\le j\le r_i\\ 1\le k\le m_i^2-n}}, \tag{3.36}$$

$$h^{a_i}_{j,i} = \sum_{k=1}^{m_i^1-n} \tau^i_{jk} x^{a_i}_{k,i}, \quad h^{b_i}_{j,i} = \sum_{k=1}^{m_i^2-n} \epsilon^i_{jl} x^{a_i}_{k,i}, \quad i=1,\dots,m. \tag{3.37}$$

Then $h_{j,i}\in D[T(M_i)]$, where

$$h_{j,i} = \begin{cases} h^{a_i}_{j,i} & \text{in} \quad (a_i,c_i], \\ h^{b_i}_{j,i} & \text{in} \quad [c_i,b_i), \quad j=1,\dots,r_i;\ i=1,\dots,m, \end{cases}$$

and $\psi_{j,i}$ $(j=1,\dots,r_i)\in D[T(M_i)]$,

$$\psi_{j,i} = \begin{cases} \psi^{a_i}_{j,i} & \text{in} \quad (a_i,c_i] \\ \psi^{b_i}_{j,i} & \text{in} \quad [c_i,b_i), \ i=1,\dots,m, \end{cases}$$

such that for $a_i'\in(a_i,c_i)$ and $k=1,\dots,n$

$$\begin{gathered}(\psi^{a_i}_{j,i})^{[k-1]}(c_i)=0, \quad (\psi^{a_i}_{j,i})^{[k-1]}(a_i') = (h^{a_i}_{j,i})^{[k-1]}(a_i'), \\ \psi^{a_i}_{j,i} = h^{a_i}_{j,i} \text{ on } (a_i,a_i'],\ j=1,\dots,r_i;\ i=1,\dots,m,\end{gathered} \tag{3.38}$$

and for $b_i'\in(c_i,b_i)$,

$$\begin{gathered}(\psi^{b_i}_{j,i})^{[k-1]}(c_i)=0, \quad (\psi^{b_i}_{j,i})^{[k-1]}(b_i') = (h^{b_i}_{j,i})^{[k-1]}(b_i'), \\ \psi^{a_i}_{j,i} = h^{a_i}_{j,i} \text{ on } (b_i',b_i],\ j=1,\dots,r_i;\ i=1,\dots,m.\end{gathered} \tag{3.39}$$

The functions $\phi_{k,i}$, $k=r_i+1,\dots,m_i$ and $\psi_{j,i}$, $j=1,\dots,r_i$, $i=1,\dots,m$ satisfy conditions (i) and (ii) in Theorem 3.2 and the last part of the theorem is immediate, see [2, Theorem 5.1] and [10].

The converse of Theorem 3.7 is

Theorem 3.8. *Let S_i, $i=1,\dots,m$ be regularly solvable with respect to $T(M_i)$ and $T_0(M_i^+)$, let $\lambda\in\Pi[T_0(M_i),T_0(M_i^+)]\cap\Delta_4(S_i)$, let r_i, s_i and m_i be defined by* (3.2) *and* (3.3) *and suppose that* (3.22) *is satisfied. Then there* *exist numerical matrices* $K^i_{r_i\times(m_i^1-n)}$, $L^i_{r_i\times(m_i^2-n)}$, $M^i_{s_i\times(m_i^1-n)}$ *and* $N^i_{s_i\times(m_i^2-n)}$, $i=1,\dots,m$ *such that conditions* (i) *and* (ii) *in Theorem* 3.7 *are satisfied and* $D(S_i)$ *is the set of* $u\in D(M_i)$ *satisfying* (3.30) *while* $D(S_i^*)$ *is the set of* $v\in D(M_i^+)$ *satisfying* (3.31).

Proof. Let $\{\psi_{j,i}, j=1,\dots,r_i\}\subset D(M_i)$, $\{\phi_{k,i}, k=r_i+1,\dots,m_i\}\subset D(M_i^+)$, $i=1,\dots,m$, and set

$$\psi_{j,i}=\begin{cases}\psi^{a_i}_{j,i} & \text{in} \quad (a_i,c_i]\\ \psi^{b_i}_{j,i} & \text{in} \quad [c_i,b_i)\end{cases} \qquad \text{and} \quad \phi_{j,i}=\begin{cases}\phi^{a_i}_{j,i} & \text{in} \quad (a_i,c_i]\\ \phi^{b_i}_{j,i} & \text{in} \quad [c_i,b_i)\end{cases}$$

are satisfying the second part of Theorem 3.2. From (3.28) and (3.29) we have that

$$\phi^{a_i}_{k,i}=y^{a_i}_{0k,i}+\sum_{j=1}^{n}\zeta^i_{kj}(z^{a_i}_{j,i})^+ + \sum_{j=n+1}^{m_i^1}\alpha^i_{kj}(y^{a_i}_{j,i}), \tag{3.40}$$

$k=r_i+1,\dots,m_i$; $i=1,\dots,m$ for some $y^{a_i}_{0k,i}\in D[T_0(M_i^+;a_i)]$ and complex constants ζ^i_{kj}, α^i_{kj} and

$$\phi^{b_i}_{k,i}=y^{b_i}_{0k,i}+\sum_{j=1}^{n}\xi^i_{kj}(z^{b_i}_{j,i})^+ + \sum_{j=n+1}^{m_i^2}\beta^i_{kj}(y^{b_i}_{j,i}), \tag{3.41}$$

