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STABILITY CONDITIONS AND SPECTRA*

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In this paper we introduce a new noncommutative Hodge Spectrum. We derive it from the moduli space of stability conditions. Several applications are described.

In this talk we introduce a new Noncommutative Hodge Structure associated to both A and B side of the Homological Mirror Symmetry.

We connect these new Hodge structures with the moduli spaces of Stability conditions. As an artifact we give an idea how to define a Kaehler metric on the moduli spaces of stability conditions.

We also obtain certain combinatorial invariants of these new Hodge Structures.

At the end we discuss applications to classical questions of Birational Geometry. Full details will appear in [17].

We begin with the following definition. Consider $\mathbb{CP}^1$ with coordinate z and an involution $p : z \to \dfrac{1}{z}$ i.e. real algebraic variety.

Definition 1. $\mathbb{R}$ – *nc HS* (*of weight* $= 0$) *is a vector bundle on* $(\mathbb{CP}^1, p)$ *with a connection* ∇ *and a pole of order* 2 *at* $u = 0$.

<u>Axioms.</u> Abstract bundle (without connection)

$$\mathcal{H}^{(\mathbb{R})} \geq \sum_{n_j} \mathcal{O}(n_j)$$

$$\Leftrightarrow \mathbb{R} \text{ vector space}$$

$$H := \Gamma(\mathbb{CP}^1, p)\mathcal{H}^{(\mathbb{R})}$$

together with 2 operators

$$U, D \in End(H) \otimes \mathbb{C} \text{ s.t. } \bar{D} = -D$$

$$\mathcal{H}^{(\mathbb{R})} = H \otimes \mathcal{O}(\mathbb{CP}^1, p)$$

$$\nabla = d + (z^{-1}U - D - z\bar{U})\frac{dz}{z}$$

$$\Leftrightarrow \nabla_{iz\frac{\partial}{\partial z}} = iz\frac{\partial}{\partial z} + (iz^{-1}U - iD - iz\bar{U}) \text{ real for } |z| = 1.$$

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Definition 2. *Polarization on* $\mathbb{C}$ – *nc HS.* $H/\mathbb{C}$, 3 *operators,* U, U^+, D + *positive Hermitian for on* $H/\mathbb{C}$ *s.t.* h *adjoint*$(U) = U^+$, h *adjoint*$(D) = D^+$ + *gluing*$\left(z, \frac{1}{z}\right)$ *data opposedness axiom.*

$$\Downarrow$$

$$\textit{holomorphic bundle}/|z| \leq 1$$

∇ – *connection with 2nd order pole at* $z = 0$ *together with Hermitian pairing.*

$$\mathcal{H}_{|z} \otimes \bar{\mathcal{H}}_{|z} \to \mathbb{C}$$

$$|z| = 1, z \in S^1$$

s.t. gluing a bundle on $\mathbb{C}\ ^1$ with $\mathcal{H}_{|z}$ hermitian conjugate to $\mathcal{H}_{|-\frac{1}{z}}$.

We get a holomorphic bundle

$$H \otimes \mathcal{O}_{\mathbb{CP}^1}$$

$$\mathcal{H} = \Gamma(\mathbb{CP}^1, \text{glued bundle}).$$

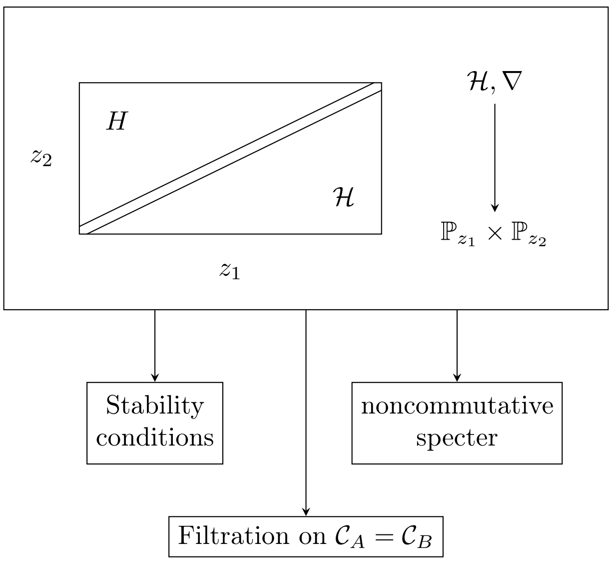
In addition we require:

Positivity Axiom | Hermitian form on $\mathcal{H} > 0$.

