

Provided for non-commercial research and educational use.  
Not for reproduction, distribution or commercial use.

# Serdica

## Mathematical Journal

### Сердика

### Математическо списание

---

The attached copy is furnished for non-commercial research and education use only.  
Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.  
Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on  
Serdica Mathematical Journal  
which is the new series of  
Serdica Bulgaricae Mathematicae Publicationes  
visit the website of the journal <http://www.math.bas.bg/~serdica>  
or contact: Editorial Office  
Serdica Mathematical Journal  
Institute of Mathematics and Informatics  
Bulgarian Academy of Sciences  
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49  
e-mail: [serdica@math.bas.bg](mailto:serdica@math.bas.bg)

## A REMARK ON S.M. BATES' THEOREM

Petr Hájek

*Communicated by S. L. Troyanski*

In his paper [1], Bates investigates the existence of nonlinear, but highly smooth, surjective operators between various classes of Banach spaces. Modifying his basic method, he obtains the following striking results.

**Theorem 1** (Bates). *Every infinite dimensional Banach space  $X$  admits a  $C^1$  Fréchet smooth and Lipschitz map onto any separable Banach space.*

**Theorem 2** (Bates). *If  $X^*$  contains a normalized, weakly null Banach-Saks sequence, then  $X$  admits a  $C^\infty$  Fréchet smooth mapping onto any separable Banach space.*

Recall that a weakly null sequence  $\{x_n\}_{n=1}^\infty$  is called Banach-Saks if for any subsequence  $\{y_n\}_{n=1}^\infty$  of  $\{x_n\}_{n=1}^\infty$  the sequence of arithmetic means  $\frac{1}{n} \left\| \sum_{i=1}^n y_i \right\| \rightarrow 0$  as  $n \rightarrow \infty$ .

In our present note we give a short proof of a variant of Theorem 2. Our version works under somewhat stronger assumptions on  $X$  (or  $X^*$ ) than Bates'. It suffices, e.g. if  $X^*$  contains a normalized sequence  $\{x_n\}_{n=1}^\infty$  with an upper  $q$ -estimate:

$$\left\| \sum_{n=1}^{\infty} \alpha_n x_n \right\| \leq C \left( \sum_{n=1}^{\infty} |\alpha_n|^q \right)^{\frac{1}{q}} \quad \text{for some } C > 0.$$

In particular ([2]), spaces whose dual has nontrivial type (so for example all spaces with nontrivial type) satisfy our assumption. The conclusion, on the other hand, seems to be considerably stronger, as our surjections are homogeneous polynomials of fixed degree (depending only on  $X$ ).

It should be noted that Theorem 2 fails for  $X = c_0$  and related spaces ([3, 4]). For definitions and facts on polynomials we refer to [5].

---

1991 *Mathematics Subject Classification*: 58C25

*Key words*: Separable Banach space, smooth surjection, homogeneous polynomial surjection, noncompact operator

Given  $r \in (0, 1]$  we put  $K_r = \{(x_i) \in B_{\ell_1}, x_i \geq 0, \sum x_i^r \leq 1\}$ . The following is a slight improvement of a well known fact.

**Lemma 3.** *Let  $r \in (0, 1]$ ,  $Y$  be a separable Banach space. Then there exists a linear operator  $T : \ell_1 \rightarrow Y, \|T\| \leq 1$  such that  $\frac{1}{2}B_Y \subset T(K_r)$ .*

**Proof.** Choose a dense sequence  $\{x_i\}_{i=1}^\infty$  in  $B_Y$ . Put  $T(e_i) = x_i$ . Let  $x \in Y, \|x\| = 1$ . Find inductively a sequence  $i_n$  in  $\mathbb{N}$  such that

$$\|Te_{i_1} - x\| < 2^{-\frac{1}{r}}$$

$$\left\| \sum_{n=1}^k 2^{-(n-1)\frac{1}{r}} Te_{i_n} - x \right\| < 2^{-\frac{k}{r}}$$

Put  $y = \sum_{n=1}^\infty 2^{-(n-1)\frac{1}{r}} e_{i_n}$ . Clearly,  $Ty = x$  and  $\frac{1}{2}y \in K_r$ .  $\square$

For the rest of the note, given  $p > 0$ , let us define  $\bar{p} = \min\{k \in \mathbb{N}, k \geq p\}$ .

**Theorem 4.** *Let  $X$  be a separable Banach space. Assume there exists a noncompact operator  $O : X \rightarrow \ell_p, 1 \leq p < \infty$ . Then for every separable Banach space  $Y$  there exists a  $\bar{p}$ -homogeneous polynomial surjection  $P : X \rightarrow Y$ .*

**Proof.** By standard argument, we can assume WLOG that the unit vectors  $e_n$  of  $\ell_p$  lie in  $O(B_X)$ . By convexity,  $\sum_{n=1}^\infty a_n e_n \in O(B_X)$  whenever  $a_n \geq 0, \sum_{n=1}^\infty a_n \leq 1$ . Define a bounded polynomial  $\tilde{P} : \ell_p \rightarrow \ell_1$  as  $\tilde{P}((a_1, a_2, \dots)) = (a_1^{\bar{p}}, a_2^{\bar{p}}, \dots)$ . Clearly,  $K_{\frac{1}{\bar{p}}} \subset \tilde{P}(O(B_X))$ . Put  $P = T \circ \tilde{P} \circ O$ , where  $T$  comes from Lemma 3 applied when  $r = \frac{1}{\bar{p}}$ .  $\square$

## REFERENCES

- [1] S. M. BATES. On smooth nonlinear surjections of Banach spaces. *Israel J. Math.* **100**, (1997), 209-220.
- [2] J. FARMER, W. B. JOHNSON. Polynomial Schur and polynomial Dunford-Pettis properties. *Contemp. Math.* **144**, (1993), 95-105.
- [3] P. HÁJEK. Smooth functions on  $c_0$ . *Israel J. Math.* **104**, (1998), 17-27.
- [4] P. HÁJEK. Smooth functions on  $C(K)$ . *Israel J. Math.* **107**, (1998), 237-252.
- [5] J. MUJICA. Complex analysis in Banach spaces. North Holland Math., Studies **120**, 1986.

*Mathematical Institute  
Czech Academy of Sciences  
Žitná 25, 115 67 Praha 1  
Czech Republic*

*Received June 7, 1999*