$k=r_i+1,\dots,m_i$; $i=1,\dots,m$ for some $y^{b_i}_{0k,i}\in D[T_0(M_i^+;b_i)]$ and complex constants ξ^i_{kj}, β^i_{kj}. Since $y^{a_i}_{0k,i}\in D[T_0(M_i^+;a_i)]$ and $y^{b_i}_{0k,i}\in D[T_0(M_i^+;b_i)]$ then $y_{0k,i}\in D[T_0(M_i^+)]$, where

$$y_{0k,i}=\begin{cases}y^{a_i}_{0k,i} & \text{in} \quad (a_i,c_i],\\ y^{b_i}_{0k,i} & \text{in} \quad [c_i,b_i).\end{cases}$$

Hence, for all $u\in D[T(M_i)]$,

$$[u,y^{a_i}_{0k,i}]_i(a_i)=[u,y^{a_i}_{0k,i}]_i(c_i)=0 \qquad \text{and}$$
$$[u,y^{b_i}_{0k,i}]_i(b_i)=[u,y^{b_i}_{0k,i}]_i(c_i) \ i=1,\dots,m.$$

Also

$$[u,(z^{a_i}_{i,j})^+]_i(a_i)=[u,(z^{b_i}_{i,j})^+]_i(b_i)=0, \quad j=1,\dots,n;\ i=1,\dots,m.$$

Let

$$(3.42)\qquad M^i_{s_i\times(m_i^1-n)}=\left(\overline{\alpha^i_{jk}}\right)_{\substack{r_i+1\le j\le m_i\\ n+1\le k\le m_i^1}},\quad N^i_{s_i\times(m_i^2-n)}=\left(\overline{\beta^i_{jk}}\right)_{\substack{r_i+1\le j\le m_i\\ n+1\le k\le m_i^2}}.$$

Then from (3.40),

$$\begin{bmatrix}[u,\phi^{a_i}_{r_i+1,i}]_i(a_i)\\ \vdots\\ [u,\phi^{a_i}_{m_i,i}]_i(a_i)\end{bmatrix}=\begin{bmatrix}\left[u,\sum_{k=n+1}^{m_i^1}\alpha^i_{r_i+1,k}y^{a_i}_{k,i}\right]_i(a_i)\\ \vdots\\ \left[u,\sum_{k=n+1}^{m_i^1}\alpha^i_{m_i,k}y^{a_i}_{k,i}\right]_i(a_i)\end{bmatrix}$$

$$=M^i_{s_i\times(m_i^1-n)}\begin{bmatrix}[u,y^{a_i}_{n+1,i}]_i(a_i)\\ \vdots\\ [u,y^{a_i}_{m_i^1,i}]_i(a_i)\end{bmatrix}.$$

Similarly, from (3.41)

$$\begin{bmatrix}[u,\phi^{b_i}_{r_i+1,i}]_i(b_i)\\ \vdots\\ [u,\phi^{b_i}_{m_i,i}]_i(b_i)\end{bmatrix}=N^i_{s_i\times(m_i^2-n)}\begin{bmatrix}[u,y^{b_i}_{n+1,i}]_i(b_i)\\ \vdots\\ [u,y^{b_i}_{m_i^2,i}]_i(b_i)\end{bmatrix}.$$

Therefore, we have shown that the boundary conditions (3.30) coincides with those in (3.5); similarly (3.31) and the conditions in (3.6) can be shown to coincide if we choose,

$$(3.43)\qquad K^i_{r_i\times(m_i^1-n)}=\left(\tau^i_{jk}\right)_{\substack{1\le j\le r_i\\ 1\le k\le m_i^1-n}},\quad L^i_{s_i\times(m_i^2-n)}=\left(\epsilon^i_{jk}\right)_{\substack{1\le j\le r_i\\ 1\le k\le m_i^2-n}}.$$

where the τ^i_{jk} and ϵ^i_{jk} are the constants uniquely determined by the decompositions,

$$(3.44)\qquad \begin{cases}\psi^{a_i}_{j,i}=x^{a_i}_{j0,i}+\sum_{k=1}^{n}\zeta^i_{jk}z^{a_i}_{k,i}+\sum_{k=1}^{m_i^1-n}\tau^i_{jk}x^{a_i}_{k,i},\\ \psi^{b_i}_{j,i}=x^{b_i}_{j0,i}+\sum_{k=1}^{n}\xi^i_{jk}z^{b_i}_{k,i}+\sum_{k=1}^{m_i^1-n}\epsilon^i_{jk}x^{b_i}_{k,i},\end{cases}$$

$j=1,\ldots,r_i$; $i=1,\ldots,m$ derived from Lemma 3.6.

The conditions (i) and (ii) are consequences of conditions (i) and (ii) in Theorem 3.2, see [2, Theorem 5.2] and [10].

Remark 3.9. Assume that M_i, $i = 1, \dots, m$ is formally J-symmetric, that is $M_i^+ = JM_iJ$, $i = 1, \dots, m$ where J is complex conjugation. The operator $T_0(M_i)$ is then J-symmetric and $T_0(M_i)$ and $T_0(M_i^+) = J[T_0(M_i)]J$ form an adjoint pair with $\Pi[T_0(M_i), T_0(M_i^+)] = \Pi[T_0(M_i)]$, $i = 1, \dots, m$. Since $M_i^+[u] = \overline{\lambda} w\overline{u}$ if and only if $M_i[v] = \lambda wv$, it follows from (3.4) that for all $\lambda \in \Pi[T_0(M_i)]$, $\mathrm{def}[T_0(M_i) - \lambda I] = \mathrm{def}[T_0(M_i^+) - \overline{\lambda} I]$ is constant ℓ_i, say, and so in (3.2), $r_i = s_i = \ell_i$ with $0 \le \ell_i \le n$, $i = 1, \dots, m$.

