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ALTERNATIVE CHARACTERIZATION OF THE CLASS k-$\mathcal{UCV}$ AND RELATED CLASSES OF UNIVALENT FUNCTIONS

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Abstract. In this paper an alternative characterization of the class of functions called k -uniformly convex is found. Various relations concerning connections with other classes of univalent functions are given. Moreover a new class of univalent functions, analogous to the 'Mocanu class' of functions, is introduced. Some results concerning this class are derived.

1. Introduction. Denote by $\mathcal{H}$ the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

analytic in the unit disk $\mathcal{U}$. By $\mathcal{S}$ we denote the subclass of $\mathcal{H}$ consisting of functions *univalent* in $\mathcal{U}$. Also, let $\mathcal{UCV}$, $\mathcal{ST}$ denote the classes of *uniformly*

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convex and *uniformly starlike* functions respectively (cf. [2] and [3]). The main feature of the elements of these classes is the fact that they map circular arcs with center at any point ζ from the unit disk on convex arcs or arcs starlike with respect to $f(\zeta)$, respectively. The classes $\mathcal{UCV}$, $\mathcal{ST}$ are defined by the following analytic conditions

$$\mathcal{UCV} = \left\{ f \in \mathcal{S} : \operatorname{Re}\left(1 + \frac{(z-\zeta)f''(z)}{f'(z)}\right) > 0,\ (z,\zeta) \in \mathcal{U} \times \mathcal{U} \right\}, \tag{1.2}$$

$$\mathcal{UST} = \left\{ f \in \mathcal{S} : \operatorname{Re} \frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)} > 0,\ (z,\zeta) \in \mathcal{U} \times \mathcal{U} \right\}. \tag{1.3}$$

The class $\mathcal{UCV}$ was characterized by a more applicable, one-variable condition (cf. [7], [9]).

$$\mathcal{UCV} = \left\{ f \in \mathcal{S} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z))}\right) > \left|\frac{zf''(z)}{f'(z)}\right|,\ z \in \mathcal{U} \right\}. \tag{1.4}$$

Very recently the geometric notion of uniform convexity was extended to the case $\zeta \in \mathbf{C}$ (see [4] and [5]). There the class of functions k-$\mathcal{UCV}$, with the property that each circular arc with center at the point $\zeta \in \mathbf{C}$, $|\zeta| \le k$ $(0 \le k < \infty)$, is mapped on a convex arc, was introduced. A two-variable characterization of that class is the following

$$k\text{-}\mathcal{UCV} = \left\{ f \in \mathcal{S} : \operatorname{Re}\left(1 + \frac{(z-\zeta)f''(z)}{f'(z)}\right) > 0,\ z \in \mathcal{U},\ |\zeta| \le k \right\} \tag{1.5}$$

and its one-variable equivalent

$$k\text{-}\mathcal{UCV} = \left\{ f \in \mathcal{S} : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > k \left|\frac{zf''(z)}{f'(z)}\right|,\ z \in \mathcal{U} \right\}. \tag{1.6}$$

Using the familiar Alexander relation the class k-$\mathcal{ST}$ was also introduced [6]

$$k\text{-}\mathcal{ST} = \left\{ f \in \mathcal{S} : \operatorname{Re}\frac{zf'(z)}{f(z)} > k \left|\frac{zf'(z)}{f(z)} - 1\right|,\ z \in \mathcal{U} \right\}. \tag{1.7}$$

Note that in the case $k = 0$ the classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ coincide with the usual classes of convex ($\mathcal{CV}$) and starlike ($\mathcal{ST}$) functions, respectively.