So we have

$$\mathcal{H} = \oplus \mathcal{O}(n)$$

We will use both A, B sides of HMS and define a new type of HS with many applications. So we combine $\mathcal{C}_A$, $\mathcal{C}_B$ to define a new Hodge structure.



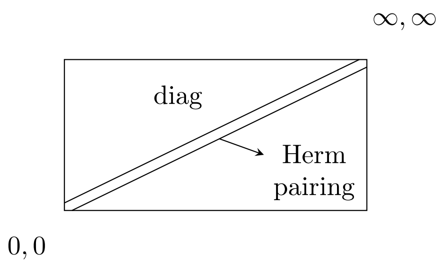
So we have:

Definition 3 (NC A,B Hodge Structure)**.** *We have* 2 $\mathbb{Q}/\mathbb{R}$ *nc HS on* $\mathbb{P}^1_{z_1}$ *and* $\mathbb{P}^1_{z_2}$ + *non-symmetric pairing*

$$H^{\mathbb{Q}}_{z_1} \otimes H^{\mathbb{Q}}_{-z_1} \to \mathbb{Q}$$
$$H^{\mathbb{Q}}_{z_2} \otimes H^{\mathbb{Q}}_{-z_2} \to \mathbb{Q}$$

$$+ \nabla = d + (z_1^{-1}U_1 - D_1 - z_1\bar{U}_1)\frac{dz_1}{z_1} + (z_2^{-1}U_2 - D_2 - z_2\bar{U}_2)\frac{dz_2}{z_2}$$
$$\nabla^2 = 0 \Leftrightarrow [U_1, D_1 \text{ or } \bar{U}_1, U_2, D_2 \text{ or } \bar{U}_2] = 0.$$

This gives compatible monodromy and pairing. We define a polarization on $\mathbb{CP}^1_{\text{diag}} \overset{\text{sym. pair}}{=} H^{\mathbb{Q}}_{z_1=z_2=z} \otimes H^{\mathbb{Q}}_{z_1=z_2=-z} \to \mathbb{Q}$. So we have:



$$z_1 \to \frac{1}{z_1}$$
$$z_2 \to \frac{1}{z_2}$$

1. Out of $H_{\mathbb{P}^1\times\mathbb{P}^1}$ we get a stability condition.

   Step 1 We get $\alpha\mathbb{C}^1 \subset$ fiber $z = 0$ a volume form in H_{z_1}

   Step 2 On $\mathbb{P}^1_{\text{diag}}$ we consider the Image of $K_{0,alg}$ in H. In such a way we obtain a Hermitian form on $\mathbb{C}H_0$.

   Theorem 4. *Moduli space of stability conditions is a Kähler manifold. There exists a potential for this Kähler metric.*

2. We consider $\displaystyle\int_{\Gamma'(0)} \alpha_{(0)} \sim$ (Asymptotics). In such a way at $z = 0$ we obtain the whole non-commutative spectrum.

Example 5 (Consider the category A_n)**.** *It is well known that the differntials* $x^j e^{\frac{p()}{u}} dx$ *are the stability conditions.*

Step 1 $\alpha = dx$ *is the special volume form.*

Step 2 *We will use the integrals* $K_{ij}(u,\bar{u}) = \displaystyle\iint_{\mathbb{C}} x^i x^j e^{\frac{p}{u} - \frac{\bar{p}}{\bar{u}}} dx d\bar{x}$ *in order to define a Kähler metric.*

$$\Phi : |u| \leq 1\tilde{G}L(n+1, \mathbb{C})$$
$$\forall |u| = 1, \Phi(u)\Phi^t(u) = K_{ij}.$$

We define Hermitian form

$$H(u) = \Phi(u)\Phi^t(u)$$

which produces a Kähler metric.

In addition

$$Asymptotics \int x^i e^{\frac{p}{u}} dx$$

which coinside with the non-commutative spectrum and the classical Arnold spectrum.

We briefly recall the notion of a non-commutative spectrum.