4. The General Theorem. In this section, the domains of regularly solvable operators on the interval (a, b) are determined in terms of "boundary conditions". These conditions involve the expressions on the various subintervals I_i, J_i, K_i and L_i of (a, b). We denote by $T_0(M)$ and $T_0(M^+)$ the maximal and minimal operators on (a, b). We see from (2.23) and Lemma 2.4 that, $T_0(M) \subset T(M) = [T_0(M^+)]^*$ and hence $T_0(M)$ and $T_0^+(M)$ form an adjoint pair of closed densely-defined operators in $L^2_w(a, b)$.

For $\lambda \in \Pi[T_0(M), T_0(M^+)]$ we define r, s and m as follows:

$$(4.1)\quad \begin{cases} r = r(\lambda) := \mathrm{def}[T_0(M) - \lambda I] = \\ \quad = \sum_{i=1}^{m} \mathrm{def}[T_0(M_i) - \lambda I] + \sum_{i=1}^{n} \mathrm{def}[T_0(M_i) - \lambda I] + \sum_{i=1}^{p} \mathrm{def}[T_0(M_i) - \lambda I] + \\ \quad + \sum_{i=1}^{q} \mathrm{def}[T_0(M_i) - \lambda I] = \sum_{i=1}^{m} r_i + \sum_{i=1}^{n} r_i + \sum_{i=1}^{p} r_i + \sum_{i=1}^{q} r_i, \\ s = s(\lambda) := \mathrm{def}[T_0(M^+) - \overline{\lambda} I] = \\ \quad = \sum_{i=1}^{m} \mathrm{def}[T_0(M_i^+) - \overline{\lambda} I] + \sum_{i=1}^{n} \mathrm{def}[T_0(M_i^+) - \overline{\lambda} I] + \sum_{i=1}^{p} \mathrm{def}[T_0(M_i^+) - \overline{\lambda} I] + \\ \quad + \sum_{i=1}^{q} \mathrm{def}[T_0(M_i^+) - \overline{\lambda} I] = \sum_{i=1}^{m} s_i + \sum_{i=1}^{n} s_i + \sum_{i=1}^{p} s_i + \sum_{i=1}^{q} s_i, \\ \text{and} \\ m := r + s = \\ \quad = \sum_{i=1}^{m} (r_i + s_i) + \sum_{i=1}^{n} (r_i + s_i) + \sum_{i=1}^{p} (r_i + s_i) + \sum_{i=1}^{q} (r_i + s_i) = \\ \quad = \sum_{i=1}^{m} m_i + \sum_{i=1}^{n} m_i + \sum_{i=1}^{p} m_i + \sum_{i=1}^{q} m_i \\ \quad = m_1 + m_2 + m_3 + m_4 \end{cases}$$

where,

$$\begin{array}{rcccll} 0 & \leq & m_1 & \leq & 2nm, & \text{on} \quad I_i, \quad i=1,\dots,m, \\ n^2 & \leq & m_2 & \leq & 2n^2, & \text{on} \quad J_i, \quad i=1,\dots,n, \\ np & \leq & m_3 & \leq & 2np, & \text{on} \quad K_i, \quad i=1,\dots,p, \\ & & m_4 & = & 2qn, & \text{on} \quad L_i, \quad i=1,\dots,q. \end{array}$$

By Lemma 3.1, m is constant on $\Pi[T_0(M), T_0(M^+)]$ and

$$(n+p)n \leq m \leq 2n(m+n+p+q). \tag{4.2}$$

For $\Pi[T_0(M), T_0(M^+)] \neq \emptyset$ the operators which are regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ are characterized by the following theorem:

Theorem 4.1. *For $\lambda \in \Pi[T_0(M), T_0(M^+)]$, let r and m be defined by (4.1) and let $\underset{\sim}{\psi}_j$ $(j=1,\dots r)$ and $\underset{\sim}{\phi}_k$ $(k=r+1,\dots,m)$ be arbitrary functions satisfying:*

(i) $\{\underset{\sim}{\psi}_j : j=1,\dots,r\} \subset D(M)$ is linearly independent modulo $D_0(M)$ and $\{\underset{\sim}{\phi}_k : k=r+1,\dots,m\} \subset D(M^+)$ is linearly independent modulo $D_0(M^+)$;

(ii)
$$\begin{aligned} \left[\underset{\sim}{\psi}_j, \underset{\sim}{\phi}_k\right] &= \sum_{i=1}^{m} \left\{[\psi_{ji}, \phi_{ki}]_i(b_i) - [\psi_{ji}, \phi_{ki}]_i(a_i)\right\} + \\ &+ \sum_{i=1}^{n} \left\{[\psi_{ji}, \phi_{ki}]_i(b_i) - [\psi_{ji}, \phi_{ki}]_i(a_i)\right\} + \\ &+ \sum_{i=1}^{p} \left\{[\psi_{ji}, \phi_{ki}]_i(b_i) - [\psi_{ji}, \phi_{ki}]_i(a_i)\right\} + \\ &+ \sum_{i=1}^{q} \left\{[\psi_{ji}, \phi_{ki}]_i(b_i) - [\psi_{ji}, \phi_{ki}]_i(a_i)\right\} \end{aligned}$$

where $\underset{\sim}{\psi}_j = \{\psi_{j1},\dots,\psi_{jm}; \psi_{j1},\dots,\psi_{jn}; \psi_{j1},\dots,\psi_{jp}; \psi_{j1},\dots,\psi_{jq}\}$ and $\underset{\sim}{\phi}_k = \{\phi_{k1},\dots,\phi_{km}; \phi_{k1},\dots,\phi_{kn}; \phi_{k1},\dots,\phi_{kp}; \phi_{k1},\dots,\phi_{kq}\}$, $j=1,\dots,r$;

