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A DIFFERENTIAL GAME DESCRIBED BY A HYPERBOLIC SYSTEM

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Abstract. An antagonistic differential game of hyperbolic type with a separable linear vector pay-off function is considered. The main result is the description of all ε-Slater saddle points consisting of program strategies, program ε-Slater maximins and minimaxes for each $\varepsilon \in \mathbb{R}^N_>$ for this game. To this purpose, the considered differential game is reduced to find the optimal program strategies of two multicriterial problems of hyperbolic type. The application of approximation enables us to relate these problems to a problem of optimal program control, described by a system of ordinary differential equations, with a scalar pay-off function. It is found that the result of this problem is not changed, if the players use positional or program strategies. For the considered differential game, it is interesting that the ε-Slater saddle points are not equivalent and there exist two ε-Slater saddle points for which the values of all components of the vector pay-off function at one of them are greater than the respective components of the other ε-saddle point.

Introduction. The present paper is a continuation of the results of [11] and [12], related to the antagonistic multicriterial differential games, described by a hyperbolic dynamic system.

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Key words: differential game, ε-Slater saddle point, ε-Slater maximin and minimax, hyperbolic dynamic system, hyperbolic boundary-value problem, approximat model (scheme).

Games with a separable linear vector pay-off function (see [12, Section 2]) are considered here. For such games, in the case when the players are using only program strategies, the structure of the ε-Slater saddle points and the ε-Slater maximins and minimaxes for each $\varepsilon \in \mathbb{R}^N_>$ (i.e. $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N) \mid \forall \varepsilon_i > 0$ for $i = 1, \dots, N$), is found (Section 1). These results enable us to describe the set of all ε-Slater saddle points consisting of program strategies, in the more general case, when the sets of strategies of the players do not consist only of program strategies. The approximation model, which reduces the given differential antagonistic game described by a hyperbolic boundary-value problem to a control problem of more simple type, described by a system of ordinary differential equations, is presented (Section 2). When the conditions of regularity are satisfied [6, p. 132], the maximizing strategy of this control problem is a program strategy.

As an illustration of these results, given in Section 1 and Section 2, a model example of a dynamic system, described by the following hyperbolic boundary-value problem

$$
\begin{array}{lll}
 & \partial^2 y/\partial t^2 = \partial^2 y/\partial x^2 & \text{in } G = (0,1) \times (0,\pi) \\
(0.1) & y(0,x) = (\partial y/\partial t)(0,x) = 0 & \text{for } x \in \Omega = (0,\pi) \\
 & -(\partial y/\partial x)(t,0) = u(t) + v(t), (\partial y/\partial x)(t,\pi) = 0 & \text{for } t \in (0,1),
\end{array}
$$

is considered (Section 3). The following bicriterial differential antagonistic game is introduced:

$$
\langle \Xi, \{\mathcal{U}_0^1, \mathcal{V}_0^1\}, \{\rho_1(h(1)), \rho_2(h(1))\}\rangle. \tag{0.2}
$$

As strategies of (0.2) the players are using the scalar, measurable functions $u(t)$ and $v(t)$, subject to the conditions $|u(t)| \leqq 1$, $|v(t)| \leqq 1$, $\forall t \in [0,1]$, i.e. the sets of program strategies are

$$
\begin{aligned}
\mathcal{U} = \mathcal{U}_0^1 = \{U \div u(.) \mid |u(t)| \leqq 1, \forall t \in [0,1]\}, \\
\mathcal{V} = \mathcal{V}_0^1 = \{V \div v(.) \mid |v(t)| \leqq 1, \forall t \in [0,1]\}.
\end{aligned}
$$

The controllable system Ξ of (0.2) is described by (0.1) and the vector pay-off function has two components

$$
\rho(h(T)) = (\rho_1(h(T)), \rho_2(h(T))) = \left(\int_0^\pi y'(1,x)dx, -\int_0^{\pi/2} y(1,x)dx \right),
$$

i.e.

$$\rho_1(h(T)) = \int_0^\pi y'(1,x)dx, \;\; \rho_2(h(T)) = -\int_0^{\pi/2} y(1,x)dx,$$

where $y'(t,x) = (\partial y/\partial t)(t,x)$ and $h(t) = (y(t,.), y'(t,.)) \in \mathcal{H} = L_2(0,\pi) \times (H_2^1(0,\pi))^*$, $\forall t \in [t_0,T]$, see [11, Theorem 1]. We should note that these components are linear and strongly continuous functionals in $\mathcal{H}$, [12]. Without any restrictions, suppose that the first player choosing the strategy $U \in \mathcal{U}$ strives to smaller possible values of all criteria $\rho_i(h(T))$, $i = 1,2$; the second player using a strategy $V \in \mathcal{V}$, strives to their maximization. Each player chooses a strategy of his own which is independent of the other player's strategy.

We shall look for all ε-Slater saddle points consisting of program strategies in game (0.2) for $\forall\varepsilon \in \mathbb{R}^2_>$. Hence, from the considerations of Section 1 and Section 2, we can suppose that the players are using only program strategies of $\mathcal{U}_0^1$ and $\mathcal{V}_0^1$.

For this specific antagonistic differential game (0.2), the sets of all ε-Slater saddle points, ε-Slater maximins and minimaxes for $\forall\varepsilon \in \mathbb{R}^2_>$, are found in Section 3. Also we shall describe the set $\rho(h(1;0,0,0,U^*,V^*))$, i.e. the set of the values of the vector functional $\rho(h(.))$ for $T=1$, $t=0$ and initial conditions $(0,0)$ of problem (0.1) for all ε-Slater saddle points (U^*,V^*) of (0.2) $\forall\varepsilon \in \mathbb{R}^2_>$ and the set of all ε-Slater maximins and minimaxes of (0.2). All these sets are subsets of $\rho(D(T;p_0))$, where $D(T;p_0) = \{h(1;0,0,0,U,V) | U \in \mathcal{U}_0^1, V \in \mathcal{V}_0^1\}$, see [11], is the domain of attainment of the controllable system Ξ of (0.2) from the initial position $p_0 = (0,0,0)$ and $T=1$, and this set $\rho(D(T;p_0))$ also will be constructed in Section 3.

Game (0.2) is interesting with these constructed sets, since the ε-Slater saddle points of (0.2) are not equivalent [11] and there exist two ε-Slater saddle points for which the values of all components of the vector pay-off function at one of them are greater than the respective components of the other ε-saddle point.

1. ε-Slater saddle points, maximins and minimaxes for games with a separable linear pay-off function. The following more general multi-criterial antagonistic differential game with a vector pay-off function is considered:

$$(1.1) \qquad \langle \Xi, \{\mathcal{U},\mathcal{V}\}, \{\rho_i(h(T))\}_{i\in\mathbb{N}} \rangle,$$

where $\mathbb{N} = \{1, \ldots, N\}, N \geqq 1$ is the number of criteria and Ξ is described by the following boundary-value problem of hyperbolic type:

$$\begin{array}{ll} \partial^2 y/\partial t^2 = Ay + b_1 u_1 + c_1 v_1 + f_1 & \text{in } G = (t_0, T) \times \Omega, \\ y_{|t=t_0} = y_0, \quad \partial y/\partial t_{|t=t_0} = y_1 & \text{in } \Omega \\ \sigma_1 \partial y/\partial \nu_A + \sigma_2 y = b_2 u_2 + c_2 v_2 + f_2 & \text{in } \Sigma = (t_0, T) \times \Gamma, \end{array} \tag{1.2}$$

where the numbers $\sigma_i \in \{0, 1\}$, $i = 1, 2$, $\sigma_1 + \sigma_2 \geqq 1$.
All parameters in (1.2) satisfy the corresponding conditions from [11, 12]. All notations and concepts of [11, 12] will be used too. It is supposed that each component of the vector pay-off function $\rho(h(T)) = (\rho_1(h(T)), \ldots, \rho_N(h(T)))$ is a linear strongly continuous (s.-continuous) functional in $\mathcal{H}$, where $\mathcal{H} = L_2(\Omega) \times (H_2^1(\Omega))^*$ for $\sigma_1 = 1$ and $\mathcal{H} = H_2^{-1}(\Omega) \times (H_{2,0}^2(\Omega))^*$, (where $H_{2,0}^2(\Omega) = H_2^2(\Omega) \times H_0^1(\Omega)$) for $\sigma_1 = 0$ and $h(t) = (y(t, .), y'(t, .)) \in \mathcal{H}$, $\forall t \in [t_0, T]$, [11, Theorem 1]. The aim of the first player is to minimize all the components of $\rho(h(T))$ choosing a strategy $U \in \mathcal{U}$. The aim of the second player is the opposite – to maximize these components by means of the strategy $V \in \mathcal{V}$. The strategies of the players are defined by the functions u_i and v_i, $i = 1, 2$, (see [11, 12]). It is supposed in addition, that these functions satisfy the conditions: $u_i = u_i(t) \in P_i(t) \subset \mathbb{R}^{r_i}$, $v_i = v_i(t) \in Q_i(t) \subset \mathbb{R}^{m_i}$, $i = 1, 2$, where the sets $P_i(t)$ and $Q_i(t)$, $i = 1, 2$ are uniformly bounded with respect to t, $\forall t \in [t_0, T]$. Let us remind that the solution of the boundary-value problem of the type (1.2) is considered as in [11, Lemma 1]. The following two multicriterial dynamic problems will correspond to the controllable system Ξ of (1.1) [12, Section 2]:

