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TOPOLOGICAL DICHOTOMY AND UNCONDITIONAL CONVERGENCE

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Abstract. In this paper, we give a criterion for unconditional convergence with respect to some summability methods, dealing with the topological size of the set of choices of sign providing convergence. We obtain similar results for boundedness. In particular, quasi-sure unconditional convergence implies unconditional convergence.

0. Introduction. In this paper, X will always be a Banach space and Ω the Cantor group $\{-1,1\}^{\mathbb{N}}$ equipped with its usual topology and its usual probability. We shall denote by $r_n(\alpha)$, the n^{th} coordinate of $\alpha\in\Omega$. The infinite matrix $A=(a_{n,p})$ indexed by $\mathbb{N}\times\mathbb{N}$ with entries in $\mathbb{C}$ will be supposed to verify the following condition

$$\text{for all } p\geq 0,\quad \lim_{n\to+\infty} a_{n,p} =: l_p \text{ exists.}$$

Given a sequence $(x_p)_{p\geq 0}$ in X and $\alpha\in\Omega$, we assume that

$$\sigma_n^\alpha := \sum_{p\geq 0} a_{n,p} r_p(\alpha) x_p \text{ exists for each integer } n.$$

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If for almost every α, σ_n^α tends to a limit as n tends to infinity, we say that the series (x_p) is almost surely A-convergent.

The probabilistic point of view is well known. In [3, p. 12], the usual case $l_p = 1$ is studied and it is shown that the series (x_p) is almost surely A-convergent if and only if $\sum_p r_p(\alpha)x_p$ is almost surely convergent. The same result holds for boundedness instead of convergence.

In this paper, we shall study this kind of results but in a topological framework. In this context, the concept of almost-sure event (having measure 1) is replaced by quasi-sure event (containing a dense G_δ) and we shall see that this gives stronger results. Actually, concerning both convergence and boundedness, the results of this work are also stronger than the application of the topological $0-1$ law of Christensen [2]. We recall that, in this context, the topological $0-1$ law asserts that, given a sequence (x_p), the set of α such that $\sum_p r_p(\alpha)x_p$ converges is either meager or residual. We recall that a subset of Ω is meager when it is included in a countable union of closed subsets of Ω with empty interior and that it is residual when its complementary is meager.

The first part is devoted to the results concerning convergence. We shall be interested by the three classical ways to express unconditionality: change of signs, summability of any subsequence (which is the same as affecting 0 or 1 as coefficients) and permutations of indexes. We shall prove that these three points of view have their quasi-sure analogs. More precisely, the following assertions are equivalent

i) $\sum_{p\geq 0} x_p$ is unconditionally convergent.

ii) $\sum_{p\geq 0} \varepsilon_p x_p$ is convergent for quasi-every $(\varepsilon_p)_p \in \{-1,1\}^{\mathbb{N}}$.

iii) $\sum_{p\geq 0} \varepsilon_p x_p$ is convergent for quasi-every $(\varepsilon_p)_p \in \{0,1\}^{\mathbb{N}}$.

iv) $\sum_{p\geq 0} x_{\pi(p)}$ is convergent for quasi-every bisection π on $\mathbb{N}$.

The second part deals with boundedness, where the results are similar. In the third part, we present some applications to the geometry of Banach spaces and to harmonic analysis.

Let us recall that:

a) the usual convergence corresponds to the triangular matrix $a_{n,p} = 1$ if $0 \leq p \leq n$ and $a_{n,p} = 0$ if $p > n$;

b) the Cesaro convergence corresponds to $a_{n,p} = (1 - \frac{p}{n})$ if $0 \leq p \leq n$ and $a_{n,p} = 0$ if $p > n$ and

c) the Poisson convergence to $a_{n,p} = t_n^p$, where $t_n \in]0,1[$ and $t_n \to 1$.