Consider the Frobenius manifold $\mathcal{M} \subset H^{\mathrm{even}}(X, \mathbb{C})/\ H^2(X, 2\pi i\mathbb{Z})$, which admits a partial compactification $\overline{\mathcal{M}}$ by "partially-infinite"Kähler classes. In particular, for the class $-\infty\cdot[\omega^{1,1}]$ one gets the classical limit

$$\frac{d}{du} + \frac{1}{u^2}c_1(T_X)\wedge\cdot + \frac{1}{u}G, \quad G_{|H^i(X)} = \frac{i - \dim_{\mathbb{C}} X}{2}$$

for which the quantum spectrum is $\{0\}$, and the maximal growth of solutions is $\sim u^{-\dim X/2}$. If we start at a point in the log-algebraic part $\overline{\mathcal{M}}^{\mathrm{alg}}$ of the Frobenius manifold), and move a little bit, so that the deformed eigenvalues of K split, then the corresponding Serre dimension can only decrease. A purely algebraic version is false: Serre dimension of a s.o.d. could be smaller than those of pieces. Together with the blow-up conjecture, this will give a very strong criterion of non-rationality:

$\Rightarrow$ if for at least one elementary pieces of s.o.d. of $\mathrm{D^b}(Coh(X))$, corresponding to a generic point of $\mathcal{M}^{\mathrm{alg}}$, its Serre dimension is greater than $\dim X - 2$, then X is **not rational**.

Corollary 6 (one of many)**.** *Any odd-dimensional complete intersection of degrees* $(d_1, \ldots, d_r)$ *in* $\mathbb{P}^n$ *is not rational if* $d_{\mathrm{sum}} + d_{\mathrm{max}} > n + 1$. (*There is one low-dimensional exception: conic in* $\mathbb{P}^2$, $2 + 2 \geq 3$).

One can try to formulate everything in terms of variations of **nc** Hodge structures. Ignoring lattice (coming from Γ-conjecture), one gets the following structure:

(H, ∇): holomorphic vector bundle on germ $|u| \ll 1$
with connection ∇ which has second order pole at $u = 0$

In a trivialization with fiber vector space V, connection ∇ is $\nabla_{\frac{d}{du}} = \dfrac{d}{du} + \sum_{i \geq -2} K_i u^i$, $K_i \in End(V)$.

Definition 7. *Such a connection is of* ***exponential type*** *if over* $\mathbb{C}[[u]]$ *it is isomorphic to the direct sum* $\oplus_a e^{\frac{z_a}{u}} \cdot regular_i$.

Numbers Z_a are eigenvalues of K_{-2}. There are connections with "wrong growth", e.g. $\sim \exp(u^{-1/2}$.

For a connection of exponential type one can define its growth exponent s_j for every $z_a \in \operatorname{Spec} K_{-2}$ as

$$S_a := \min\{s \in \mathbb{R} \mid \exists \text{ solution } \sim u^{s+it}\log(u)^k e^{\frac{z_a}{u}} + \cdots \text{ for some } k \in \mathbb{Z}_{\geq 0}\}$$

Definition 8. *A* ***log-isomonodromic deformation*** *is a meromorphic flat connection in two variables u, q close to 0, such that both* $\nabla_{u^2\frac{d}{du}}$ *and* $\nabla_{uq\frac{d}{dq}}$ *are operators without poles.*

Conjecture 9. *For a log-isomonodromic deformation, if for $q = 0$ the connection in u-plane is of exponential type and all growth exponents are $\geq s_0 \in \mathbb{R}$, then the same is true for any $q \neq 0$.*

Let us now assume that (H, ∇) is a connection with second order pole and regular singularity (i.e. all solutions have polynomial growth). Then the order of growth defines a filtration by subbundles, preserved by connection ∇, the indices form the **growth spectrum**.

Real life example: consider hypersurface $X \subset \mathbb{P}^n$ of Calabi-Yau/general type. The connection on the image of $H^\bullet(\mathbb{P}^n)$ in $H^\bullet(X)$ under restriction map, i.e. the span of powers of $c_1(\mathcal{O}(1)) \in H^2(X)$:

$$\nabla_{\frac{d}{du}} = \frac{d}{du} + \frac{1}{u^2}K + \frac{1}{u}G, K = \text{ classical product with } c_1(T_X)$$

the growth spectrum is $(-\dim X/2, -\dim X/2, \ldots)$ for general type, and $(-\dim X/2, 1-\dim X/2, \ldots, +\dim X/2)$ for Calabi-Yau. In general, at conformal point when $K_{-2} = 0$ the growth spectrum is $-\mathrm{Spec}\, K_{-1}$.