$$\langle \Xi^{(1)}, \mathcal{U}, \{-\rho_i(h^{(1)}(T))\}_{i \in \mathbb{N}} \rangle \tag{1.3}$$

and

$$\langle \Xi^{(2)}, \mathcal{V}, \{\rho_i(h^{(2)}(T))\}_{i \in \mathbb{N}} \rangle, \quad h^{(j)} = (y^{(j)}, \partial y^{(j)}/\partial t), \ j = 1, 2, \tag{1.4}$$

described by the corresponding boundary-value problems

$$\left| \begin{array}{ll} \partial^2 y^{(1)}/\partial t^2 = Ay^{(1)} + b_1 u_1 + 0,5 f_1 & \text{in } G = (t_0, T) \times \Omega, \\ y^{(1)}_{|t=t_0} = 0,5 y_0, \ \partial y^{(1)}/\partial t_{|t=t_0} = 0,5 y_1 & \text{in } \Omega \\ \sigma_1 \partial y^{(1)}/\partial \nu_A + \sigma_2 y^{(1)} = b_2 u_2 + 0,5 f_2 & \text{in } \Sigma = (t_0, T) \times \Gamma \end{array} \right. \tag{1.5}$$

and

$$
(1.6) \qquad \left|\begin{array}{ll}
\partial^2 y^{(2)}/\partial t^2 = Ay^{(2)} + c_1 v_1 + 0,5 f_1 & \text{in } G = (t_0, T) \times \Omega, \\
y^{(2)}_{|t=t_0} = 0,5 y_0,\ \partial y^{(2)}/\partial t_{|t=t_0} = 0,5 y_1 & \text{in } \Omega \\
\sigma_1 \partial y^{(2)}/\partial \nu_A + \sigma_2 y^{(2)} = c_2 v_2 + 0,5 f_2 & \text{in } \ \Sigma = (t_0, T) \times \Gamma.
\end{array}\right.
$$

It is supposed that the sets $\mathcal{U}$ and $\mathcal{V}$ consist only of program strategies. The following assertion is valid:

Lemma 1.1 [12, Lemma 2.2]**.** *Let the fixed vector $\varepsilon \in \mathbb{R}^N_{\geq}$ be given and let $U^\varepsilon \in \mathcal{U}$ and $V^\varepsilon \in \mathcal{V}$ be program strategies. Then the situation $(U^\varepsilon, V^\varepsilon)$ is an ε-Slater saddle point of* (1.1) *if and only if U^ε and V^ε are ε-Slater maximal strategies of problems* (1.3) *and* (1.4)*, respectively.*

Further we shall apply Lemma 1.1 for game (0.2).

Let the vector $\varepsilon \in \mathbb{R}^N_{>}$ be fixed and let $\mathcal{U}^\varepsilon$ and $\mathcal{V}^\varepsilon$ be the sets of program ε-Slater maximal strategies U^ε and V^ε of multicriterial problems (1.3) and (1.4), respectively. We denote

$$
\begin{aligned}
Y^{(1)} &= \bigcup_{U^\varepsilon \in \mathcal{U}^\varepsilon} \rho(h^{(1)}(T; t_0, 0,5y_0, 0,5y_1, U^\varepsilon)), \\
Y^{(2)} &= \bigcup_{V^\varepsilon \in \mathcal{V}^\varepsilon} \rho(h^{(2)}(T; t_0, 0,5y_0, 0,5y_1, V^\varepsilon)),
\end{aligned}
$$

where $h^{(j)}(.) = (y^{(j)}(.), \partial y^{(j)}(.)/\partial t)$, $j = 1, 2$, and $y^{(1)}(.; t_0, 0,5y_0, 0,5y_1, U^\varepsilon)$ and $y^{(2)}(.; t_0, 0,5y_0, 0,5y_1, V^\varepsilon)$ are the solutions of boundary-value problems (1.5) and (1.6) for $U^\varepsilon \div u^\varepsilon(t)$ and $V^\varepsilon \div v^\varepsilon(t), t_0 \leqq t \leqq T$, respectively. Let $Y^{(1)} + Y^{(2)} = \{a + b \mid a \in Y^{(1)}, b \in Y^{(2)}\}$. The following assertion is valid:

Lemma 1.2. *Let the vector $\varepsilon \in \mathbb{R}^N_{>}$ be fixed. Then*

1. *The set of the program ε-Slater minimaxes for game* (1.1) *coincides with the set of the ε-Slater minimal points of the set $Y^{(1)} + Y^{(2)}$.*

2. *The set of the program ε-Slater maximins for game* (1.1) *coincides with the set of the ε-Slater maximal points of the set $Y^{(1)} + Y^{(2)}$.*

Lemma 1.2 is proved by analogy to [9, p. 82-85, Assertion 8.2] or [10, Theorem 2], taking into account that the sets $Y^{(1)}$ and $Y^{(2)}$ are bounded [11, Theorem 1].

In connection with problem (1.4), the following auxiliary problem of optimal control is considered:

$$\langle \Xi^{(2)}, \mathcal{V}, \rho_\beta(h(T))\rangle, \tag{1.7}$$

where the controllable system $\Xi^{(2)}$ is described by boundary-value problem (1.6), where for simplification of the symbols, the upper index (2) at $y^{(2)}$ and $h^{(2)}$ will be omitted, $\mathcal{V}$ is the set of the strategies V of the second player in game (1.1), the functional $\rho_\beta(h(T)) = \sum\limits_{i\in\mathbb{N}} \beta_i \rho_i(h(T))$ is linear and s.-continuous in $\mathcal{H}$ and the vector $\beta = (\beta_1, \ldots, \beta_N) \in \overline{M} = \left\{\beta \in \mathbb{R}^N_{\geq} \mid \sum\limits_{i\in\mathbb{N}} \beta_i = 1\right\}$.

We should note that for multicriterial problem (1.3), the corresponding auxiliary problem of optimal control is of the form

$$\langle \Xi^{(1)}, \mathcal{U}, -\rho_\beta(h(T))\rangle. \tag{1.8}$$

Let us remind that if $\rho = (\rho_1, \ldots, \rho_N)$ is a vector functional and V is a positional (in the general case) strategy, then $\underset{\delta\to 0}{\underline{\mathrm{LIM}}}\, \rho(h_\Delta[T; t_0, y_0, y_1, V]) =$

$$\left(\lim_{\delta\to 0} \inf_{\substack{h_\Delta[.],\\ \delta(\Delta)\leqq\delta}} \rho_1(h_\Delta[T; t_0, y_0, y_1, V]), \ldots, \lim_{\delta\to 0} \inf_{\substack{h_\Delta[.],\\ \delta(\Delta)\leqq\delta}} \rho_N(h_\Delta[T; t_0, y_0, y_1, V])\right),$$

where lim inf is taken over all the step motions $h_\Delta[.] = h_\Delta[T; t_0, y_0, y_1, V]$, which is caused by the strategy V, the arbitrary partition $\Delta \in \mathbf{\Delta}$ of the interval $[t_0, T]$ with $\delta(\Delta) \leqq \delta$ and the initial position $p_0 = \{t_0, y_0, y_1\}$, see [11].

($\underset{\delta\to 0}{\overline{\mathrm{LIM}}}\, \rho(h_\Delta[T; t_0, y_0, y_1, V])$ is defined by analogy, [11, 12]).

Definition 1.1. *Let $\gamma > 0$ be a fixed number. The strategy $V^\gamma \in \mathcal{V}$ is called a γ-maximal strategy of problem of optimal control* (1.7), *if there exists a constant $\delta(\gamma) > 0$ such that for $\forall V \in \mathcal{V}$ and for $\forall h_\Delta[.] = h_\Delta[.; t_0, 0,5y_0, 0,5y_1, V]$ with $\delta(\Delta) \leqq \delta(\gamma)$, the following inequality is valid:*

$$\rho_\beta(h_\Delta[T]) - \gamma \leqq \lim_{\delta\to 0} \inf_{\substack{h_\Delta[.],\\ \delta(\Delta)\leqq\delta}} \rho_\beta(h_\Delta[T; t_0, 0,5y_0, 0,5y_1, V^\gamma])$$

$$= \underset{\delta\to 0}{\underline{\mathrm{LIM}}}\, \rho_\beta(h_\Delta[T; t_0, 0,5y_0, 0,5y_1, V^\gamma]).$$

The number $c^\gamma = \underset{\delta\to 0}{\underline{\mathrm{LIM}}}\, \rho_\beta(h_\Delta[T; t_0, 0,5y_0, 0,5y_1, V^\gamma])$ *is called* γ*-maximum of the functional* $\rho_\beta(h(T))$*.*

Let us point out that in Definition 1.1, the strategy $V^\gamma \in \mathcal{V}$ is positional (in the general case) and ρ_β is a scalar functional.

Hence we should note that the concept for γ-maximal strategy of problem (1.7) given in Definition 1.1 follows from the definition for γ-Slater maximal strategy of problem (1.4) for the scalar functionals ($N = 1$), i.e. for $\mathbb{N} = \{1\}$, see [12, Definition 2.1].