We shall suppose the following condition on the matrix A and the sequence $x = (x_p)$ for all $n \in \mathbb{N}$ and all $\chi \in X^*$:

$$(U_{n,\chi}(x)) \qquad \lim_{r\to+\infty} \sum_{p>r} |a_{n,p}\chi(x_p)| = 0$$

Let us point out that, for bounded sequences, a sufficient condition (which is clearly satisfied for usual, Cesaro and Poisson convergence) about the matrix only is the following:

$$(U_n) \qquad \lim_{r\to+\infty} \sum_{p>r} |a_{n,p}| = 0.$$

1. Convergence. Fix a sequence $x = (x_n)_{n\geq 0}$ in X. For $\varepsilon > 0$, $m, m' \in \mathbb{N}$ and $\chi \in X^*$, consider the following sets

$$F_\varepsilon^{m,m'}(\chi) = \Big\{\omega \in \Omega; \Big|\sum_{p\geq 0} (a_{m',p} - a_{m,p}) r_p(\omega)\chi(x_p)\Big| \leq \varepsilon\Big\}$$

$$\text{and} \quad F_\varepsilon^{m}(\chi) = \Big\{\omega \in \Omega; \Big|\sum_{p\geq 0} a_{m,p} r_p(\omega)\chi(x_p)\Big| \leq \varepsilon\Big\}.$$

Lemma. *The sets $F_\varepsilon^{m,m'}(\chi)$ and $F_\varepsilon^{m}(\chi)$ are closed.*

Proof. The map θ from Ω to $\mathbb{R} : \omega \mapsto \sum_{p\geq 0} (a_{m',p} - a_{m,p}) r_p(\omega)\chi(x_p)$ is continuous since the conditions $(U_{m,\chi}(x))$ ensure that θ is the sum of a uniformly convergent series of continuous functions on Ω. Hence $F_\varepsilon^{m,m'}(\chi) = \theta^{-1}([0,\varepsilon])$ is closed.

The set $F_\varepsilon^{m}(\chi)$ is closed by the same very easy argument. □

Now, we have

Theorem 1.1. *Let $\Omega_c = \{\alpha \in \Omega | (\sigma_n^\alpha)_{n\geq 0}$ converges in $X\}$. Either Ω_c is meager in Ω or $\sum_{p\geq 0} l_p x_p$ is unconditionally convergent.*

Proof. Let us assume that Ω_c is not meager. Fix $\varepsilon > 0$. For every $q \geq 1$, set:

$$F_q = \Big\{\omega \in \Omega | \forall m' \geq m \geq q, \Big\|\sum_{p\geq 0} (a_{m',p} - a_{m,p}) r_p(\omega) x_p\Big\| \leq \varepsilon\Big\}.$$

It is easy to see that F_q is closed using the previous lemma and writing:

$$F_q = \bigcap_{m' \geq m \geq q} \bigcap_{\substack{\chi \in X^* \\ \|\chi\| \leq 1}} F_\varepsilon^{m,m'}(\chi).$$

The definition of Ω_c gives

$$\Omega_c \subset \bigcup_{q \in \mathbb{N}^*} F_q,$$

and, since Ω_c is not meager, there exists $q \geq 1$ such that $\overset{\circ}{F}_q \neq \emptyset$. So, there exist an $\omega_0 \in \Omega$ and an integer P such that the equalities $r_p(\omega) = r_p(\omega_0)$ for $p \leq P$ imply $\omega \in F_q$.

Fix $\omega \in \Omega$ and define ω_1 by

$$\begin{cases} r_p(\omega_1) = r_p(\omega_0) & \text{if} \quad p \leq P \\ r_p(\omega_1) = r_p(\omega) & \text{if} \quad p \geq P+1. \end{cases}$$

We have $\omega_1 \in F_q$.

Then, we obtain

$$\begin{aligned} \Big\| \sum_{p \geq 0} (a_{m',p} - a_{m,p}) r_p(\omega) x_p \Big\| &\leq \Big\| \sum_{p \geq 0} (a_{m',p} - a_{m,p}) r_p(\omega_1) x_p \Big\| \\ &\quad + \Big\| \sum_{p=0}^{P} (a_{m',p} - a_{m,p})(r_p(\omega) - r_p(\omega_1)) x_p \Big\| \\ &\leq \varepsilon + 2 \sum_{p=0}^{P} |a_{m',p} - a_{m,p}| . \|x_p\|. \end{aligned}$$