$k = r+1, \dots, m$. *Then the set,*

$$\begin{aligned}(4.3)\qquad \Big\{u : u \in D(M), [u, \underset{\sim}{\phi_k}] &= \sum_{i=1}^{m} \{[u, \phi_{ki}]_i(b_i) - [u, \phi_{ki}]_i(a_i)\} + \\ &+ \sum_{i=1}^{n} \{[u, \phi_{ki}]_i(b_i) - [u, \phi_{ki}]_i(a_i)\} + \\ &+ \sum_{i=1}^{p} \{[u, \phi_{ki}]_i(b_i) - [u, \phi_{ki}]_i(a_i)\} + \\ &+ \sum_{i=1}^{q} \{[u, \phi_{ki}]_i(b_i) - [u, \phi_{ki}]_i(a_i)\} = 0\Big\}, \\ &\qquad (k = r+1, \dots, m)\end{aligned}$$

is the domain of an operator S *which is regularly solvable with respect to* $T_0(M)$ *and* $T_0(M^+)$ *and*

$$\begin{aligned}(4.4)\qquad \Big\{v : v \in D(M^+), [\underset{\sim}{\psi_j}, v] &= \sum_{i=1}^{m} \left\{[\psi_{ji}, v]_i(b_i) - [\psi_{ji}, v]_i(a_i)\right\} + \\ &+ \sum_{i=1}^{n} \left\{[\psi_{ji}, v]_i(b_i) - [\psi_{ji}, v]_i(a_i)\right\} + \\ &+ \sum_{i=1}^{p} \left\{[\psi_{ji}, v]_i(b_i) - [\psi_{ji}, v]_i(a_i)\right\} + \\ &+ \sum_{i=1}^{q} \left\{[\psi_{ji}, v]_i(b_i) - [\psi_{ji}, v]_i(a_i)\right\} = 0\Big\}, \\ &\qquad (j = 1, \dots, r)\end{aligned}$$

is the domain of S^*, *moreover* $\lambda \in \Delta_4(S)$.

Conversely, if S *is regularly solvable with respect to* $T_0(M)$ *and* $T_0(M^+)$ *and* $\lambda \in \Pi[T_0(M), T_0(M^+)] \cap \Delta_4(S)$, *then with* r *and* m *defined by* (4.1), *there exist functions* $\underset{\sim}{\psi_j}$, $(j = 1, \dots, r)$, $\underset{\sim}{\phi_k}$, $(k = r+1, \dots, m)$ *which satisfy* (*i*) *and* (*ii*) *and are such that* (4.3) *and* (4.4) *are the domains of* S *and* S^*.

S *is self-adjoint if and only if* $M = M^+$, $r = s$ *and* $\underset{\sim}{\phi_k} = \underset{\sim}{\psi_{k-r}}$, $(k = r+1, \dots, m)$; S *is* J*-self-adjoint if and only if* $M = JM^+J$, $r = s$ *and* $\underset{\sim}{\phi_k} = \underset{\sim}{\psi_{k-r}}$ $(k = r+1, \dots, m)$.

P r o o f. The proof is entirely similar to that of Theorem 3.2 and [2, Theorem 3.2] and therefore omitted. □

The regularly solvable operators are determined by boundary conditions imposed at the end-points of various subintervals I_i, J_i, K_i and L_i. The type of these boundary conditions depends on the nature of the problem in the various subintervals. We have the following cases:

Case (i): $I_i = (a_i, b_i)$, $i = 1, \ldots, m$, i.e., the case of two singular end-points of I_i. The boundary conditions in this case on the functions $v \in D(M_i^+)$ and $u \in D(M_i)$ are given by (3.30) and (3.31) respectively which determine the domains of the regularly solvable with respect to $T_0(M_i)$ and $T_0(M_i^+)$ for each i.

Case (ii): $J_i = [a_i, b_i)$, $i = 1, \ldots, n$, i.e., the case of the problem with the left-hand end-point of J_i assumed to be regular and the right-hand end-point is singular. For $\lambda \in \Pi[T_0(M_i), T_0(M_i^+)]$, we define r_i, s_i and m_i as follows:

$$\text{(4.5)} \qquad \begin{cases} r_i := r_i(\lambda) = \operatorname{def}[T_0(M_i) - \lambda I], \\ s_i := s_i(\lambda) = \operatorname{def}[T_0(M_i^+) - \overline{\lambda} I], \ i = 1, \ldots, n, \\ \text{and} \\ m_i := r_i + s_i, \ i = 1, \ldots, n, \end{cases}$$

where,

$$\text{(4.6)} \qquad n \leq m_i \leq 2n, \ i = 1, \ldots, n.$$

Let $\{\psi_{j,i} : j = 1, \ldots, s_i; i = 1, \ldots, n\}$, $\{\phi_{k,i} : k = s_i + 1, \ldots, m_i; i = 1, \ldots, n\}$, be basis for $N[T_0(M_i) - \lambda I]$ and $N[T_0(M_i^+) - \overline{\lambda} I]$ respectively. Thus $\psi_{j,i}, \phi_{k,i} \in L^2_w(J_i)$, $(j = 1, \ldots, s_i;\ k = s_i + 1, \ldots, m_i)$ and

$$M_i[\psi_{j,i}] = \lambda w \psi_{j,i}, \qquad M_i^+[\phi_{k,i}] = \overline{\lambda} w \phi_{k,i}, \qquad i = 1, \ldots, n.$$