The connection between the solutions of problems (1.4) and (1.7) is given by the following assertions:

Lemma 1.3. *Let* $\varepsilon \in \mathbb{R}^N_>$ *be a fixed vector and let* $V^\varepsilon \in \mathcal{V}$ *be an* ε*-Slater maximal strategy of problem* (1.4)*,* [12, Definition 2.1]*. Then there exists a vector* $\beta \in \overline{M}$ *such that* V^ε *is a* γ*-maximal strategy of problem* (1.7)*, where* $\gamma \geqq \sum\limits_{i\in\mathbb{N}} \beta_i\varepsilon_i$*.*

Corollary 1.3. *Let one and the same strategy* $V^* \in \mathcal{V}$ *be an* ε*-Slater maximal strategy of problem* (1.4) $\forall\varepsilon \in \mathbb{R}^N_>$*. Then there exists a vector* $\beta \in \overline{M}$ *such that* V^* *is a* γ*-maximal strategy of problem* (1.7) *for each number* $\gamma > 0$*.*

We have to point out that similar assertions for the parabolic case are obtained by Matveev [8]. For the proof of Lemma 1.3, at first we give the following definition:

Definition 1.2. *The set* $Y \subseteq \mathbb{R}^m$ *is called effective convex, if the set* $Y_* = Y - \mathbb{R}^m_{\geqq} = \{y_* \in \mathbb{R}^m \mid y_* \leqq y \text{ for some } y \in Y\}$ *is convex,* [4, p. 104].

Now, the following set is considered:

$$D_{\delta(\varepsilon)}(T; t_0, 0,5y_0, 0,5y_1) \overset{\text{def}}{=\!=} \{h_\Delta[T; t_0, 0,5y_0, 0,5y_1, V] \mid V \in \mathcal{V}, \delta(\Delta) \leqq \delta(\varepsilon)\}.$$

From the definitions of program and positional strategies and step motions, see [11, p. 27], it follows that

$$D_{\delta(\varepsilon)}(T; t_0, 0,5y_0, 0,5y_1) = D(T; t_0, 0,5y_0, 0,5y_1)$$

$$= \{h(T) = h(T; t_0, 0,5y_0, 0,5y_1, v(.)) \mid v(.) \in Q(t_0, T]\}.$$

Since the set $Q(t_0,T]$ is convex and $\rho_i(.), \forall i \in \mathbb{N}$ are linear and s.-continuous functionals, then

$$\rho(D_{\delta(\varepsilon)}(T;t_0,0,5y_0,0,5y_1)) - \varepsilon = \{\rho(h) - \varepsilon \mid h \in D_{\delta(\varepsilon)}(T;t_0,0,5y_0,0,5y_1)\}$$

is an effective convex set in $\mathbb{R}^N$, (Definition 1.2) and

$$\underset{\delta\to 0}{\underline{\mathrm{LIM}}}\, \rho(h_\Delta[T;t_0,0,5y_0,0,5y_1,V^\varepsilon])$$

is an weak effective Slater maximal vector [4, p. 33] for this set. According to [4, p. 104, Theorem 1 (U)], there exists a vector $\beta \in \overline{M}$ such that

$$\begin{aligned} &\sum_{i\in\mathbb{N}} [\beta_i \rho_i(h_\Delta[T]) - \beta_i\varepsilon_i] \\ &\leqq \sum_{i\in\mathbb{N}} \beta \lim_{\delta\to 0} \inf_{\substack{h_\Delta[.],\\ \delta(\Delta)\leqq\delta}} \rho_i(h_\Delta[T;t_0,0,5y_0,0,5y_1,V^\varepsilon]) \\ &\leqq \lim_{\delta\to 0} \inf_{\substack{h_\Delta[.],\\ \delta(\Delta)\leqq\delta}} \rho_\beta(h_\Delta[T;t_0,0,5y_0,0,5y_1,V^\varepsilon]), \end{aligned} \tag{1.9}$$

$\forall V \in \mathcal{V}$ and $h_\Delta[.] \in h_\Delta[.;t_0,0,5y_0,0,5y_1,V]$ with $\delta(\Delta) \leqq \delta(\varepsilon)$.

If $\gamma \geqq \gamma_1 = \sum_{i\in\mathbb{N}} \beta_i\varepsilon_i > 0$, from (1.9),

$$\rho_\beta(h_\Delta[T]) - \gamma \leqq \lim_{\delta\to 0} \inf_{\substack{h_\Delta[.],\\ \delta(\Delta)\leqq\delta}} \rho_\beta(h_\Delta[T;t_0,0,5y_0,0,5y_1,V^\varepsilon])$$

and from Definition 1.1 it follows that V^ε is a γ-maximal strategy of problem (1.7), hence Lemma 1.3 is obtained.

For the proof of Corollary 1.3, we take into account that for $\forall \varepsilon_0 \in \mathbb{R}^N_>$, the set $\bigcup_{0_N<\varepsilon<\varepsilon_0} \{\rho(D_{\delta(\varepsilon)}(T;t_0,0,5y_0,0,5y_1)) - \varepsilon\}$ is effective convex and the vector $\underset{\delta\to 0}{\underline{\mathrm{LIM}}}\, \rho(h_\Delta[T;t_0,0,5y_0,0,5y_1,V^*])$ is weak effective Slater maximal for this set.

Let us point out that Lemma 1.3 and Corollary 1.3 are valid in the case, when the set $\mathcal{V}$ does not consist only of program strategies.

Remark. Let the sets $\mathcal{U}$ and $\mathcal{V}$ of game (1.1) do not consist only of program strategies. Then we can describe the set of all ε-Slater saddle points consisting of program strategies of game (1.1) for $\forall \varepsilon \in \mathbb{R}^N_>$, by means of Lemma 1.1, Lemma 1.3, Corollary 1.3 and [11, Lemma 4 and Corollary 4], since these assertions are valid in this more general case.

2. An approximation model. In this section we consider a method of constructing of ε-Slater saddle point of linear problem of type (1.1) with a scalar pay-off function

$$\langle \Xi, \{\mathcal{U}, \mathcal{V}\}, \{\rho_1(h(T))\}\rangle, \tag{2.1}$$

which is based on finite dimensional approximation models. Approximation schemes are applied to dynamical systems described by parabolic or hyperbolic boundary-value problems with a scalar pay-off function in [3, 8, 11]. The approximation of game (2.1) by using [3] will be considered in this section. The obtained results are analogous to [1] and [8].

At first, the following sequence of antagonistic differential games with a scalar pay-off function is introduced:

$$\Gamma_k = \langle \Xi_k, \{\mathcal{U}^k, \mathcal{V}^k\}, \rho^k(h^k[T])\rangle, \quad k = 1, 2, \dots \, . \tag{2.2}$$

In Γ_k, the controllable system Ξ_k is described by the following system of ordinary differential equations:

$$\begin{gathered} \frac{d^2 y_j^k}{dt^2} + \lambda_j y_j^k = \langle b_1 u_1 + c_1 v_1 + f_1, \omega_j\rangle_{L_2(\Omega)} + \langle b_2 u_2 + c_2 v_2 + f_2, F(\omega_j)\rangle_\Gamma, \\ y_j^k(t_0) = \langle y_0, \omega_j\rangle, \quad \frac{dy_j^k}{dt}(t_0) = \langle y_1, \omega_j\rangle, \ j = 1, \dots, k, \end{gathered} \tag{2.3}$$

where $F(\varphi) \overset{\text{def}}{=\!=} \begin{cases} -\partial\varphi/\partial\nu_A & \text{for} \quad \sigma_1 = 0 \\ \varphi & \text{for} \quad \sigma_1 = 1 \end{cases}$ and $\{\lambda_j, \omega_j\}$, $j = 1, 2, \dots$ is the solution of the spectral problem $A\omega_j = -\lambda\omega_j$ in Ω, $\sigma_1 \partial\omega_j/\partial\nu_A + \sigma_2\omega_j = 0$ in Γ. Obviously that system (2.3) can be presented as a linear system of ordinary differential equations in relation to $h_j^k = (y_j^k, dy_j^k/dt)$, $j = 1, \dots, k$, $\forall k = 1, 2, \dots$. Furthermore for $\forall k = 1, 2, \dots$, the position of the controllable system Ξ_k in the moment $t \in [t_0, T]$ is described by the phase vector $h^k(t) = (h_1^k(t), \dots, h_k^k(t))$ of $\mathbb{R}^{2k}$. A positional strategy of the first player $U^k = U^k(t_1, t_2, h^k(t_1))$ is a mapping, for which to every ordered triplet $(t_1, t_2, h^k(t_1)) \in [t_0, T) \times (t_1, T] \times \mathbb{R}^{2k}$ there corresponds a unique, measurable function $u \in P(t_1, t_2]$. The set of these strategies of the game Γ_k is denoted by $\mathcal{U}^k$. The positional strategies of the game Γ_k of the second player are defined by analogy and the set of these strategies is denoted by $\mathcal{V}^k$.