For all $p \in \{0, \dots, P\}$, the sequence $(a_{n,p})_n$ converges as n tends to $+\infty$ so there exists $q' \geq q$ (independent from ω) such that for all $m, m' \geq q'$:

$$2 \sum_{p=0}^{P} |a_{m',p} - a_{m,p}| . \|x_p\| \leq \varepsilon.$$

This leads to the uniform inequality in $\omega \in \Omega$:

$$(UI) \qquad \text{for all } m, m' \geq q', \quad \Big\| \sum_{p \geq 0} (a_{m',p} - a_{m,p}) r_p(\omega) x_p \Big\| \leq 2\varepsilon.$$

To conclude, let us fix $s' \geq s \geq 0$ and $n \geq q'$. Then

$$\Big\|\sum_{p=s}^{s'} a_{n,p} r_p(\omega) x_p\Big\| \leq \Big\|\sum_{p=s}^{s'} (a_{n,p} - a_{q',p}) r_p(\omega) x_p\Big\| + \Big\|\sum_{p=s}^{s'} a_{q',p} r_p(\omega) x_p\Big\|.$$

We use a standard symmetrization principle: consider $\beta \in \Omega$ defined by $r_p(\beta) = r_p(\omega)$ if $p \in \{s, \dots, s'\}$ and $r_p(\beta) = -r_p(\omega)$ if $p \notin \{s, \dots, s'\}$ and write

$$\begin{aligned}\Big\|\sum_{p=s}^{s'} (a_{n,p} - a_{q',p}) r_p(\omega) x_p\Big\| &= \Big\|\sum_{p\geq 0} (a_{n,p} - a_{q',p}) \frac{r_p(\omega) + r_p(\beta)}{2} x_p\Big\| \\ &\leq \sup_{\alpha\in\Omega} \Big\|\sum_{p\geq 0} (a_{n,p} - a_{q',p}) r_p(\alpha) x_p\Big\|\end{aligned}$$

which is less than 2ε by (UI).

As $\sum_{p\geq 0} a_{q',p} r_p(\alpha) x_p$ converges for all $\alpha \in \Omega$, there exists $s_\omega \in \mathbb{N}$ such that $\Big\|\sum_{p=s}^{s'} a_{q',p} r_p(\omega) x_p\Big\| \leq \varepsilon$ for all $s' \geq s \geq s_\omega$.

Then, we obtain

$$\Big\|\sum_{p=s}^{s'} a_{n,p} r_p(\omega) x_p\Big\| \leq 3\varepsilon \qquad (s' \geq s \geq s_\omega,\ n \geq q').$$

Finally, letting n tend to $+\infty$ gives $\Big\|\sum_{p=s}^{s'} l_p r_p(\omega) x_p\Big\| \leq 3\varepsilon$, for $s' \geq s \geq s_\omega$. Since X is complete, $\sum_{p\geq 0} l_p r_p(\omega) x_p$ converges in X. Since ω is arbitrary, this concludes the proof. $\square$

The following corollary, obtained with $a_{n,p} = 1$ for $0 \leq p \leq n$, and 0 else, worth to be pointed out:

Corollary 1.2. *Given a sequence $(x_n)_{n\geq 0}$ in X, set*

$$\Omega_C = \Big\{\alpha \in \{-1, 1\}^{\mathbb{N}} | \sum_{p\geq 0} r_p(\alpha) x_p \text{ converges in } X\Big\}.$$

Then either Ω_C is meager or $\sum_p x_p$ is unconditionally convergent in X.

Remark 1.3. The proof of Theorem 1.1 actually shows that if Ω_C is not meager, the convergence is even uniform in $\omega \in \Omega$ because (UI) is nothing but the Cauchy criterion for the convergence in X of $\sum_p r_p(\omega)x_p$. More precisely, if Ω_C is not meager, then (UI) can be written, $\varepsilon > 0$ fixed,

$$\exists q' \in \mathbb{N},\ \forall m' > m \geq q', \forall \omega \in \Omega,\ \Big\| \sum_{p=m+1}^{m'} r_p(\omega)x_p \Big\| \leq 2\varepsilon.$$

Corollary 1.4. *Assume that the sequence $x = (x_n)_{n\geq 0}$ in X has the following property:*
there exists a dense G_δ set Ω_x in $\Omega = \{-1,1\}^{\mathbb{N}}$ such that $\sum_{n\geq 0} r_n(\alpha)x_n$ converges in X for all $\alpha \in \Omega_x$.