We can therefore define the following $x_{j,i}$, $y_{j,i}$, $j = 1, \ldots, m_i$; $i = 1, \ldots, n$,

$$\text{(4.7)} \qquad \begin{cases} x_{j,i} := \psi_{j,i} \quad (j = 1, \ldots, m_i), \\ [T(M_i) - \lambda I] x_{j,i} := \phi_{j,i} \quad (j = s_i + 1, \ldots, m_i), \end{cases}$$

$$\text{(4.8)} \qquad \begin{cases} [T(M_i^+) - \overline{\lambda} I] y_{j,i} := \psi_{j,i} \quad (j = 1, \ldots, s_i), \\ y_{j,i} := \phi_{j,i} \quad (j = s_i + 1, \ldots, m_i), \end{cases}$$

and these functions are arranged to satisfy (3.22) for each i. Let

$$M^i_{s_i \times n} J^{-1}_{n \times n} = -i^n \left(\alpha^i_{jk}\right)_{\substack{r_i+1 \leq j \leq m_i \\ 1 \leq k \leq n}}, \quad N^i_{s_i \times (m_i - n)} = \left(\beta^i_{jk}\right)_{\substack{r_i+1 \leq j \leq m_i \\ n+1 \leq k \leq m_i}}$$

and set

$$g_{j,i} := \sum_{k=n+1}^{n} \overline{\beta^i_{jk}} y_{k,i}, \quad (j = r_i + 1, \ldots, m_i;\ i = 1, \ldots, n).$$

Then $g_{j,i} \in D(M_i^+)$ and, by [17, Theorem 8] we may choose $\phi_{j,i}$ $(j{=}r_i{+}1,\dots,m_i) \in D(M_i^+)$ such that for $k = 1,\dots,n$ and some $c_i \in (a_i, b_i)$,

$$(\phi_{j,i})_+^{[k-1]}(a_i) = \overline{\alpha_{jk}^i}, \quad (\phi_{j,i})_+^{[k-1]}(c_i) = (g_{j,i})_+^{[k-1]}(c_i),$$
$$\phi_{j,i} = g_{j,i} \quad \text{on } [c_i, b_i),\ i = 1,\dots,n.$$

Similarly,

$$K^i_{r_i\times n} J^{-1}_{n\times n} = -i^n \left(\tau^i_{jk}\right)_{\substack{1\le j\le r_i\\ 1\le k\le n}}, \quad L^i_{r_i\times(m_i-n)} = \left(\epsilon^i_{jk}\right)_{\substack{1\le j\le r_i\\ 1\le k\le m_i-n}}$$

$$h_{j,i} := \sum_{k=1}^{m_i-n} \epsilon^i_{jk} x_{k,i}, \quad (j = 1,\dots,r_i;\ i = 1,\dots,n),$$

and $\psi_{j,i}$ $(j = 1,\dots,r_i; i = 1,\dots,n) \in D(M_i)$ such that,

$$(\psi_{j,i})_+^{[k-1]}(a_i) = \tau^i_{jk}, \quad \psi_{j,i}^{[k-1]}(c_i) = h_{j,i}^{[k-1]}(c_i),$$
$$\psi_{j,i} = h_{j,i} \quad \text{on } [c_i, b_i).$$

Then the boundary conditions in this case on the function $u \in D(M_i)$ are,

$$(4.9)\quad B_i(u, J_i) = M^i_{s_i\times n}\begin{bmatrix} u(a_i)\\ \vdots \\ u_i^{[n-1]}(a_i)\end{bmatrix} - N^i_{s_i\times(m_i-n)}\begin{bmatrix} [u, y_{n+1,i}]_i(b_i)\\ \vdots \\ [u, y_{m_i,i}]_i(b_i)\end{bmatrix} = O_{s_i\times 1},$$

and on the function $v \in D(M_i^+)$ are

$$(4.10)\quad B_i^*(v, J_i) = K^i_{r_i\times n}\begin{bmatrix} \overline{v}(a_i)\\ \vdots \\ (\overline{v}_i)_+^{[n-1]}(a_i)\end{bmatrix} - L^i_{r_i\times(m_i-n)}\begin{bmatrix} [x_{1,i}, v]_i(b_i)\\ \vdots \\ [x_{m_i-n,i}, v]_i(b_i)\end{bmatrix} = O_{r_i\times 1},$$

which determine the domains of the operators which are regularly solvable with respect to $T_0(M_i)$ and $T_0(M_i^+)$ for each i, where $M^i_{s_i\times n}$, $N^i_{s_i\times(m_i-n)}$, $K^i_{r_i\times n}$ and $L^i_{r_i\times(m_i-n)}$ are numerical matrices which satisfy the following conditions:

$$(4.11)\quad \begin{cases} \text{Rank}\{K^i_{r_i\times n} \oplus L^i_{r_i\times(m_i-n)}\} = r_i,\\ \text{Rank}\{M^i_{s_i\times n} \oplus N^i_{s_i\times(m_i-n)}\} = s_i,\end{cases}$$

$$\begin{aligned}(4.12)\qquad &\Big\{L^i_{r_i\times(m_i-n)}E^{1,2}_{(m_i-n)\times(m_i-n)}\left(N^i_{s_i\times(m_i-n)}\right)^\top+\\ &+(-i)^nK^i_{r_i\times n}J_{n\times n}\left(M^i_{s_i\times n}\right)^\top\Big\}=0\quad(i=1,\dots,n);\end{aligned}$$

see [2, Theorems 5.1 and 5.2] and [10] for more details.