Following [3], we define the operators $S^k : H \to \mathbb{R}^k$ and $A^k : \mathbb{R}^k \to H$ such that $S^k y = (\langle y, \omega_1 \rangle, \ldots, \langle y, \omega_k \rangle)$, $k = 1, 2, \ldots$ and $A^k y^k = \sum_{j=1}^{k} y_j^k \omega_j$, where $y^k \stackrel{\text{def}}{=} (y_1^k(t), \ldots, y_k^k(t))$, $k = 1, 2, \ldots$, $H = L_2(\Omega)$ for $\sigma_1 = 1$ and $H = H_2^{-1}(\Omega)$ for $\sigma_1 = 0$. The result of game (2.2) for fixed number $k = 1, 2, \ldots$ is evaluated by the functional $\rho^k(.) : \mathbb{R}^{2k} \to \mathbb{R}$, which is defined such that $\rho^k(h^k[T]) = \rho_1(A^k y^k[T], \frac{d}{dt} A^k y^k[T]) = \rho_1(A^k h^k[T])$, where $\rho_1(.)$ is the functional of (2.1).

For each $k = 1, 2, \ldots$, $\rho^k(.)$ is a continuous functional in $\mathbb{R}^{2k}$, since the operator $(A^k(.), \frac{d}{dt} A^k(.)) : \mathbb{R}^{2k} \to \mathcal{H}$ and the functional $\rho_1(.) : \mathcal{H} \to \mathbb{R}$ are continuous. Then for every fixed natural number k, there exists a saddle point $(U^{0k}, V^{0k}) \in \mathcal{U}^k \times \mathcal{V}^k$ with a value c_0^k in game (2.2), where U^{0k} and V^{0k} are positional strategies, see for example [5, p. 76-79].

Now the results of [3] will be used, since the parameters of game (2.1) satisfy the corresponding conditions of [3]. In [11, p. 32, proof of Theorem 2] was proved that all conditions of [3, Theorem 2.1] are satisfied, so that Theorem 2.1 of [3] can be applied with respect to the set $M = M_1(c_0) = \{h \in \mathcal{H} | \rho_1(h) \leqq c_0\}$, where c_0 is the value of game (2.1), see [11, proof of Theorem 2]. Moreover, the evasion problem from the set $M_1(c_0)$ can be solved by a strategy from the type $V(t_1, t_2, h) = V_k^*(t_1, t_2, A_k^*(t_1) h) = V_k^*(t_1, t_2, (S^k y, S^k y'))$, see [3, p. 1014], where V_k^* is the corresponding "extremal" strategy, connected with system (2.3), $S^k = S^k(t_1)$ and the defined in [3] operator A_k^* is analogous to the defined above operator S^k, (for simplification of the symbols, further we denote S^k instead of $S^k(t_1)$). Analogous assertion is true for the ε-approach problem with $M_1(c_0)$, which is solved by the corresponding extremal strategy $U_k^*(.)$. Thus, using the results of [3] and the fact that $\|\rho^k(h^k[T]) - \rho_1(h[T])\| \to 0$ for $k \to \infty$ since the functional $\rho_1(.)$ is strong continuous in $\mathcal{H}$, we obtain the following assertion (analogous assertions with respect to parabolic dynamic system are proved in [1] and [8]):

Lemma 2.1. *Let c_0^k and $(U^{0k}, V^{0k}) \in \mathcal{U}^k \times \mathcal{V}^k$ be the value and the saddle point of game* (2.2) *respectively $\forall k = 1, 2, \ldots$ and let c_0 be the value of game* (2.1). *Then*

1. *For each number $\gamma > 0$, $\exists k_0 : \forall k \geqq k_0$ (k and k_0 are natural numbers), there exists a constant $\delta(k, \gamma) > 0$ such that $\forall h_\Delta[.] = h_\Delta[.; p_0, U^*, V^*]$ with $\delta(\Delta) \leqq \delta(k, \gamma)$, the following inequality is valid: $|c_0 - \rho_1(h_\Delta[T])| \leqq \gamma$, where the*

strategies $U^* \in \mathcal{U}$ *and* $V^* \in \mathcal{V}$ *in game* (2.1) *are defined as follows:* $U^*(t_1, t_2, h) = U^{\text{ok}}(t_1, t_2, (S^k y, S^k y'))$ *and* $V^*(t_1, t_2, h) = V^{\text{ok}}(t_1, t_2, (S^k y, S^k y'))$,

2. *Moreover* $c_0^k \to c_0$ *for* $k \to \infty$.

We shall apply this assertion for the sequence of problems

$$\Gamma_k^{(2)} = \langle \Xi_k^{(2)}, \mathcal{V}^k, \rho^k(h^k[T]) \rangle, \quad k = 1, 2, \ldots . \tag{2.4}$$

The sequence of problems (2.4), related to problem (1.7) is constructed by analogy to the sequence of games (2.2), related to game (2.1). The difference is that in (2.4) there is only one (maximizing) player. From Lemma 2.1 and [11, Theorem 2] there exist a maximal strategy $V^{\text{ok}} \in \mathcal{V}^k$ maximizing $\rho^k(h^k[T])$ and numbers $c^{(2)}$ and $c_k^{(2)}$, which are the maximal values of $\rho_\beta(h[T])$ and $\rho^k(h^k[T])$ respectively for problems (1.7) and $\Gamma_k^{(2)}$ of (2.4), $\forall k = 1, 2, \ldots$. Thus from Lemma 2.1, the following assertion is obtained:

Corollary 2.1. *Let* $c_k^{(2)}$ *be the maximal value of* $\rho^k(h^k[T]), V^{\text{ok}} \in \mathcal{V}^k$ *– the corresponding maximizing strategy of problem* (2.4) *and let* $c^{(2)}$ *be the maximum of* $\rho_\beta(h[T])$ *in (1.7). Then*

1. *For each number* $\gamma > 0$, $\exists k_0 : \forall k \geqq k_0$ *(*k *and* k_0 *are natural numbers), there exists a constant number* $\delta(k, \gamma) > 0$ *such that*

$$\forall h_\Delta[.] = h_\Delta[.; p_0, V^*] \quad \textit{with} \quad \delta(\Delta) \leqq \delta(k, \gamma),$$

the following inequality is valid: $|c^{(2)} - \rho_\beta(h_\Delta[T])| \leqq \gamma$, *where the strategy* $V^* \in \mathcal{V}$ *in problem* (1.7) *is defined as follows:* $V^*(t_1, t_2, h) = V^{\text{ok}}(t_1, t_2, (S^k y, S^k y'))$,

2. *Moreover* $c_k^{(2)} \to c^{(2)}$ *for* $k \to \infty$.

Thus, the solution of problem (1.7) in the space $\mathcal{H}$ is approximated (with arbitrary level of exactness, given in advance) by the sequence of problems (2.4) in $\mathbb{R}^{2k}, k \geqq 1$, the solution of which is “less complicated” than the solution of (1.7).

The game Γ_k of (2.2) is considered for some fixed natural number k. The sufficient conditions, for which the usage of program strategies leads to the same result as the usage of positional strategies, are obtained in [6, p. 129-133]. In particular, if the controllable system is described by a linear (with respect to the phase variable and the control functions) system of ordinary differential equations with convex set of values of the control and linear scalar pay-off function $\rho^k(h^k[T])$, then these sufficient conditions are satisfied. Indeed, let the function

$\tilde{\rho}(l) \stackrel{\text{def}}{=} \sup\{h'l - \rho^k(h) \mid h \in \mathbb{R}^{2k}\}$ and $L = \operatorname{dom}\tilde{\rho}(.) = \{l \in \mathbb{R}^{2k} \mid \tilde{\rho}(l) < \infty\}$, see [6, p. 130, (5.8)]. Since the functional $\rho^k(.)$ is linear and continuous, the set L consists only of one point $l_0 \in \mathbb{R}^{2k}$, moreover $\tilde{\rho}(l_0) = 0$, i.e. $L = \{l_0\}$ and hence the requirements of [6, p. 132, Corollary] are satisfied.

Now the problem of optimal control $\Gamma_k^{(2)}$ from (2.4) is considered. Then the maximum of the linear and s.-continuous functional $\rho^k(h^k[T])$ on the set of positional strategies coincides with the maximum of the same functional on the set of program strategies. Therefore, further we shall look for the control $V^{\text{ok}} \in \mathcal{V}^k$, which is maximizing the functional $\rho^k(h^k[T])$ in the set of program strategies. This approach to a large extent will make considerably easier the finding of V^{ok} and c_0^k.

From Corollary 2.1, the solution of problem (2.4) approximates (with arbitrary given level of exactness) the auxiliary problem of optimal control (1.7). It means that $\forall\gamma > 0$, using the above described approximation scheme, there can be constructed a γ-maximal solution of problem (1.7), for which this solution is attained for program strategies.