Then $\sum_{n\geq 0} x_n$ is unconditionally convergent in X.

Proof. By Baire's theorem, Ω_x cannot be meager and a dense G_δ. The previous theorem then gives the result. □

With the obvious changing, if we consider

$$\Omega_0 = \{\alpha \in 0,1^{\mathbb{N}} | (\sigma_n^\alpha)_n \text{ converges in } X\},$$

we obtain a theorem which is linked to a "subsets" point of view of unconditional convergence as Theorem 1.1 is linked to unconditional convergence with respect to any choice of signs. It is a classical fact that the two notions are the same. The following result, which is obvious with the proof of Theorem 1.1, shows that this remains true in our "non meager" framework.

Theorem 1.5. *Either Ω_0 is meager in $\{0,1\}^{\mathbb{N}}$ or $\sum_{p\geq 0} l_p x_p$ is unconditionally convergent in X.*

We are now interested in the permutations of indexes. Denote by $\mathcal{S}$ the group of all permutations $\pi : \mathbb{N} \to \mathbb{N}$. The topology is that induced by the product topology of $\mathbb{N}^{\mathbb{N}}$. Let us assume that, the sequence (x_p) is bounded and define

$$\sigma_n^\pi := \sum_{p\geq 0} a_{n,p} x_{\pi(p)} \quad \text{(assuming convergence for each } n),$$

$$\mathcal{S}_c = \{\pi; (\sigma_n^\pi)_n \text{ converges in } X\}$$

$$\text{and } \mathcal{S}_C = \Big\{\pi; \sum_{p\geq 0} x_{\pi(p)} \text{ converges in } X\Big\}.$$

The condition $(U_{n,\chi}(x))$ could be replaced by the condition

$$\lim_{r\to+\infty}\ \sup_{\pi\in\mathcal{S}}\Big|\sum_{p>r} a_{n,p}\chi(x_{\pi(p)})\Big| = 0$$

but practically, we may, and do, assume that the condition (U_n) holds for each n.

Theorem 1.6. *Either $\mathcal{S}_C$ is meager in $\mathcal{S}$ or $\sum_{p\geq 0} x_p$ is unconditionally convergent.*

P r o o f. It is a direct application of the following proposition with the triangular matrix corresponding to usual convergence.

Proposition 1.7. *Either $\mathcal{S}_c$ is meager in $\mathcal{S}$ or $\mathcal{S}_c = \mathcal{S}$; more precisely, in the latter case:*

$$\forall\varepsilon > 0,\ \exists q_0 \in \mathbb{N},\ \forall m, m' \geq q_0,\quad \sup_{\pi\in\mathcal{S}}\Big\|\sum_{p\geq 0}(a_{m',p} - a_{m,p})x_{\pi(p)}\Big\| \leq \varepsilon.$$

P r o o f o f t h e p r o p o s i t i o n. By hypothesis, there exists $K \in \mathbb{R}$ such that $\|x_p\| \leq K$. Let us assume that $\mathcal{S}_c$, is not meager. Fix $\varepsilon > 0$. For $q \geq 1$, set:

$$F_q = \Big\{\pi \in \mathcal{S} | \forall m', m \geq q,\ \Big\|\sum_{p\geq 0}(a_{m',p} - a_{m,p})x_{\pi(p)}\Big\| \leq \varepsilon\Big\}.$$

The definition of $\mathcal{S}_c$ gives:

$$\mathcal{S}_c \subset \bigcup_{q\in\mathbb{N}^*} F_q.$$

Let us show that F_q is closed. We notice that $F_q = \theta^{-1}([0,\varepsilon])$, where $\theta(\pi) = \sup_{m'\geq m\geq q}\|\theta_m(\pi) - \theta_{m'}(\pi)\|$ with $\theta_m(\pi) = \sum_{p\geq 0} a_{m,p}x_{\pi(m)}$. The condition (U_m) ensures the continuity of $\theta_m : \mathcal{S} \to X$. The map θ is then lower semi-continuous and F_q is closed.