Similar behavior happens for Calabi-Yau when we replace multiplication by $c_1 T_X = 0$, by the multiplication by an inhomogeneous class $c_1(T_X) + \sum_{i \neq 2} (2-i)\gamma_i$, $\gamma_i \in H^i(X)$, $i \in 2\mathbb{Z}$.

We see that the growth spectrum outside conformal point is more narrow than the one at the conformal point, has the **same** fractional parts (coming from eigenvalues of the monodromy around) and the **same** extremal growth (the most negative exponent).

All these suggest the following:

Consider the space of germs of connections with second order pole at $u = 0$ and regular singularity. On the set of gauge equivalence classes introduce equivalence relation $\nabla \cong \nabla'$ if there exist a flat connection on a bundle on $C^\infty[0,1]_t \widehat{\otimes} \mathbb{C}[[u]]\}$ such that operators $\nabla_{u^2 \frac{d}{du}}$ and $\nabla_{u \frac{d}{dt}}$ are regular, and the induced connections in -direction obtained by the restriction at $t = 0$, $t = 1$, are equal to ∇, ∇' respectively.

Conjecture 10. *$\exists$ a complete exhaustive decreasing **weight filtration** $(W_s)_{s \in \mathbb{R}}$ invariant under monodromy and the above equivalence relation, which satisfies*

(1) *the monodromy operator on Gr_s^W has eigenvalues in $e^{is}\mathbb{R}_{>0}$,*

(2) *at the conformal point (i.e. if $K_{-2} = 0$) the weight filtration coincides with the filtration by the growth (and hence induces the filtration by real parts of eigenvalues of K_{-1} on the fiber at $u = 0$),*

(3) *outside conformal point (i.e. $K_{-2} \neq 0$), the growth filtration is squeezed from $W_\bullet$ by shifts by $\mathbb{Z}_{\leq 0}$,*

(4) *the lowest value of such that $\mathcal{H} = W_s \mathcal{H}$ is the order of growth of a typical non-zero solution.*

In the case of Landau-Ginzburg model, this hypothetical weight filtration seems to be the same as the one defined by Steenbrink, generalizing those introduced by Arnold for isolated singularities.

A simple example. Consider weighted projective space $\mathbb{P}^{\omega_0, \ldots, \omega_n}$ and generic complete intersection X of hypersurfaces of degrees $d_1, \ldots, d_m$. We will predict **centered weight**

filtration for the Kuznetsov component. Such a complete intersection is **well-formed** iff (here unions are understood **with multiplicities**)

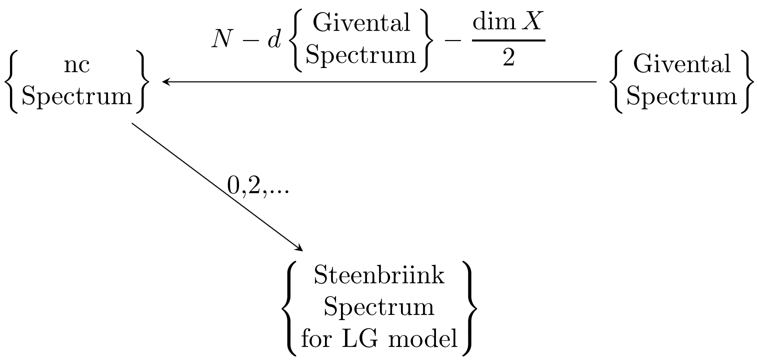
$$(\star) \qquad \bigcup_i \left\{ \frac{1}{\omega_i}, \ldots, \frac{\omega_i - 1}{\omega_i} \right\} \subset \bigcup_j \left\{ \frac{1}{d_j}, \ldots, \frac{d_j - 1}{d_j} \right\}.$$

It is well-known that in this case generic X is smooth, and does not meet singularities of $\mathbb{P}^{\omega_0,\ldots,\omega_n}$. Assume that X is a Fano variety, i.e. $\sum_i \omega_i > \sum_j d_j \to$ Givental's hypergeometric operator:

$$\prod_i \omega_i^{\omega_i} \cdot \partial^{\dim X} - \prod_j d_j^{d_j} \cdot q \cdot \frac{\prod_j (\partial + \frac{1}{d_j}) \cdots (\partial + \frac{d_j - 1}{d_j})}{\prod_i (\partial + \frac{1}{\omega_i}) \cdots (\partial + \frac{\omega_i - 1}{\omega_i})}, \quad \partial := q\frac{d}{dq}, u = c \cdot q^{-\frac{1}{\sum_i \omega_i - \sum_j d_j}}.$$