3. Solution of game (0.2). Let us return to the considerations of game (0.2). The controllable system $\Xi^{(2)}$ for the considered problem, analogous to (1.4) is described by the following boundary-value problem of hyperbolic type:

$$\begin{aligned} &\partial^2 y/\partial t^2 = \partial^2 y/\partial x^2 && \text{in } G = (0,1)\times(0,\pi) \\ &y(0,x) = (\partial y/\partial t)(0,x) = 0 && \text{in } \Omega = (0,\pi) \\ &-(\partial y/\partial x)(t,0) = v(t), (\partial y/\partial x)(t,\pi) = 0 && \text{for } t \in (0,1), \end{aligned} \tag{3.1}$$

where the upper index (2) at $y^{(2)}$ and $h^{(2)}$ will be omitted. The aim of the second player is (by means of the choice of the program strategy $V \in \mathcal{V}_0^1$) the attainment of possible larger values of the two components of the vector functuonal

$$\rho(h(T)) = (\rho_1(h(1)), \rho_2(h(1))) = \left(\int_0^\pi y'(1,x)dx, -\int_0^{\pi/2} y(1,x)dx\right).$$

This multicriterial problem will be denoted by

$$\langle \Xi^{(2)}, \mathcal{V}_0^1, \{\rho_1(h^{(2)}(1)), \rho_2(h^{(2)}(1))\}\rangle \tag{3.2}$$

further on. By analogy, the multicriterial problem of the type (1.3) for the first player will be denoted by

$$\langle \Xi^{(1)}, \mathcal{U}_0^1, \{-\rho_1(h^{(1)}(1)), -\rho_2(h^{(1)}(1))\}\rangle. \tag{3.3}$$

The problem of optimal control, analogous to (1.7) for (3.2), is of the form

$$\Gamma_\beta = \langle \Xi \div (3.1), \mathcal{V}_0^1, \rho_\beta(h(T))\rangle,$$

where Ξ is described by (3.1) (i.e. by the dynamic system of (3.2)), and the scalar criterion is presented by the functional

$$\rho_\beta(h(T)) = \beta \int_0^\pi y'(1,x)dx - (1-\beta)\int_0^{\pi/2} y(1,x)dx,$$

where the scalar parameter $\beta \in [0,1]$.

Multicriterial problem (3.2) will be solved if we obtain all the γ-maximal strategies and the respective γ-maximums for the problem Γ_β for each $\gamma > 0$ and $\beta \in [0,1]$. Indeed, from [11, Corollary 4] and Corollary 1.3, the program strategy $V^* \in \mathcal{V}_0^1$ is γ-maximal $\forall\gamma > 0$ of the problem Γ_β for some $\beta \in [0,1]$, if and only if V^* is ε-Slater maximal $\forall\varepsilon \in \mathbb{R}^2_>$ of problem (3.2).

Hence we shall look for all the γ-maximal program strategies of $\Gamma_\beta, \forall\gamma > 0$ for each $\beta \in [0,1]$. To this end, the corresponding sequence of problems $\Gamma_k^{(2)}$ from (2.4) is considered. In the considered case the eigenvalues $\lambda_j = j^2, j = 0,1,2,\ldots$ and the eigenfunctions $\omega_0(x) = \sqrt{1/\pi}$, $\omega_j(x) = \sqrt{2/\pi}\cos jx$, $j = 1,2,\ldots$. Then the approximating system is of the form

$$\left| \begin{array}{l} d^2y_0^k/dt^2 = \sqrt{1/\pi}v \\ y_0^k(0) = 0 \\ (y_0^k)'(0) = 0 \end{array} \right. \iff \left| \begin{array}{l} dy_0^k/dt = z_o^k \\ dz_0^k/dt = \sqrt{1/\pi}v \\ y_0^k(0) = z_0^k(0) = 0 \end{array} \right.$$

$$\left| \begin{array}{l} d^2y_j^k/dt^2 = -j^2y_j^k + \sqrt{2/\pi}v \\ y_j^k(0) = 0 \\ (y_j^k)'(0) = 0 \end{array} \right. \iff \left| \begin{array}{l} dy_j^k/dt = z_j^k \\ dz_j^k/dt = -j^2y_j^k + \sqrt{2/\pi}v \\ y_j^k(0) = z_j^k(0) = 0 \end{array} \right. \tag{3.4}$$

$$j = 1,2,\ldots,k$$

and the linear s.-continuous functional

$$\rho_\beta^k(h^k(T)) = \beta\sqrt{\pi}(y_0^k)'(1) - (1-\beta)\Big[\frac{\sqrt{\pi}}{2}y_0^k(1) + \sqrt{2/\pi}(y_1^k(1) - \frac{1}{3}y_3^k(1) + \frac{1}{5}y_5^k(1) - \ldots)\Big],$$

where the number of the terms with multiplier $(1-\beta)\sqrt{2/\pi}$ is $k/2$ or $[k/2]+1$, for even or odd number k respectively. (Here $[k/2]$ is the entire part of the number $k/2$). The solution of (3.4) is of the form

$$y_0^k(t) = \frac{1}{\sqrt{\pi}}\int_0^t\int_0^\tau v(\xi)d\xi d\tau,\ y_j^k(t) = j^{-1}\sqrt{2/\pi}\int_0^t v(\tau)\sin j(t-\tau)d\tau,$$

$j = 1, 2, \ldots, k$ and the value of the functional $\rho_\beta^k(.)$ for this solution is

$$\begin{aligned}\rho_\beta^k(h^k(1)) &= \beta\int_0^1 v(\tau)d\tau - (1-\beta)\Big[\frac{1}{2}\int_0^1\int_0^t v(\tau)d\tau dt \\ &+\frac{2}{\pi}\int_0^1 v(\tau)(\sin(1-\tau) - \frac{1}{3^2}\sin 3(1-\tau) + \frac{1}{5^2}\sin 5(1-\tau) - \ldots)d\tau\Big],\end{aligned} \tag{3.5}$$

and the number of the terms in the last integral in (3.5) is as it is indicated above. Preliminary, some properties about the derivative of the function

$$C_{2k}(t) \stackrel{\text{def}}{=\!=} \sin t - \frac{1}{3^2}\sin 3t + \frac{1}{5^2}\sin 5t - \ldots + \frac{(-1)^{k-1}}{(2k-1)^2}\sin(2k-1)t \tag{3.6}$$

will be proved.

Lemma 3.1. *Let $k \geqq 1$ be a natural number and* $B_{2k}(t) = \cos t - \frac{1}{3}\cos 3t + \frac{1}{5}\cos 5t - \ldots + \frac{(-1)^{k-1}}{2k-1}\cos(2k-1)t$. *Then*

a) $|2/\pi B_{2k}(t) - 1/2| < 1/k,\ \forall t \in [0,1],\ \forall k \geqq 1,$

b) $|2/\pi B_{2k}(t) - 1/2| \leqq \frac{2}{\pi}\cdot\frac{1,095}{2k} < \frac{0,7}{2k},\ \forall t \in [0,1]$ *for* $k \geqq 100.$

P r o o f. The following equalities are valid:

$$\begin{aligned}B_{2k}(t) &= \int_{\pi/2}^t [-\sin\tau + \sin 3\tau - \ldots + (-1)^k\sin(2k-1)\tau]d\tau \\ &= \int_{\pi/2}^t (-1)^k\frac{\sin 2k\tau}{2\cos\tau}d\tau = \frac{1}{2}\int_0^{\pi/2-t}\frac{\sin 2k\tau}{\sin\tau}d\tau \\ &= \frac{1}{2}\int_0^{\pi/2-t}\frac{\sin 2k\tau}{\tau}d\tau + \frac{1}{2}\int_0^{\pi/2-t}\left(\frac{\tau-\sin\tau}{\tau\sin\tau}\right)\sin 2k\tau d\tau.\end{aligned} \tag{3.7}$$

Let the function $\varphi(t) = \dfrac{t - \sin t}{t\sin t}$ for $t \in (0, \pi/2]$ and $\varphi(0) = 0$. It will be proved that $\varphi(t) \in C^1[0, \pi/2]$ is monotonously increasing function with respect to $t \in [0, \pi/2]$ and the following inequalities are valid:

$$0 \leqq \varphi'(t) < 1/2, \ \forall t \in [0, \pi/2], \qquad 0 \leqq \varphi(t) \leqq 1 - 2/\pi, \ \forall t \in [0, \pi/2], \tag{3.8}$$

where the respective equalities are reached only for $t = 0$ or $t = \pi/2$. Indeed, taking into account the inequalities $\sin t \geqq t - t^3/6$, $\forall t \geqq 0$ and $\cos t \leqq 1 - t^2/2 + t^4/24$, $\forall t \geqq 0$, we obtain that $\varphi'(t) = \dfrac{\sin^2 t - t^2 \cos t}{t^2 \sin^2 t} = \dfrac{(\sin t/t)^2 - \cos t}{\sin^2 t} \geqq$ $\geqq \dfrac{1}{\sin^2 t}\left[\left(1 - \dfrac{t^2}{6}\right)^2 - 1 + \dfrac{t^2}{2} - \dfrac{t^4}{24}\right] = \dfrac{t^2}{6\sin^2 t}(1 - t^2/12) \geqq 0$ for $t \in [0, \pi/2]$, and $\varphi'(t) \leqq \dfrac{\sin^2 t - \sin^2 t \cos t}{t^2 \sin^2 t} = \dfrac{1 - \cos t}{t^2} = \dfrac{2\sin^2(t/2)}{t^2} \leqq \dfrac{2(t/2)^2}{t^2} = \dfrac{1}{2}$, which proves the first inequality of (3.8). Hence $0 = \varphi(0) \leqq \varphi(t) \leqq \varphi(\pi/2) = 1 - 2/\pi$, $\forall t \in [0, \pi/2]$, which proves (3.8).