$\mathcal{S}_c$ is not meager hence there exists $q \geq 1$ such that $\overset{\circ}{F}_q \neq \emptyset$. So, there exist a $\pi_0 \in \mathcal{S}$ and an integer P such that the condition $\pi(p) = \pi_0(p)$ for every $p \leq P$ implies $\pi \in F_q$.

Fix $\pi \in \mathcal{S}$ and define π_1 by

$$\begin{cases} \pi_1(p) = \pi_0(p) & \text{if} \quad p \leq P \\ \pi_1(p) = \pi(p) & \text{if} \quad p \geq P+1. \end{cases}$$

We have $\pi_1 \in F_q$.

Then, we obtain

$$\begin{aligned}\Big\|\sum_{p\geq 0}(a_{m',p}-a_{m,p})x_{\pi(p)}\Big\| &\leq \Big\|\sum_{p\geq 0}(a_{m',p}-a_{m,p})x_{\pi_1(p)}\Big\| \\ &\quad +\Big\|\sum_{p=0}^{P}(a_{m',p}-a_{m,p})(x_{\pi(p)}-x_{pi_1(p)})\Big\| \\ &\leq \varepsilon+2K\sum_{p=0}^{P}|a_{m',p}-a_{m,p}|.\end{aligned}$$

For all $p\in\{0,\dots,P\}$, the sequence $(a_{n,p})_n$ converges when n tends to $+\infty$, so there exists $q'\geq q$ (independent from π) such that for all $m,m'\geq q'$:

$$2K\sum_{p=0}^{P}|a_{m',p}-a_{m,p}|\leq\varepsilon.$$

This leads to the following uniform inequality in $\pi\in\mathcal{S}$:

$$\text{for all } m,m'\geq q',\quad \Big\|\sum_{p\geq 0}(a_{m',p}-a_{m,p})x_{\pi(p)}\Big\|\leq 2\varepsilon.$$ □

2. Boundedness. We consider the following set

$$\Omega_b=\{\alpha\in\Omega=\{-1,1\}^{\mathbb{N}}\mid(\sigma_n^\alpha)_n\ \text{ is bounded in } X\}.$$

Theorem 2.1. *Either Ω_b is meager in Ω or the formal series $\sum_p l_p x_p$ is weakly unconditionally convergent.*

Proof. Let us assume that Ω_b is not meager. For $q\geq 1$, set:

$$F_q=\Big\{\omega\in\Omega\mid\forall n\geq 0,\quad \Big\|\sum_{p\geq 0}a_{n,p}r_p(\omega)x_p\Big\|\leq q\Big\}.$$

It is easy to see that F_q is closed using the lemma and writing:

$$F_q=\bigcap_{n\geq 0}\ \bigcap_{\substack{\chi\in X^*\\ \|\chi\|\leq 1}}F_q^n(\chi).$$

The definition of Ω_b gives:

$$\Omega_b\subset\bigcup_{q\in\mathbb{N}^*}F_q.$$

Ω_b is not meager. So there exists $q \geq 1$ such that $\overset{\circ}{F}_q \neq \emptyset$.

Hence, there exist $\omega_0 \in \Omega$ and $P \geq 1$ such that $r_p(\omega) = r_p(\omega_0)$ for each $p \leq P$ implies $\omega \in F_q$.

Set $M = q + 2 \sup\limits_n \sum\limits_{p=0}^{P} |a_{n,p}|.\|x_p\|$ and fix $\omega \in \Omega$. Define ω_1 by

$$\begin{cases} r_p(\omega_1) = r_p(\omega_0) & \text{if} \quad p \leq P \\ r_p(\omega_1) = r_p(w) & \text{if} \quad p \geq P+1. \end{cases}$$

Then, we obtain:

$$\sum_{p\geq 0} a_{n,p} r_p(\omega) x_p = \sum_{p \geq 0} a_{n,p} r_p(\omega_1) x_p + \sum_{p=0}^{P} a_{n,p}(r_p(\omega) - r_p(\omega_0)) x_p.$$

Hence, as $\omega_1 \in F_q$,

$$\Big\| \sum_{p=0}^{N} a_{n,p} r_p(\omega) x_p \Big\| \leq q + 2 \sum_{p=0}^{P} |a_{n,p}|.\|x_p\| \leq M.$$