Growth spectrum is $-\frac{\dim X}{2} + \{\text{complement in } (\star)\} \cdot \left(\sum_i \omega_i - \sum_j d_j \right) \to$ numbers $s_0 \leq s_1 \leq \cdots$. Steenbrink spectrum is $(s_0, s_1 + 1, s_2 + 2, \ldots)$. Small miracle: symmetric with center at 0.

E.g. for complete intersection $d_1 = 2$, $d_2 = 4$ in $\mathbb{P}^6 = \mathbb{P}^{1111111}$. Growth spectrum is $\left(-\frac{7}{4}, -\frac{6}{4}, -\frac{6}{4}, -\frac{5}{4}\right)$ (i.e. solutions grow as $u^{-\frac{7}{4}}$, $\log(u)u^{-\frac{6}{4}}$, $u^{-\frac{6}{4}}$, $u^{-\frac{5}{4}}$). Adding $(0, 1, 2, 3)$ obtain $\left(-\frac{7}{4}, -\frac{1}{2}, +\frac{1}{2}, +\frac{7}{4}\right)$.



The uppersemicontinuity conjecture. Consider $\mathcal{D}_{u,t}$ a D-module depending on 2 parameters u, t.

(1) $\nabla_{u\frac{\partial}{\partial u}} = u\frac{\partial}{\partial u} + U^{-1}M$, $M \in Mat(N \times N) \otimes \mathbb{C}[[u,t]]$.

(2) $\nabla_{t\frac{\partial}{\partial t}} = t\frac{\partial}{\partial t} + U^{-1} \otimes \mathbb{C}[[u,t]]$.

Conjecture 11. $\delta\left(\nabla_{u\frac{\partial}{\partial u}} | t = 0\right) \geq \delta\left(\nabla_{u\frac{\partial}{\partial u}} | t \neq 0\right)$.

The proof of this conjecture is consequence of positivity of the noncommutative polarized Hodge structure defined above.

We now propose a Hypothetical formula for Serre dimension.

The answer is given purely in terms of differential equation $\left(u\partial_u + \frac{1}{u}K + G\right)\psi(u)$:

Conjecture 12.

$$\dim_{\text{Serre}} C_{z_i} = -2\min\{s \in \mathbb{Q}_{\leq 0} \mid \exists \textit{ solution } \sim \psi_{s,k}\cdot u^s \log(u)^k e^{\frac{z_i}{u}} + \cdots\}.$$

Evidence: We have checked 100s of examples of complete intersections in projective spaces. If X is a Fano complete intersection of hypersurfaces of degrees $d_1, \ldots, d_r$ in $\mathbb{P}^{N-1}$, then the corresponding semiorthogonal decomposition is

$$\mathrm{D^b}(X) = \langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(N-1-d_{\text{tot}}),\ \text{Kuznetsov component}\rangle$$

where $d_{\text{sum}} := \sum_i d_i$, the spectrum is $\{0\} \cup \mu_{N-d_{\text{tot}}}$ and the predicted Serre dimension for Kuznetsov component $C_{z=0}$ is

$$\dim_{\text{Serre}} C_{z=0} = (N-r-1) - 2\frac{N-\sum_i d_i}{\max_i d_i} = \dim X - 2\frac{N-d_{\text{sum}}}{d_{\max}}.$$

Moreover, there is always a very striking equality

$$\begin{aligned}&\max\{i \in 2\mathbb{Z} + \dim X \mid i \leq \dim_{\text{Serre}} C_{z=0}\}\\ =&\max\{k \in \mathbb{Z} \mid HH_k(C_{z=0}) \neq 0\}\\ =&\max\{p-q \mid H^{p,q}(X) \neq 0\}\end{aligned}$$

Analogy: For any smooth projective variety X we have:

$$\dim X \geq \max\{p-q \mid H^{p,q}(X) \neq 0\}.$$

The decomposition $\langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(N-1-d_{\text{tot}}),\ \text{Kuznetsov component } C_{z=0}\rangle$ corresponds to a very special point in the Frobenius manifold, with the trivial correction to the ample class. In general, we should deform it by other algebraic cycles in X. The spectrum can a priori split into a finer one, and elementary pieces will be decomposed into smaller pieces. This motivates the following:

1. Semi-continuity conjecture. If we deform a bit parameters (a point in the algebraic part $\mathcal{M}^{\text{alg}}$ of the Frobenius manifold), so that the deformed eigenvalues of K split, then the corresponding Serre dimension can only decrease.

