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A REMARK ON S.M. BATES' THEOREM

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In his paper [1], Bates investigates the existence of nonlinear, but highly smooth, surjective operators between various classes of Banach spaces. Modifying his basic method, he obtains the following striking results.

Theorem 1 (Bates). *Every infinite dimensional Banach space X admits a C^1 Fréchet smooth and Lipschitz map onto any separable Banach space.*

Theorem 2 (Bates). *If X^* contains a normalized, weakly null Banach-Saks sequence, then X admits a C^∞ Fréchet smooth mapping onto any separable Banach space.*

Recall that a weakly null sequence $\{x_n\}_{n=1}^\infty$ is called Banach-Saks if for any subsequence $\{y_n\}_{n=1}^\infty$ of $\{x_n\}_{n=1}^\infty$ the sequence of arithmetic means $\frac{1}{n}\|\sum_{i=1}^n y_n\| \to 0$ as $n \to \infty$.

In our present note we give a short proof of a variant of Theorem 2. Our version works under somewhat stronger assumptions on X (or X^*) than Bates'. It suffices, e.g. if X^* contains a normalized sequence $\{x_n\}_{n=1}^\infty$ with an upper q-estimate:

$$\|\sum_{n=1}^\infty \alpha_n x_n\| \le C(\sum_{n=1}^\infty |\alpha_n|^q)^{\frac{1}{q}} \quad \text{for some} \quad C > 0.$$

In particular ([2]), spaces whose dual has nontrivial type (so for example all spaces with nontrivial type) satisfy our assumption. The conclusion, on the other hand, seems to be considerably stronger, as our surjections are homogeneous polynomials of fixed degree (depending only on X).

It should be noted that Theorem 2 fails for $X = c_0$ and related spaces ([3, 4]). For definitions and facts on polynomials we refer to [5].

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Given $r \in (0,1]$ we put $K_r = \{(x_i) \in B_{\ell_1}, x_i \geq 0, \sum x_i^r \leq 1\}$. The following is a slight improvement of a well known fact.

Lemma 3. *Let $r \in (0,1]$, Y be a separable Banach space. Then there exists a linear operator $T : \ell_1 \to Y$, $\|T\| \leq 1$ such that $\frac{1}{2}B_Y \subset T(K_r)$.*

Proof. Choose a dense sequence $\{x_i\}_{i=1}^{\infty}$ in B_Y. Put $T(e_i) = x_i$. Let $x \in Y$, $\|x\| = 1$. Find inductively a sequence i_n in $\mathbb{N}$ such that

$$\|Te_{i_1} - x\| < 2^{-\frac{1}{r}}$$

$$\|\sum_{n=1}^{k} 2^{-(n-1)\frac{1}{r}} Te_{i_n} - x\| < 2^{-\frac{k}{r}}$$

Put $y = \sum_{n=1}^{\infty} 2^{-(n-1)\frac{1}{r}} e_{i_n}$. Clearly, $Ty = x$ and $\frac{1}{2}y \in K_r$. $\square$

For the rest of the note, given $p > 0$, let us define $\overline{p} = \min\{k \in \mathbb{N}, k \geq p\}$.

Theorem 4. *Let X be a separable Banach space. Assume there exists a noncompact operator $O : X \to \ell_p$, $1 \leq p < \infty$. Then for every separable Banach space Y there exists a $\overline{p}$-homogeneous polynomial surjection $P : X \to Y$.*

Proof. By standard argument, we can assume WLOG that the unit vectors e_n of ℓ_p lie in $O(B_X)$. By convexity, $\sum_{n=1}^{\infty} a_n e_n \in O(B_X)$ whenever $a_n \geq 0$, $\sum_{n=1}^{\infty} a_n \leq 1$. Define a bounded polynomial $\tilde{P} : \ell_p \to \ell_1$ as $\tilde{P}((a_1, a_2, \ldots)) = (a_1^{\overline{p}}, a_2^{\overline{p}}, \ldots)$. Clearly, $K_{\frac{1}{\overline{p}}} \subset \tilde{P}(O(B_X))$. Put $P = T \circ \tilde{P} \circ O$, where T comes from Lemma 3 applied when $r = \dfrac{1}{\overline{p}}$. $\square$

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