In the first integral of (3.7) we change the variables $\tau \to 2k\tau$, and in the second integral of (3.7) we integrate by parts, through which

$$B_{2k}(t) = \frac{1}{2}\int_0^{2k(\pi/2 - t)} \frac{\sin\tau}{\tau} d\tau - \frac{1}{4k}\frac{(\pi/2 - t) - \sin(\pi/2 - t)}{(\pi/2 - t)\sin(\pi/2 - t)}\cos 2k(\pi/2 - t)$$

$$+\frac{1}{4k}\int_0^{\pi/2 - t} \varphi'(\tau)\cos 2k\tau d\tau = \frac{1}{2}\int_0^{+\infty} \frac{\sin\tau}{\tau} d\tau + \frac{1}{2}\int_{2k(\pi/2 - t)}^{+\infty} \frac{d(\cos\tau)}{\tau}$$

$$-\frac{1}{4k}\varphi(\pi/2 - t)\cos 2k(\pi/2 - t) + \frac{1}{4k}\int_0^{\pi/2 - t} \varphi'(\tau)\cos 2k\tau d\tau$$

is obtained. Integrating by parts once more and taking into account that $\displaystyle\int_0^{\infty} \frac{\sin\tau}{\tau} d\tau = \pi/2$, we obtain that

$$B_{2k}(t) = \frac{\pi}{4} - \frac{1}{4k(\pi/2 - t)}\cos 2k(\pi/2 - t)\left[1 + \frac{(\pi/2 - t) - \sin(\pi/2 - t)}{\sin(\pi/2 - t)}\right]$$

$$+\frac{1}{4k}\int_0^{\pi/2 - t} \varphi'(\tau)\cos 2k\tau d\tau + \frac{1}{2}\int_{2k(\pi/2 - t)}^{+\infty} \frac{\cos\tau}{\tau^2} d\tau$$

$$= \frac{\pi}{4} - \frac{\cos 2k(\pi/2 - t)}{4k\sin(\pi/2 - t)} + \frac{1}{4k}\int_0^{\pi/2 - t} \varphi'(\tau)\cos 2k\tau d\tau + \frac{1}{2}\int_{2k(\pi/2 - t)}^{+\infty} \frac{\cos\tau}{\tau^2} d\tau.$$

Hence

$$\left|B_{2k}(t)-\frac{\pi}{4}\right| \leqq \frac{1}{4k}\left[\frac{1}{\sin(\pi/2-t)}+\frac{1}{2}\left(\frac{\pi}{2}-t\right)\right]+\frac{1}{2}\,2\,\frac{1}{4k^2(\pi/2-t)^2}$$
$$\leqq \frac{2,15}{4k}+1/k^2. \tag{3.9}$$

The inequalities of Lemma 3.1 follow from (3.9). Lemma 3.1 is proved.

Taking into account that $C_{2k}(1-t)=-\int_1^t B_{2k}(1-\tau)d\tau$ and Lemma 3.1, the following assertion is obtained:

Corollary 3.1. *Let $k \geqq 1$ be an arbitrary natural number. Then the following inequalities are valid:*

a) $\left|\frac{2}{\pi}C_{2k}(1-t)-0,5(1-t)\right| < 1/k,\ \forall t \in [0,1],\ \forall k \geqq 1,$

b) $\left|\frac{2}{\pi}C_{2k}(1-t)-0,5(1-t)\right| \leqq \frac{2}{\pi}\cdot\frac{1,095}{2k} < \frac{0,7}{2k},\ \forall t \in [0,1]$ *for* $k \geqq 100.$

Now we return to equality (3.5). Since $\rho_\beta^{2k-1}(h^{2k-1}(1)) = \rho_\beta^{2k}(h^{2k}(1))$, without restriction it can be considered that k is an even number, i.e. in (3.5) will be considered $\rho_\beta^{2k}(h^{2k}(1))$. Then

$$\rho_\beta^{2k}(h^{2k}(1)) = \int_0^1 \left[\beta-(1-\beta)\frac{2}{\pi}C_{2k}(1-t)\right]v(t)dt-\frac{1-\beta}{2}\int_0^1\int_0^t v(\tau)d\tau dt$$
$$=(1-\beta)\left[\int_0^1\left(\beta_1-\frac{2}{\pi}C_{2k}(1-t)\right)v(t)dt-\frac{1}{2}\int_0^1 V(t)dt\right], \tag{3.10}$$
$$\beta_1=\beta/(1-\beta) \text{ for } \beta\neq 1,$$

where $C_{2k}(t)$ was defined in (3.6) and

$$g_{2k}(t) \overset{\text{def}}{=\!=} \beta_1-\frac{2}{\pi}C_{2k}(1-t), \qquad V(t) \overset{\text{def}}{=\!=} \int_0^t v(\tau)d\tau. \tag{3.11}$$

When $\beta\neq 1$, $\rho_\beta^{2k}(h^{2k}(1)) = (1-\beta)\left[\int_0^1 g_{2k}(t)v(t)dt-\frac{1}{2}\int_0^1 V(t)dt\right]$ and the maximization of $\rho_\beta^{2k}(h^{2k}(1))$ is equivalent to the minimization of the functional

$\dfrac{1}{2}\displaystyle\int_0^1 V(t)dt - \int_0^1 g_{2k}(t)v(t)dt$, where $g_{2k}(t)$ and $V(t)$ are defined in (3.11), the function $v(t) \in \mathcal{V}_0^1 = \{V \div v(.) \mid |v(t)| \leqq 1, \forall t \in [0,1]\}$ and $V'(t) = v(t)$, $V(0) = 0$. This problem can be solved using the methods of [7, p. 91-106]. According to the symbols accepted there, in our case $f^0(V,v,t) = \dfrac{1}{2}V - g_{2k}(t)v$, $\Phi(V) \equiv 0$, the adjoint problem [7, p. 100, (36)] satisfies the conditions $\psi'(t) = \dfrac{1}{2}$, $\psi(1) = 0$, hence $\psi(t) = 0,5t - 0,5$. In this case all the conditions of [7, p. 101, Theorem 4] are satisfied. Then the optimal control $\overline{v}_\beta^k(.)$ is found from the inequality [7, p. 101, (38)]:

$$\int_0^1 (-g_{2k}(t) - 0,5t + 0,5)(v(t) - \overline{v}_\beta^k(t))dt \geqq 0, \quad \text{i.e.}$$

$$\int_0^1 (-g_{2k}(t) - 0,5t + 0,5)\overline{v}_\beta^k(t))dt \leqq \int_0^1 (-g_{2k}(t) - 0,5t + 0,5)v(t)dt$$

for each function $v(t) \in \mathcal{V}_0^1$. Since $\overline{v}_\beta^k(t) \in \mathcal{V}_0^1$, it is obtained that

$$\overline{v}_\beta^k(t) = \operatorname{sign}(g_{2k}(t) + 0,5t - 0,5) \quad \text{for} \quad t \in [0,1], \tag{3.12}$$

where the function $g_{2k}(t)$ depends on $\beta_1 = \beta/(1-\beta)$. (If $f(t)$ is an arbitrary continuous function for $t \in [0,1]$, here $\operatorname{sign} f(t)$ denotes the function-element of $L_\infty[0,1]$, which is equal to -1 for $f(t) < 0$; 1 for $f(t) > 0$ and 0 for $f(t) = 0$). If in (3.12) a formal limited transition for $k \to \infty$ is done, from Corollary 3.1 we obtain the function

$$\overline{v}_\beta(t) = \operatorname{sign}(t + \beta_1 - 1) = \operatorname{sign}(t - \tau_0) = \begin{cases} -1 & \text{for} \quad 0 \leqq t \leqq \tau_0 \\ 1 & \text{for} \quad \tau_0 < t \leqq 1 \end{cases} \tag{3.13}$$

for $\beta \in [0, 0{,}5)$, where $\tau_0 = (1-2\beta)/(1-\beta) \in (0,1]$ and $\overline{v}_\beta(t) \equiv 1$ for $\beta \in [0{,}5, 1]$, (for $\beta = 1$ $\overline{v}_\beta(t) \equiv 1$, which is verified directly from (3.10)). We shall show that there exists a natural number $k_0 \geqq 1$ such that for $\forall k \geqq k_0$ and $\beta \in [0,1]$, the function $\overline{v}_\beta^k(t)$ is presented in one of the following three types:

$$\overline{v}_\beta^k(t) = \begin{cases} -1 & \text{for} \quad 0 \leqq t \leqq \tau_0^k \\ 1 & \text{for} \quad \tau_0^k < t \leqq 1 \end{cases}, \qquad (\tau_0^k \in [0,1]) \tag{3.14}$$
$$\text{or } \overline{v}_\beta^k(t) \equiv 1, \forall t \in [0,1] \text{ or } \overline{v}_\beta^k(t) \equiv -1, \forall t \in [0,1].$$

Indeed, from (3.11) and Lemma 3.1 a), $\left(g_{2k}(t) + \dfrac{1}{2}t - \dfrac{1}{2}\right)' = \dfrac{1}{2} + \dfrac{2}{\pi}B_{2k}(1-t) > 0$ $\forall t \in [0,1]$ and $\forall k \geqq 1$, which shows that the function $g_{2k}(t) + 0,5t - 0,5$

is strictly monotonously increasing with respect to $t \in [0,1]$. Hence there is not more than one root of the equation $g_{2k}(t) + 0,5t - 0,5 = 0$ in the interval $[0,1]$, which (if there exists) will be denoted by τ_0^k. Moreover from the definition of the function $g_{2k}(t)$ from (3.11), it can be showed that for $\forall k \geqq 5$, the function $g_{2k}(t) + 0,5t - 0,5$ is negative in a small neighbourhood of the point $t = 0$ for each $\beta_1 \leqq 0,25$ and positive in a small neighbourhood of $t = 1$ $\forall \beta_1 \geqq 0,25$. Thus the assertion about the representation of the function $\overline{v}_\beta^k$ in one of the indicated types of (3.14) is proved.