For every $N \in \mathbb{N}$, the symmetrization principle gives

$$\sup_{\omega\in\Omega} \Big\| \sum_{p=0}^{N} a_{n,p} r_p(\omega) x_p \Big\| \leq \sup_{\omega\in\Omega} \Big\| \sum_{p\geq 0} a_{n,p} r_p(\omega) x_p \Big\| \leq M.$$

We let n tend to $+\infty$ to obtain

$$\Big\| \sum_{p=0}^{N} l_p r_p(\omega) x_p \Big\| \leq M$$

this holds for any N and any ω, so

$$\sup_{\chi \in X^*} \sum_{p \geq 0} |l_p \chi(x_p)| = \sup_{\substack{N\in\mathbb{N} \\ \omega\in\Omega}} \Big\| \sum_{p=0}^{N} l_p r_p(\omega) x_p \Big\| \leq M.$$

and we have the result. $\square$

Given a sequence $(x_n)_{n\geq 0}$ in X, we set

$$\Omega_B = \Big\{ \alpha \in \{-1,1\}^{\mathbb{N}} \mid \sup_{N \geq 0} \Big\| \sum_{n=0}^{N} r_n(\alpha) x_n \Big\| < \infty \Big\}.$$

Corollary 2.2. *Either Ω_B is meager in Ω or $\sum_p x_p$ is weakly unconditionally convergent.*

Proof. If Ω_B is not meager, the sequence (x_n) is bounded. We then apply the previous theorem with the triangular matrix of usual boundedness. □

Remark 2.3. As for the convergence point of view, we have obvious similar versions for boundedness replacing Ω by $\{0,1\}^{\mathbb{N}}$ or by $\mathcal{S}$.

3. Applications. In this section, we shall freely use the previous notations.

3.1. Geometry of Banach spaces.

Theorem 3.1. *Let X be a Banach space having no subspace isomorphic to c_0. With the previous notations, suppose that Ω_b is not meager.*

Then $\sum_p l_p x_p$ is unconditionally convergent.

Proof. This is a direct consequence of Theorem 2.1 and of the classical result of Bessaga and Pełczyński [1] asserting that in such a space, every weakly unconditionally convergent series is convergent. □

Proposition 3.2. *Consider a Banach space X with a Schauder basis $(x_p)_p$ and a sequence (u_n) of vectors of X with coordinates $(a_{n,p})$ on (x_p) such that $\sum_p a_{n,p} x_p$ is unconditionally convergent for each n.*

If Ω_c is not meager, then $\sum_p l_p x_p$ is unconditionally convergent.

Proof. This is a direct consequence of Theorem 1.1. □

A natural question is about a generalization of the previous results about a sequence (x_p) for the whole space. We shall assume that the matrix A verifies $l_p = 1$. We have the following

Theorem 3.3. *Given a sequence $(e_p)_p$ of a Banach space X, set*

$$U = \{(x,\omega) \in X \times \Omega; (\sigma_n^\omega(x))_n \text{ is bounded}\}$$

where $\sigma_n^\omega(x) = \sum_{p\geq 0} a_{n,p} e_p^*(x) r_p(\omega) e_p$ and $(e_p^*)_p$ is a sequence in X^*. Thus, we assume that the condition $U_{n,\chi}((e_p)_p)$ holds for all $n \in \mathbb{N}$ and all $\chi \in X^*$ (or sufficiently U_n for all n).

If U is not meager in $X \times \Omega$, then $\sum_{p\geq 0} e_p^*(x) e_p$ is weakly unconditionally convergent for all $x \in X$.

Proof. We have obviously the following inclusion:

$$U \subset \bigcup_{k \geq 0} C_k \quad \text{where} \quad C_k = \{(x, \omega) \in X \times \Omega; \forall n \geq 0, \|\sigma_n^\omega(x)\| \leq k\}.$$

The sets C_k are closed in $X \times \Omega$ (the proof holds as in the lemma) and as U is not meager, one of these sets has non empty interior. Hence there exist $c \in X$ and $\delta > 0$ such that, for every $x \in X$ verifying $\|x - c\| \leq \delta$, the set $\{w \in \Omega; \forall n \geq 0, \|\sigma_n^\omega(x)\| \leq k\}$ has non empty interior too. A fortiori, the set $\{w \in \Omega; (\sigma_n^\omega(x))_n$ is bounded$\}$ has non empty interior hence is not meager (by Baire's theorem). Using, Theorem 2.1, $\sum_{p \geq 0} e_p^*(x) e_p$ is weakly unconditionally convergent.