Case (iii): $K_i=(a_i,b_i]$, $i=1,\dots,p$, it is similar to case (ii) with the right-hand end-point of K_i is assumed to be regular and the left-hand end-point is singular. The boundary conditions in this case on the function $u\in D(M_i)$ are

$$(4.13)\ B_i(u,K_i)=M^i_{s_i\times n}\begin{bmatrix}u(b_i)\\ \vdots\\ u_i^{[n-1]}(b_i)\end{bmatrix}-N^i_{s_i\times(m_i-n)}\begin{bmatrix}[u,x_{n+1,i}]_i(a_i)\\ \vdots\\ [u,x_{m_i,i}]_i(a_i)\end{bmatrix}=O_{s_i\times 1},$$

and on the function $v\in D(M_i^+)$ are

$$(4.14)\ B_i^*(v,K_i)=K^i_{r_i\times n}\begin{bmatrix}\overline{v}(b_i)\\ \vdots\\ (\overline{v}_i)_+^{[n-1]}(b_i)\end{bmatrix}-L^i_{r_i\times(m_i-n)}\begin{bmatrix}[x_{1,i},v]_i(a_i)\\ \vdots\\ [x_{m_i-n,i},v]_i(a_i)\end{bmatrix}=O_{r_i\times 1},$$

which determine the domains of the operators which are regularly solvable with respect to $T_0(M_i)$ and $T_0(M_i^+)$ for each i, where $M^i_{s_i\times n}$, $N^i_{s_i\times(m_i-n)}$, $K^i_{r_i\times n}$ and $L^i_{r_i\times(m_i-n)}$ are numerical matrices which satisfy (4.11) and (4.12) respectively.

Remark. All the boundary conditions in the above cases featuring $L^2_w(I_i)$, $L^2_w(J_i)$ and $L^2_w(K_i)$-solutions of the equations $M_i[u]=\lambda wu$ and $M_i^+[v]=\overline{\lambda}wv$ respectively.

Case (iv): $L_i=[a_i,b_i]$, $i=1,\dots,q$, i.e., the case of two regular end-points. In this case, we put $r_i=s_i=n$ $(i=1,\dots,q)$ in (4.5) then for each i,

$$\operatorname{def}[T_0(M_i)-\lambda I]+\operatorname{def}[T_0(M_i^+)-\overline{\lambda}I]=2n,\ \text{for}\lambda\in\Pi[T_0(M_i);T_0(M_i^+)].$$

By (3.5) and (3.6), if we put,

$$\begin{aligned}\alpha^i_{jk}&=-(\overline{\phi}_{j,i})_+^{[n-k]}(a_i),\ \ \beta^i_{jk}=(\overline{\phi}_{j,i})_+^{[n-k]}(b_i),\\ \tau^i_{jk}&=-(\psi_{j,i})^{[n-k]}(a_i),\ \ \delta^i_{jk}=(\phi_{j,i})^{[n-k]}(b_i),\end{aligned}$$

$(j,k=1,\dots,n;i=1,\dots,q)$. Then the boundary conditions in this case on the function $u\in D(M_i)$ are,

$$(4.15)\qquad B_i(u,L_i)=M^i_{n\times n}\underset{\sim}{u}(a_i)+N^i_{n\times n}\underset{\sim}{u}(b_i)=0,(i=1,\dots,q),$$

where,

$$M^i_{n\times n} = \left((-1)^k \alpha^i_{jk}\right)_{\substack{1\le j\le n\\1\le k\le n}}, \quad N^i_{n\times n} = \left((-1)^k \beta^i_{jk}\right)_{\substack{1\le j\le n\\1\le k\le n}},$$

$\underset{\sim}{u}(\cdot) = \left(u(\cdot),\dots,u^{[n-1]}(\cdot)\right)^\top$ ($\top$ for a transposed matrix), and on the function $v \in D(M_i^+)$ are,

$$(4.16) \qquad B^*_i(v, L_i) = K^i_{n\times n}\underset{\sim}{\overline{v}}(a_i) + L^i_{n\times n}\underset{\sim}{\overline{v}}(b_i) = 0, (i = 1,\dots,q),$$

where

$$K^i_{n\times n} = \left((-1)^{n+1-k} \tau^i_{jk}\right)_{\substack{1\le j\le n\\1\le k\le n}}, \quad L^i_{n\times n} = \left((-1)^{n+1-k} \delta^i_{jk}\right)_{\substack{1\le j\le n\\1\le k\le n}},$$

$\underset{\sim}{\overline{v}}(\cdot) = \left(v(\cdot),\dots,v^{[n-1]}(\cdot)\right)^\top$, and α^i_{jk}, β^i_{jk}, τ^i_{jk}, δ^i_{jk} $(j, k = 1,\dots,n;$ $i = 1,\dots,q)$ are complex numbers satisfying

$$(4.17) \qquad M^i_{n\times n} J_{n\times n} \left(K^i_{n\times n}\right)^\top = N^i_{n\times n} J_{n\times n} \left(L^i_{n\times n}\right)^\top.$$

The above boundary conditions determine the domains of the operators which are regularly solvable with respect to $T_0(M_i)$ and $T_0(M_i^+)$ for each i; see [1, Theorem III.10.6] and [9, Theorem II.2.12] for more details.