If there exists a root τ_0^k, then $\tau_0^k = 2\left(\frac{1}{2} + \frac{2}{\pi}C_{2k}(1-\tau_0^k) - \beta_1\right) \Rightarrow$

$$\begin{aligned}
\tau_0^k - \tau_0 &= 2\left(\frac{1}{2} + \frac{2}{\pi}C_{2k}(1-\tau_0^k) - \beta_1\right) - \tau_0 = \frac{4}{\pi}C_{2k}(1-\tau_0^k) - \beta_1 \\
&= \left(\frac{4}{\pi}C_{2k}(1-\tau_0^k) - (1-\tau_0^k)\right) + (1 - \tau_0^k - \beta_1) \\
&= \left(\frac{4}{\pi}C_{2k}(1-\tau_0^k) - (1-\tau_0^k)\right) - (\tau_0^k - \tau_0).
\end{aligned}$$

Hence $\tau_0^k - \tau_0 = \left(\frac{2}{\pi}C_{2k}(1-\tau_0^k) - 0,5(1-\tau_0^k)\right)$. From Corollary 3.1, $|\tau_0^k - \tau_0| < 1/k$, $\forall k \geqq 1$ and $|\tau_0^k - \tau_0| < \dfrac{0,7}{2k}$ for $k \geqq 100$. In the case, when $\beta = 1$, it is verified directly that the functional $\rho_\beta^{2k}(h^{2k}(1))$ is maximized for $v_\beta^k(t) \equiv 1$ and in this case $\overline{v}_\beta^k(t) \equiv \overline{v}_\beta(t) \equiv 1$, $\forall t \in [0,1]$. All this proves that $\overline{v}_\beta^k(t) \to \overline{v}_\beta(t)$ at $L_2(0,1)$ norm for $k \to \infty$. Then

$$\left(\sum_{j=1}^k \overline{y}_j^k(t)\omega_j(x), \sum_{j=1}^k (\overline{y}_j^k(t))'\omega_j(x)\right) - \left(\sum_{j=1}^k \overline{y}_j(t)\omega_j(x), \sum_{j=1}^k \overline{y}_j'(t)\omega_j(x)\right) \to 0$$

at $C([0,1],\mathcal{H})$ norm [11, Theorem 1], where $\mathcal{H} = L_2(0,\pi) \times (H_2^1(0,\pi))^*$, $\overline{y}_j^k$ and $\overline{y}_j$ are the solutions of the system of ordinary differential equations (3.4) for $\overline{v}_\beta^k(t)$ and $\overline{v}_\beta(t)$ respectively and the functional $\rho(.)$ is s.-continuous in $\mathcal{H}$. Hence

$$\lim_{k\to\infty} \rho\left(\sum_{j=k+1}^\infty y_j(t)\omega_j(x), \sum_{j=k+1}^\infty y_j'(t)\omega_j(x)\right) = 0,$$

where $y_j(t)$ are the solutions of system (3.4) for an arbitrary function $v(t) \in \mathcal{V}_0^1$ and moreover this convergence is uniform with respect to $v(t) \in \mathcal{V}_0^1$, see [11, Theorem 1]. Hence $\overline{v}_\beta(t)$, $\forall\beta\in[0,1]$ are all the γ-maximal strategies of problem Γ_β $\forall\gamma>0$ and then $\overline{v}_\beta(t)$ $\forall\beta\in[0,1]$ are all the ε-Slater maximal strategies $\forall\varepsilon\in\mathbb{R}^2_>$ of problem (3.2). Moreover, every function of the type (3.13) where $\tau_0\in[0,1]$ is an arbitrary real number, is an ε-Slater maximal program strategy of problem (3.2) $\forall\varepsilon\in\mathbb{R}^2_>$. On the other hand, for every vector $\varepsilon\in\mathbb{R}^2_>$ there exists a natural number $k_0\geqq 1$ such that $\overline{v}^k_\beta(t)$ is an ε-Slater maximal strategy of problem (3.2), $\forall k\geqq k_0$.

Now the set $\rho(D)$ of the values of the vector functional in the space of the criteria $\mathbb{R}^2$ for problem (3.2) will be constructed. (Here D is the domain of attainment of the controllable system $\Xi^{(2)}$ of (3.2)). The set $\rho(D)$ consists of the vectors $\rho(h(T))$ for all possible program strategies $V\div v(t)\in\mathcal{V}^1_0$. Using the representation (3.5) and Corollary 3.1, it is proved that $\rho(D)$ is the set of all vectors $\rho=(\rho_1,\rho_2)\in\mathbb{R}^2$, for which

$$(\rho_1,\rho_2)=\left(\int_0^1 v(\tau)d\tau,-\frac{1}{2}\int_0^1\int_0^t v(\tau)d\tau dt-\frac{1}{2}\int_0^1(1-\tau)v(\tau)d\tau\right)\ \Big|\ v(t)$$

is an arbitrary function of $\mathcal{V}^1_0$).
(Let us remind that the set $\mathcal{V}^1_0$ was defined in the Introduction). From Corollary 1.3 and [11, Corollary 4], all the ε-Slater maximal values of the vector functional $\forall\varepsilon\in\mathbb{R}^2_>$ form the set of all the vectors $\rho(h(1))$, for which $h(.)$ are calculated for $v=\overline{v}_\beta(t)$ defined in (3.13), where $\tau_0\in[0,1]$ is an arbitrary real number. It is easy to prove that all these ε-Slater maximal vectors $\forall\varepsilon\in\mathbb{R}^2_>$ are of the form $\rho=(1-2\tau_0,-\tau_0^2+2\tau_0-0,5)$, $\forall\tau_0\in[0,1]$.

In Figure 3.1 the set $\rho(D)$ is represented. The north-eastern boundary of this domain, which is marked with a dark line, corresponds to the values of the vector functional for $v=\overline{v}_\beta(t)$.

In Table 3.1 are given the values of six ε-Slater maximums $\forall\varepsilon\in\mathbb{R}^2_>$ of bicriterial problem (3.2), corresponding to the points A,B,C,D,E,F and the respective values of the parameters $\beta\in[0,1]$ and τ_0. Here the number τ_0 defines the function $\overline{v}_\beta(t)$, defined from (3.13).

Now the method of calculating of the approximating strategies $\overline{v}^k_\beta(.)$ and the corresponding ε-maximums $\rho^{2k}(h^{2k}(1))$ will be shown. For that purpose a program is prepared, which calculates approximately τ^k_0 as a unique root of the equation $g_{2k}(t)+0,5t-0,5=0$ in the interval $[0,1]$ (if the root exists), where $g_{2k}(t)$ is given from (3.11) and $\beta_1=\beta/(1-\beta)$. For thus defined number τ^k_0, which determines the corresponding ε-Slater maximal strategy $\overline{v}^k_\beta(t)$ by means of

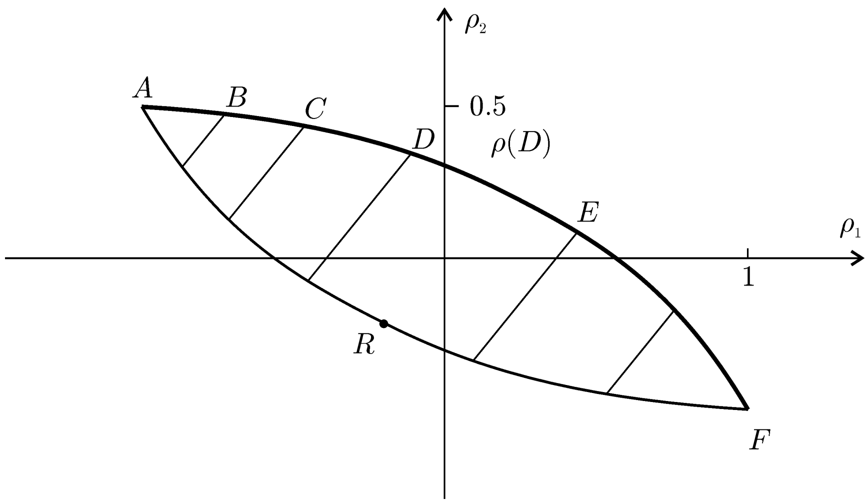

Fig. 3.1

Table 3.1

Points	Coordinates	τ_0	β
A	$(-1; 0,5)$	1	0
B	$(-0,77\ldots; 0,4876543)$	8/9	0,1
C	$(-0,5; 0,4375)$	0,75	0,2
D	$(-0,14286; 0,3163265)$	4/7	0,3
E	$(0,33\ldots; 0,055\ldots)$	1/3	0,4
F	$(1; -0,5)$	0	$\geqq 0,5$