To conclude, just notice that, $x \neq 0$ given, $y = c + \frac{\delta}{\|x\|} x$ verifies $\|y - c\| \leq \delta$. So $\sum_{p \geq 0} e_p^*(y) e_p$ is weakly unconditionally convergent. As $\sum_{p \geq 0} e_p^*(c) e_p$ is weakly unconditionally convergent, we get the claim. $\square$

An immediate corollary is

Corollary 3.4. *Let $(e_p)_p$ be a basis of a Banach space X and U the set*

$$U = \{(x, \omega) \in X \times \Omega; (\sigma_n^\omega(x))_n \text{ is bounded in } X\}$$

where $\sigma_n^\omega(x) = \sum_{p \geq 0} a_{n,p} e_p^*(x) r_p(\omega) e_p$ and $(e_p^*)_p \subset X^*$ is the sequence biorthogonal to $(e_p)_p$.

If U is not meager in $X \times \Omega$, then $(e_p)_p$ is an unconditional basis.

Definition 3.5. *A Banach space X has the Orlicz property if $\sum_p \|x_p\|^2 < +\infty$, for any weakly unconditionally convergent series $(x_p)_p$.*

Definition 3.6. *A Banach space X has cotype s ($s \geq 2$) if there is a constant $C > 0$ such that for any finite family of vectors (x_p),*

$$\Big(\sum_p \|x_p\|^s\Big)^{\frac{1}{s}} \leq C \int_\Omega \Big\|\sum_p r_p(\omega) x_p\Big\| d\omega.$$

Equivalently: for all sequences $(x_p)_p$ such that Ω_B has measure 1, we have $\sum_p \|x_p\|^s < +\infty$.

The following theorem is rather trivial with the previous notations and Theorem 2.1 but worth mentioning

Theorem 3.7. *If the space X has cotype s and if Ω_B is not meager, then*

$$\sum_p \|x_p\|^s < \infty.$$

Remark 3.8. There exists a Banach space X and a sequence $(x_p)_p$ of X such that Ω_B is meager but has measure 1 and such that $\sum\limits_p \|x_p\|^2 = +\infty$.

In fact, take for X the spectacular space constructed by Talagrand ([6]): having the Orlicz property but not having cotype 2; therefore there exist a series $\sum\limits_p x_p$ which is almost surely convergent (i.e. Ω_b has measure 1) such that $\sum\limits_p \|x_p\|^2 = +\infty$. As the space has the Orlicz property, the series is not weakly unconditionally convergent and by the previous results, Ω_B is meager.

3.2. Harmonic analysis. Let us recall some classical definitions of harmonic analysis. Let G be a compact abelian group, Γ its dual group, $M(G)$ the dual space of $C(G)$, the set of continuous functions on G. The Fourier transform of $\mu \in M(G)$ is, for $\gamma \in \Gamma$:

$$\hat{\mu}(\gamma) = \int_G \overline{\gamma}(g) d\mu(g).$$

For $B \subset M(G)$ and $\Lambda \subset \Gamma$, set:

$$B_\Lambda = \{f \in B | \forall \gamma \notin \Lambda, \quad \hat{f}(\gamma) = 0\}.$$

B_Λ is the set of elements of B whose spectrum is contained in Λ.

Definition 3.9. *A subset Λ of Γ is a Sidon set if*

$$\text{there is } C > 0 \text{ such that } \sum_{\gamma \in \Lambda} |\hat{f}(\gamma) \leq C \|f\|_\infty \text{ for all } f \in C_\Lambda(G).$$

Using Theorem 2.1, we have the immediate following corollary.