Next, the characterization of all operators which are regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ in terms of boundary conditions featuring $L^2_w(a_i, b_i)$-solutions of $M_i[u] = \lambda wu$ and $M_i^+[v] = \overline{\lambda} wv$ for various subintervals I_i, J_i, K_i and L_i is covered by the following theorems.

Theorem 4.2. *Let $\lambda \in \Pi[T_0(M), T_0(M^+)]$ and let r, s and m be as in (4.2). Then the set of all $u \in D(M)$ such that,*

$$(4.18) \qquad \sum_{i=1}^{m} B_i(u, I_i) + \sum_{i=1}^{n} B_i(u, J_i) + \sum_{i=1}^{p} B_i(u, K_i) + \sum_{i=1}^{q} B_i(u, L_i) = 0$$

is the domain of an operator S which is regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$ and $D(S^*)$ is the set of all $v \in D(S^*)$ which are such that,

$$(4.19) \qquad \sum_{i=1}^{m} B^*_i(v, I_i) + \sum_{i=1}^{n} B^*_i(v, J_i) + \sum_{i=1}^{p} B^*_i(v, K_i) + \sum_{i=1}^{q} B^*_i(v, L_i) = 0.$$

In (4.18) and (4.19), $B_i(u, I_i)$ and $B^*_i(v, I_i)$, $i = 1,\dots,m$ are given by (3,30) and (3.31); $B_i(u, J_i)$ and $B^*_i(v, J_i)$, $i = 1,\dots,n$ are given by (4.9) and (4.10);

$B_i(u,K_i)$ and $B_i(v,K_i)$, $i=1,\dots,p$ are given by (4.13) *and* (4.14)*; $B_i(u,L_i)$ and $B_i^*(v,L_i)$, $i=1,\dots,q$ are given by* (4.15) *and* (4.16) *respectively.*

The converse of Theorem 4.2 is

Theorem 4.3. *Let S be regularly solvable with respect to $T_0(M)$ and $T_0(M^+)$, let $\lambda\in\Pi[T_0(M),T_0(M^+)]\cap\Delta_4(S)$ and $D(S)$ is the set of $u\in D(M)$ satisfying* (4.18) *while $D(S^*)$ is the set of $v\in D(M^+)$ satisfying* (4.19).

Remark. Theorems 4.1 and 4.2 follows from the following results for the case of a single interval: [10, Theorems 4.1 and 4.2] for the case when both end-points are singular, [2, Theorems 5.1 and 5.2] for the case of one singular point, [1, Theorem III.10.6] and [9, Theorem 2.2.12] for the regular problem.

5. Discussion. In this section we discuss the possibility of the regularly solvable operators which are not expressible as the direct sums of regularly solvable operators defined in the separate intervals $I_i=(a_i,b_i)$, $i=1,2,3,4$. We will refer to these operators as "new regularly solvable operators". If a_i is a regular end-point and b_i, singular, then by [1, Theorem III.10.13] the sum

$$\operatorname{def}[T_0(M_i)-\lambda I]+\operatorname{def}[T_0(M^+)-\overline{\lambda}I]=n \text{ for } \lambda\in\Pi[T_0(M_i),T_0(M^+)]$$

$(i=1,2,3,4)$ if, and only if, the term in (3.5) at the end-point b_i is zero.

By Lemma 3.1, for $\lambda\in\Pi[T_0(M_i),T_0(M^+)]$, we get in all cases,

$$0\le \operatorname{def}[T_0(M_i)-\lambda I]+\operatorname{def}[T_0(M^+)-\overline{\lambda}I]\le 8n \tag{5.1}$$

while

$$4n\le \operatorname{def}[T_0(M_i)-\lambda I]+\operatorname{def}[T_0(M_i^+)-\overline{\lambda}I]\le 8n, \tag{5.2}$$

when each interval has at least one singular end-point, and

$$\operatorname{def}[T_0(M_i)-\lambda I]+\operatorname{def}(T_0(M_i^+)-\overline{\lambda}I]=8n, \tag{5.3}$$

for the case when all end-pints are regulae. Let,

$$\operatorname{def}[T_0(M)-\lambda I]+\operatorname{def}[T_0(M^+)-\overline{\lambda}I]=d$$

and

$$\operatorname{def}[T_0(M_i)-\lambda I]+\operatorname{def}[T_0(M_i^+)-\overline{\lambda}I]=d_i, (i=1,2,3,4).$$

Then by part (c) of Lemma 2.4, we have that,

$$d = \sum_{i=1}^{4} d_i.$$

We now consider some of the possibilities:

Example 1. $d = 0$. This is the minimal case in (5.1) and can only occur when all eight end-points are singular. In this case $T_0(M)$ is itself regularly solvable and has no proper regularly solvable extensions; see [1, Capter III] and [3].

Example 2. $d = n$ with one of d_1, d_2, d_3 and d_4 is equal to n and all the others are equal to zero. We assume that $d_1 = n$ and $d_2 = d_3 = d_4 = 0$. The other possibilities are entirely similar. In this case we must have seven singular end-points and one regular. There are no new regularly solvable extensions and we have that $S = S_1 \oplus T_0(M_2) \oplus T_0(M_3) \oplus T_0(M_4)$, where S_1 is regularly solvable extension of $T_0(M_1)$, i.e., all regularly solvahle extensions of $T_0(M)$ can be obtained by forming sums of regularly solvable extensions of $T_0(M_i)$, $i = 1, 2, 3, 4$. These are obtained as in the "one interval" case.