(3.14), the respective components of the ε-maximum $\rho^{2k}(h^{2k}(1))$ are calculated. The calculations are made for $k = 100$ and $k = 1000$. Moreover from Corollary 3.1, the ε-maximums of bicriterial problem (3.2) for $\varepsilon_i = 0,01$, (respectively $\varepsilon_i = 0,001$), $i = 1,2$, where $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \mathbb{R}^2_>$ will be obtained. In Table 3.2 are given the coordinates of the points $A_k, B_k, C_k, D_k, E_k, F_k$ for $k = 100$ and $k = 1000$, calculated for the same values of the parameter β as in Table 3.1. Moreover it is obtained that for these points $|\tau_0^{100} - \tau_0| < 10^{-4}$, respectively $|\tau_0^{1000} - \tau_0| < 10^{-6}$. Using these results, it can be constructed the set $\rho^{2k}(D^{2k})$ and the corresponding set of the ε-Slater maximums for the indicated values of

Table 3.2

Points	Coordinates k=100	Coordinates k=1000	β
A_k	$(-1; 0,5)$	$(-1; 0,5)$	0
B_k	$(-0,77784; 0,48766)$	$(-0,777778; 0,4876543)$	$0,1$
C_k	$(-0,5; 0,437502)$	$(-0,5; 0,4375001)$	$0,2$
D_k	$(-0,14292; 0,31635)$	$(-0,1428564; 0,3163262)$	$0,3$
E_k	$(0,33328; 0,05559)$	$(0,3333334; 0,0555554)$	$0,4$
F_k	$(1; -0,5)$	$(1; -0,5)$	$0,5$

k. Because of the rapid convergence of $\rho^{2k}(h^{2k}(1; 0, 0, 0, \overline{v}^k_\beta))$ for $k \to \infty$, the sets $\rho^{2k}(D^{2k})$ for $k = 100$ and $k = 1000$ and the corresponding sets of the ε-Slater maximal values of the vector functional $\rho^{2k}(h^{2k}(1))$ for the same values of k, differ "very little" from the set $\rho(D)$ and the set of the ε-Slater maximal vectors $\forall\varepsilon \in \mathbb{R}^2_>$ of the functional $\rho(h(1))$ respectively, which are constructed in Fig. 3.1. Moreover, the coordinates of the corresponding points of Table 3.1 and Table 3.2 differ in absolute value no more than $0,0001$.

As the data on Table 3.2 show, for $k = 1000$ the coordinates of the points differ from the coordinates of the corresponding points on Table 3.1 less than in the case when $k = 100$, as it to be expected.

Bicriterial problem (3.3) for the first player is solved by analogy. The program ε-Slater maximal $\forall\varepsilon \in \mathbb{R}^2_>$ (ε-Slater maximal) strategies are of the form $\overline{u}_\beta(t) = -\overline{v}_\beta(t)$ ($\overline{u}^k_\beta(t) = -\overline{v}^k_\beta(t)$). Thus, for reasons of symmetry, the sets $\rho(D)$ ($\rho^{2k}(D^{2k})$) and the corresponding sets of the ε-Slater maximal $\forall\varepsilon \in \mathbb{R}^2_>$ (ε-Slater maximal) values of the vector functional $\rho(h(1))(\rho^{2k}(h^{2k}(1)))$ in the space of the criteria $\mathbb{R}^2$ for the first and the second player coincide.

Now we shall consider game (0.2). The set of all the values of the vector pay-off function $\rho(.) = (\rho_1(.), \rho_2(.))$ for game (0.2) can be obtained as an algebraic sum of two sets

$$\rho(D) + \rho(D), \tag{3.15}$$

where $\rho(D)$ is the set, represented in Fig. 3.1. The set (3.15) coincides with the set $\rho(D(T; p_0))$, see [11], where $D(T; p_0)$ is the domain of attainment of the controllable system Ξ of (0.2).

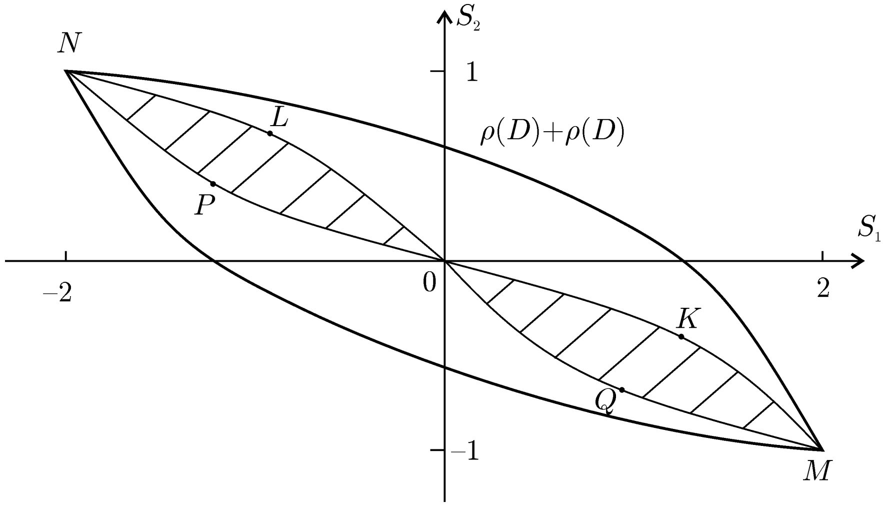

Fig. 3.2

Table 3.3

Points	Coordinates	Situations$(U^*, V^*) = (u^*(t), v^*(t))$
N	$(-2, 1)$	$u^*(t) \equiv -1, v^*(t) \equiv -1$
M	$(2, -1)$	$u^*(t) \equiv 1, v^*(t) \equiv 1$
0	$(0, 0)$	$u^*(t) \equiv 1, v^*(t) \equiv -1$
		$(u^*(t) \equiv -1, v^*(t) \equiv 1)$

From Lemma 1.1, all possible pairs of strategies $(\overline{u}_{\beta_1}(t), \overline{v}_{\beta_2}(t))$, $0 < t \leqq 1$, $\beta_1 \in [0,1]$, $\beta_2 \in [0,1]$, where $\overline{v}_\beta(t)$ is defined from (3.13) and $\overline{u}_\beta(t) = -\overline{v}_\beta(t)$, form the set of the ε-Slater saddle points of game (0.2), $\forall \varepsilon \in \mathbb{R}^2_>$. The set of the values of the vector pay-off function of game (0.2) $\rho(.) = (\rho_1(.), \rho_2(.))$, calculated for these functions $\overline{u}_{\beta_1}$ and $\overline{v}_{\beta_2}$, $\beta_1 \in [0,1]$, $\beta_2 \in [0,1]$ coincides with the set of the values of the vector pay-off function $\rho(.)$, realizing all the ε-Slater saddle points $\forall \varepsilon \in \mathbb{R}^2_>$. This set is obtained as an algebraic sum of the curves $ABCDEF$ and ARF of Fig. 3.1.

The set (3.15) is represented in Fig. 3.2, where the values of the vector pay-off function $\rho(.) = (\rho_1(.), \rho_2(.))$, realizing the ε-Slater saddle points $\forall \varepsilon \in \mathbb{R}^2_>$, are hatched.

In the same way the set $\rho^{2k}(D^{2k}) + \rho^{2k}(D^{2k})$ of the values of the vector pay-off function $\rho^{2k} = (\rho_1^{2k}, \rho_2^{2k})$ and the set of the values, realizing the ε-Slater saddle points for the corresponding values of k and $\varepsilon = \varepsilon(k)$, can be constructed. These sets will differ "very little" from the respective sets constructed in Fig. 3.2, because of which they will not be constructed separately.

As a guaranteed result of the first (second) player in game (0.2), the corresponding ε-minimax or ε-maximin can be used. From Lemma 1.2, the set of the ε-minimaxes (ε-maximins) $\forall\varepsilon \in \mathbb{R}^2_>$ forms the south-western (north-eastern) boundary of the set of the vectors, which are values of the bicomponent vector pay-off function $\rho(.)$ on the ε-Slater saddle points $\forall\varepsilon \in \mathbb{R}^2_>$. As it had been pointed out, in Fig. 3.2 the last set is hatched and the sets of the ε-minimaxes (ε-maximins) $\forall\varepsilon \in \mathbb{R}^2_>$ are represented by the points of the curve $NPOQM$ (respectively the curve $NLOKM$).

As Fig. 3.2 shows, the ε-Slater saddle points of game (0.2) $\forall\varepsilon \in \mathbb{R}^2_>$ are not equivalent. What is more, there exist two ε-Slater saddle points for which the components of the vector pay-off function $\rho(.)$ at one of them are greater than the respective components for the other ε-saddle point $\forall\varepsilon \in \mathbb{R}^2_>$. Therefore, the most acceptable situation $(U^*, V^*) \in \mathcal{U}_0^1 \times \mathcal{V}_0^1$ for both players is the situation, which is an ε-Slater saddle point $\forall\varepsilon \in \mathbb{R}^2_>$ and the value of the vector pay-off function $\rho(.)$ in this situation coincides with the ε-Slater maximin and minimax $\forall\varepsilon \in \mathbb{R}^2_>$. In game (0.2) such solutions are corresponding to the values of the vector pay-off function, which in Fig. 3.2 are represented by the points N, M, 0. The data of these points are given in Table 3.3.

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