Corollary 3.10. *Let us assume $\Lambda = (\lambda_n)_{n \geq 0} \subset \Gamma$ shares the following property:*

$$(P) \quad \begin{cases} \text{for all } f \in C_\Lambda(G), \text{ there exists a subset } \Omega_f \subset \Omega, \text{ which is not meager} \\ \text{such that for all } \alpha \in \Omega_f: \text{ there exists } f^\alpha \in L^\infty_\Lambda(G) \\ \text{with } \widehat{f^\alpha}(\lambda_n) = r_n(\alpha)\hat{f}(\lambda_n) \text{ for every } n \in \mathbb{N}. \end{cases}$$

Then Λ is a Sidon set.

Let us remark that if we replace this topological point of view by a probabilistic one in the property (P) (more precisely, if the condition "Ω_f not meager" is replaced by "Ω_f has a positive measure" (hence "Ω_f has measure one", by the $0-1$ probabilistic law)), this leads to the notion of stationary sets which is strictly more general than the notion of Sidon set. We refer to [4] where this aspect is studied. Thus a set Λ being stationary but not Sidon ($\{3^i+3^j\}_{i,j}$ for example) will give the following situation:

there is a continuous function $f \in C_\Lambda$ such that Ω_f is meager but has measure one.

Actually, there is a stronger characterization of Sidon sets.

Theorem 3.11. *Let us assume that $\Lambda = (\lambda_n)_{n\geq 0} \subset \Gamma$ has the following property: the set $\{(f,\omega) \in C_\Lambda(G) \times \Omega; f^\omega \in L^\infty_\Lambda(G)\}$ is not meager.*

Then Λ is a Sidon set.

Proof. Let F_N be the Fejer kernel of index N. The obvious inclusion

$$\{(f,\omega)\in C_\Lambda(G) \times \Omega; f^\omega \in L^\infty_\Lambda(G)\} \subset \{(f,\omega)\in C_\Lambda(G) \times \Omega; \sup_N \|F_N * f^\omega\|_\infty < +\infty\}$$

shows that the set $\{(f,\omega) \in C_\Lambda(G) \times \Omega; \sup_N \|F_N * f^\omega\|_\infty < +\infty\}$ is not meager. Applying Theorem 3.3 with $e_p = \lambda_p$ and $e^*_p(f) = \hat{f}(\lambda_p)$, we obtain the weak unconditional convergence of the series $\sum_p \hat{f}(\lambda_p)\lambda_p$, for any $f \in C_\Lambda(G)$, and therefore the convergence of $\sum_p |\hat{f}(\lambda_p)|$ for any $f \in C_\Lambda(G)$. $\square$

The previous results give a simple proof of a theorem of Littlewood (see also [5]). In the following, Ω shall denote $\{-1,1\}^{\mathbb{Z}}$.

Theorem 3.12. *Let $(c_n)_{n\in\mathbb{Z}}$ be a sequence such that for all $\omega \in \Omega$ there exists $\mu^\omega \in M(\mathbb{T})$ with $\widehat{\mu^\omega}(n) = r_n(\omega)c_n$ for all $n \in \mathbb{Z}$.*

Then $(c_n) \in \ell^2$.

Proof. We shall prove the conclusion under the weaker assumption that the set Ω_0 of ω such that $\mu^\omega \in M(\mathbb{T})$ is not meager. Set $X = M(\mathbb{T})$, $a_{n,p} = 1 - \frac{|p|}{n}$ if $|p| \leq n$ and $a_{n,p} = 0$ else; $x_p = c_p e_p$ where $e_p(t) = \exp(ipt)$. It is sufficient to observe that, with the previous notations, one has

$$(*) \qquad \Omega_0 \subset \Omega_B.$$

In fact, if $\omega \in \Omega_0$, then

$$\sigma_n^\omega = \sum_{|p|\le n}\Big(1-\frac{|p|}{n}\Big)x_p = F_n * \mu^\omega$$

so that $\|\sigma_n^\omega\| \le \|F_n\|_1.\|\mu^\omega\| \le \|\mu^\omega\|$.

As X has cotype 2, $(*)$ and Theorem 3.7 imply that $\sum_p \|x_p\|^2 < \infty$; that is $\sum_p |c_p|^2 < \infty$. $\square$

Let us point out that this remains true for any compact abelian group and not only for the torus.

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