Example 3. Six singular end-points and $d = 2n$. We consider two cases:

(i) One interval has two regular end-points, say, I_1, and each one of the others has two singular end-points. Then, $S = S_1 \oplus T_0(M_2) \oplus T_0(M_3) \oplus T_0(M_4)$, where S is regularly solvable extension of $T_0(M_i)$, generates all regularly solvable extensions of $T_0(M)$.

(ii) There are two intervals, say, I_1 and I_2 each one has one regular and one singular end-points and each one of the others has two singular end-paints. In this case $S = S_1 \oplus S_2 \oplus T_0(M_3) \oplus T_0(M_4)$, and $S_1 \oplus S_2$ generates all regularly solvable extensions of $T_0(M)$. The other possibilities in the cases (i) and (ii) are entirely similar.

Example 4. Five singular end-points and $d = 3n$. We consider two cases:

(i) There are two intervals, say, I_1 and I_2, such that I_1 has two regular end-points and I_2 has one regular and one singular end-points and each one of the others has two singular end-points. In this case $d_1 = 2n$ and $d_2 = n$, then $S = S_1 \oplus S_2 \oplus T_0(M_3) \oplus T_0(M_4)$, which is similar to case (ii) in Example 3.

(ii) There are three intervals, say, I_1, I_2 and I_3 each one has one regular and one singular end-points, and the fourth has two singular end-points. In this case, $d_1 = d_2 = d_3$ =n and $d_4 = 0$, and $S = S_1 \oplus S_2 \oplus S_3 \oplus T_0(M_4)$, then $S_1 \oplus S_2 \oplus S_3$ generates all regularly solvable extensions of $T_0(M)$. The other possibilities are entirely similar.

Example 5. Four singular end-points and $d = 4n$. We consider three cases:

(i) There are two intervals, say, I_1 and I_2 each one has two regular end-points, and each one of the others has two singular end-points. In this case $d_1 = d_2 = 2n$ and $d_3 = d_4 = 0$, then $S = S_1 \oplus S_2 \oplus T_0(M_3) \oplus T_0(M_4)$.

(ii) There are two intervals, say, I_1 and I_2 each one has one regular and one singular end-points, and the others I_3 and I_4 has two regular and two singular end-points respectively. In this case $d_1 = d_2 = n$, $d_3 = 2n$ and $d_4 = 0$, then $S = S_1 \oplus S_2 \oplus S_3 \oplus T_0(M_4)$ as the case (ii) in Example 4.

(iii) Each interval has one regular and one singular end-points. In this case $d_i = n$, $i = 1, 2, 3, 4$. Then "mixing' can occur and we get new regularly solvable extensions of $T_0(M)$. For the sake of definiteness assume that the end-points a_1, b_2, a_3 and b_4 are singular end-points and b_1, a_2, b_4 and a_4 are regular end-points. The other possibilities are entirely similar.

For $u \in D(M)$, $\underset{\sim}{\phi_i} \in D(M^+)$ with $\underset{\sim}{\phi_i} = (\phi_{i1}, \phi_{i2}, \phi_{i3}, \phi_{i4})$, condition (4.3) reads,

$$0 = \left[u, \underset{\sim}{\phi_i}\right] = \sum_{j=1}^{4} \left\{ [u, \phi_{ij}]_j(b_j) - [u, \phi_{ij}]_j(a_j),\ i = 1, \ldots, n \right\}. \tag{5.4}$$

Also, for $v \in D(M^+)$, $\underset{\sim}{\psi_k} \in D(M)$ with $\underset{\sim}{\psi_k} = (\psi_{k1}, \psi_{k2}, \psi_{k3}, \psi_{k4})$, condition (5.4) reads,

$$0 = \left[\underset{\sim}{\psi_k}, v\right] = \sum_{j=1}^{4} \left\{ [\psi_{kj}], v_j(b_j) - [\psi_{kj}, v]_j(a_j),\ i = 1, \ldots, n \right\}. \tag{5.5}$$

and condition (ii) in Theorem 4.1 reads,

$$0 = \left[\underset{\sim}{\psi_k}, \underset{\sim}{\phi_i}\right] = \sum_{j=1}^{4} \left\{ [\psi_{kj}], \phi_{ij}{}_j(b_j) - [\psi_{kj}, \phi_{ij}]_j(a_j),\ i = 1, \ldots, n \right\}. \tag{5.6}$$

By [1, Theorem III.10.13], the terms involving the singular end-points a_1 b_2, a_3 and a_4 are zero, such that (5.4), (5.5) and (5.6) reduces to,

$$[u, \phi_{i2}]_2(b_2) - [u, \phi_{i1}]_1(a_1) - [u, \phi_{i3}]_3(a_3) - [u, \phi_{i4}]_4(a_4) = 0$$

$$[\psi_{k2}, v]_2(b_2) - [\psi_{k1}, v]_1(a_1) - [\psi_{k3}, v]_3(a_3) - [\psi_{k4}, v]_4(a_4) = 0$$

and

$$[\psi_{k2}, \phi_{i2}]_2(b_2) - [\psi_{k1}, \phi_{i1}]_1(a_1) - [\psi_{k3}, \phi_{i3}]_3(a_3) - [\psi_{k4}, \phi_{i4}]_4(a_4) = 0,$$

$i, k = 1, \ldots, n$ respectively. Thus the boundary conditions are not separated for the four intervals and hence the regularly solvable operator cannot be expressed as a direct sum of regularly solvable operators defined in the separate intervals I_i, $i = 1, 2, 3, 4$.

We refer to Everitt and Zettl's papers [7] and [8] for more examples and more details.

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