Together with the blow-up conjecture, this will give a very strong criterium of non-rationality:

$\Rightarrow$ if for at least one elementary pieces of s.o.d. of $\mathrm{D^b}(X)$, corresponding to a generic point of $\mathcal{M}^{\text{alg}}$, its Serre dimension is $> \dim X - 2$, then X is not rational.

Corollary 13. *any odd-dimensional complete intersection of degrees $(d_1, \ldots, d_r)$ in $\mathbb{P}^{N-1}$ is not rational if $d_{\text{sum}} + d_{\max} > N$.*

2. Landau-Ginzburg model. Let Y be a noncompact complex manifold of dimension $n \geq 0$, and $W : Y \to \mathbb{C}$ be a holomorphic map. Denote by $Crit(W) \subset Y$ the critical locus considered as a closed analytic subspace (possibly non-reduced).

Assumptions:

(1) (the most crucial) $Crit(W)$ is compact

(2) (also important) there exists a Kähler metric on Y (the choice is not a part of the structure, only existence is required)

(3) (technical, for later convenience) $Crit(W)$ is nonempty and connected, and moreover $f(Crit(W)) = \{0\} \subset \mathbb{C}$.

What follows will not change if we replace Y by any open subset $Y' \subset Y$ containing $Crit(W)$ (alternatively, one can consider Y as a germ at $Crit(W)$).

Consider $\mathbb{Z}$-graded complex

$$(\Gamma(Y, \Omega^{\bullet}_{C^\infty}(Y))[[u]], \text{ differential } d_{\text{tot}} := \bar{\partial} + u\partial + dW \wedge \cdot\,)$$

It calculates hypercohomology $R\Gamma(Y, \Omega^{\bullet}_{Y}[[u]], ud + dW \wedge \cdot\,)$. If $\alpha \in \Gamma(Y, \Omega^{\bullet}_{C^\infty}(Y))\{u\}$ (i.e. not only a formal series in u, but a germ of analytic [forms on Y]-valued function at $u = 0$ and near the compact set $Crit(W)$, then $d_{\text{tot}}\alpha = 0$ means that

$$d(e^{\frac{W}{u}} u^{Gr}(\alpha)) = 0. \quad Gr_{|\Omega^{p,q}_{C^\infty}} := \frac{q-p}{2}$$

Conjecture 14 (Hodge-de Rham degeneration for LG models)**.** *Cohomology of* d_{tot} *is a free finite rank (equivalently, flat)* $\mathbb{C}[[u]]$*-module.*

Let us assume that Y is endowed with an everywhere non-vanishing holomorphic volume form $vol \in \Gamma(Y, \Omega^n)$, $n = \dim Y$.

Then in the case when $Crit(W)$ is connected and non-empty, there is a canonical 1-dimensional subspace at the fiber at 0:

$$\begin{aligned}
&[vol] \in \mathbb{H}^n(Y, \omega^{\bullet}_{Y}, dW \wedge \cdot\,) \\
&\to \mathbb{H}^n(Crit(W), (\Omega^{\bullet}_{Y})_{|Crit(W)}) \\
&\to H^0(Crit^{\text{red}}(W), (\Omega^{n}_{Y})) = \mathbb{C} \ni 1
\end{aligned}$$

Conjecture 15. *The leading growth of solutions appears in the exactly one-dimensional subspace of cohomology of* d_{tot}*, and its reduction at* $u = 0$ *is* $\mathbb{C}\cdot [vol]$*. This is precisely the form we have used to define a stability condition.*

In this section we compare the results of our method with previously known results. We use the simplest of invarinats – the lowest assymptotics:

$$\delta = \dim(X) - 2(N-d)/d.$$

As an immediate consequence we get

Theorem 16. *Every smooth hypersurface of degree* d *in* P^{N-1} *such that*

$$d > N/2$$

is not rational.

This result is stronger than the results of S. Schreieder who proved that very general hypersurface of degree d such that $d > \log_2(N-2) + 2$. According to our results all quardics in P^6 are not rational – $d = 4$, $N = 7$.

Similar result by L. Braune states that very general hypersurface of degree d such that

$$d > 2N/3 + 1$$

is not rational.

Obviously our method is very different and so are the results. We begin with the new results about the Fano treefolds. We summarise our results in the table bellow.

Table 1. Threefold Fano weighted complete intersections

No.	i	$\mathbb{P}$	Degrees	$-K^2$	$h^0(-K)$	$h^{1,2}$	New Result
1	1	$\mathbb{P}(1^4,3)$	6	2	4	52	$\delta = \frac{8}{3}$ all non-rational
2	1	$\mathbb{P}^4$	4	4	5	30	$\delta = \frac{5}{2}$ all non-rational
3	1	$\mathbb{P}^5$	2,3	6	6	20	$\delta = \frac{7}{3}$ all non-rational
4	1	$\mathbb{P}^6$	2,2,2	8	7	14	$\delta = 2$ all non-rational
5	2	$\mathbb{P}(1^3,2,3)$	6	8	7	21	$\delta = \frac{7}{3}$ all non-rational
6	2	$\mathbb{P}(1^4,2)$	4	16	11	10	$\delta = 2$ all non-rational
7	2	$\mathbb{P}^4$	3	24	15	5	$\delta = \frac{5}{3}$ all non-rational
8	2	$\mathbb{P}^5$	2,2	32	19	2	
9	3	$\mathbb{P}^4$	2	54	30	0	
10	4	$\mathbb{P}^3$	$\emptyset$	64	35	0	

It appears these are stronger than the previously known results recorded bellow.

Table 2. Rationality for threefold Fano weighted complete intersections

No.	Not stably rational	Non-rational	Rational	Reference
1	very general	all	none	[1],[2]
2	very general	all	none	[3],[4]
3	very general	general		[1],[5]
4	very general	all	none	[1],[5]
5	very general	all	none	[1], [6]
6	very general	all	none	[7], [8]
7		all	none	[9]
8	none	none	all	projection from a line
9	none	none	all	projection from a point
10	none	none	all	definition

Similarly we have new results for the Fano's described bellow.

Table 3. Four Fano weighted complete intersections

No.	i	$\mathbb{P}$	Degrees	K^4	$h^0(-K)$	$h^{1,3}$	$h^{2,2}$	New Result
1	1	$\mathbb{P}(1^3,2^2,3^2)$	6,6	1	3	107	503	$\delta = \frac{11}{3}$ all non-rational
2	1	$\mathbb{P}(1^4,2,5)$	10	1	4	412	1801	$\delta = \frac{19}{5}$ all non-rational
3	1	$\mathbb{P}(1^4,2^2,3)$	4,6	2	4	121	572	$\delta = \frac{11}{3}$ all non-rational
4	1	$\mathbb{P}(1^5,4)$	8	2	5	325	1452	$\delta = \frac{15}{4}$ all non-rational
5	1	$\mathbb{P}(1^5,2)$	6	3	5	156	731	$\delta = \frac{11}{3}$ all non-rational
6	1	$\mathbb{P}(1^5,2^2)$	4,4	4	5	75	378	$\delta = \frac{7}{2}$ all non-rational
7	1	$\mathbb{P}(1^6,3)$	2,6	4	6	196	912	$\delta = \frac{11}{3}$ all non-rational
8	1	$\mathbb{P}^5$	5	5	6	120	581	$\delta = \frac{18}{5}$ all non-rational
9	1	$\mathbb{P}(1^6,2)$	3,4	6	6	71	364	$\delta = \frac{7}{2}$ all non-rational
10	1	$\mathbb{P}^6$	2,4	8	7	77	394	$\delta = \frac{7}{2}$ all non-rational
11	1	$\mathbb{P}^6$	3,3	9	7	49	267	$\delta = \frac{10}{3}$ all non-rational
12	1	$\mathbb{P}^7$	2,2,3	12	8	42	236	$\delta = 2$ most non-rational
13	1	$\mathbb{P}^8$	2,2,2,2	16	9	27	166	
14	2	$\mathbb{P}(1^5,3)$	6	32	15	70	382	
15	2	$\mathbb{P}^5$	4	64	21	21	142	
16	2	$\mathbb{P}^6$	2,3	96	27	8	70	
17	2	$\mathbb{P}^7$	2,2,2	128	33	3	38	
18	3	$\mathbb{P}(1^4,2,3)$	6	81	25	24	161	$\delta = 3$ all non-rational
19	3	$\mathbb{P}(1^5,2)$	4	162	40	5	52	$\delta = \frac{5}{2}$ all non-rational
20	3	$\mathbb{P}^5$	3	243	55	1	21	$\delta = 2$ most non-rational
21	3	$\mathbb{P}^6$	2,2	324	70	0	8	$\delta = \frac{5}{2}$ all non-rational
22	4	$\mathbb{P}^5$	2	512	105	0	2	
23	5	$\mathbb{P}^4$	$\emptyset$	625	126	0	0	

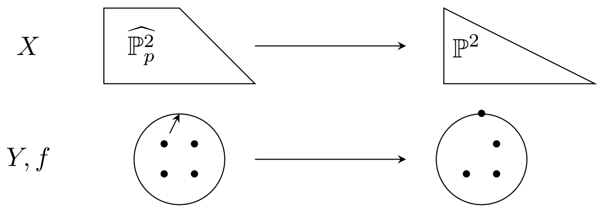
Table 4. Rationality for fourfold Fano weighted complete intersections

No.	Not stably rational	Non-rational	Rational	Reference
2	very general	very general		[10]
4	very general	very general		[10]
5	very general	very general		[10]
8	very general	all	none	[11],[12], [13]
13	very general	very general		[14]
14	very general	very general		[10]
15	very general	very general		[11],[12]
16	very general	very general		[14]
17	very general	very general	some	[15]
18	very general	very general		[10]
19	very general	very general		[16]
20			some	[18]
21	none	none	all	projection from a line
22	none	none	all	projection from a point
23	none	none	all	definition

Bellow we record known results.

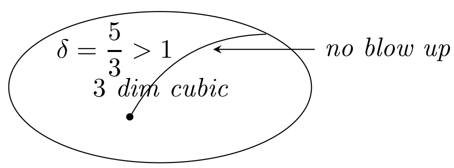
We see that Combinatorics and Non-commutative HS can be interpreted as PDE invariants. It is well known that Kähler-Ricci flow determines the birational geometry of X. Let us look at a simple example.

Example 17. *Consider $\widehat{\mathbb{P}^2_p}$. The Kähler-Ricci flow is as follows:*

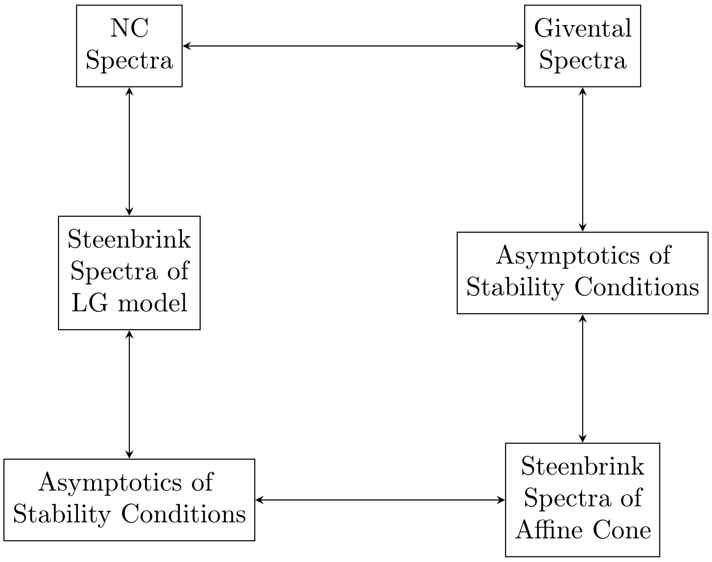


We have a theorem:

Theorem 18. *NC spectrum of $FS(Y, f)$ serves as an obstruction to Kähler-Ricci flow.*



We have correspondences:



The above correspondences connect many classical and neoclassical interpretations of spectra which will lead to new categorical filtration. It can lead to new applications.

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УСЛОВИЯТА ЗА СТАБИЛНОСТ И СПЕКТЪР НА КАТЕГОРИЯ

Людмил Кацарков

Тази статия предлага нов некомутативен спектър, получен от условията за стабилност на категория. Демонстрирани са някои приложения.