The continuous 'passage' between the usual classes of starlike and convex univalent functions is due to Mocanu. He introduced the class of α-convex functions (cf. [8]), denoted $\mathcal{M}(\alpha)$, as follows:

$$\mathcal{M}(\alpha) = \left\{ f \in \mathcal{S} : \ \mathrm{Re}\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right)\right] > 0, \quad z \in \mathcal{U} \right\}. \tag{1.8}$$

In actual fact, Mocanu defined the class $\mathcal{M}(\alpha)$ geometrically as a class of functions that map the circle centered at the origin on a α-convex arcs. He also proved the above analytic condition. We note that in the case $\alpha = 0$ the condition (1.8) reduces to the analytic condition of $\mathcal{ST}$ and when $\alpha = 1$ we get from (1.8) the characterization of the class $\mathcal{CV}$.

2. Alternative characterization of the class *k*-$\mathcal{UCV}$ and its applications. In this section we present an alternative, three-variable characterization, of the class k-$\mathcal{UCV}$. As a corollary we describe in a similar way the class k-$\mathcal{ST}$. These characterizations in a surprisingly simple way leads to results concerning different relations between the mentioned and other classes of univalent functions.

Theorem 2.1. *Let $f \in \mathcal{H}$. Then $f \in k$-$\mathcal{UCV}$ if and only if*

$$\mathrm{Re}\, F(z,\zeta,\eta) \geq 0, \quad z,\eta \in \mathcal{U}, \ |\zeta| \leq k, \tag{2.1}$$

where

$$F(z,\zeta,\eta) = \begin{cases} \dfrac{2(z-\zeta)f'(z)}{f(z)-f(\eta)} - \dfrac{z+\eta-2\zeta}{z-\eta} & for \quad z \neq \eta \\[2ex] 1 + \dfrac{(z-\zeta)f''(z)}{f'(z)} & for \quad z = \eta \end{cases}$$

P r o o f. Since

$$\lim_{\eta \to z}\left[\frac{2(z-\zeta)f'(z)}{f(z)-f(\eta)} - \frac{z+\eta-2\zeta}{z-\eta}\right] = 1 + \frac{(z-\zeta)f''(z)}{f'(z)}$$

then $F(z,\zeta,\eta)$ is continuous and hence analytic in z,η and ζ. Moreover, the condition (2.1) gives that f is starlike of order $1/2$, and so f is univalent in $\mathcal{U}$. Thus, by (1.5) it is obvious that (2.1) implies $f \in k$-$\mathcal{UCV}$.

Now suppose that $f \in k\text{-}\mathcal{UCV}$. We need to show that (2.1) holds. Clearly (2.1) holds if $z = \eta$. Then consider the case $z \neq \eta$, but $|z-\zeta| = |\eta-\zeta| = r$. Since $f \in k\text{-}\mathcal{UCV}$ then f maps each circular arc on a convex arc and so the part of the arc $z(t) = \zeta + re^{it}$ which lies inside $\mathcal{U}$ will be mapped on a convex arc containing $f(\eta)$. A convex arc is starlike with respect to each point in its interior or on its boundary, therefore

$$\operatorname{Re} \frac{(z-\zeta)f'(z)}{f(z)-f(\eta)} \geq 0.$$

This fact together with

$$\operatorname{Re} \frac{z+\eta-2\zeta}{z-\eta} = \operatorname{Re} \frac{(z-\zeta)+(\eta-\zeta)}{(z-\zeta)-(\eta-\zeta)} = 0, \quad \text{for} \quad |z-\zeta| = |\eta-\zeta| = r,\ z \neq \eta.$$

yields

$$\operatorname{Re} F(z,\zeta,\eta) \geq 0 \quad \text{when} |z-\zeta| = |\eta-\zeta| = r.$$

By the fact that the function $\operatorname{Re} F(z,\zeta,\eta)$ is a harmonic function in z for fixed ζ and η, an application of the minimum principle gives (2.1) in the case $|z-\zeta| < |\eta-\zeta|$. Similarly (2.1) holds when $|z-\zeta| > |\eta-\zeta|$, and the proof is complete. □

Corollary 2.2. *Let* $0 \leq k < \infty$. *The function* $f \in \mathcal{H}$ *belongs to* $k\text{-}\mathcal{UCV}$ *if and only if*

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)-f(\eta)} + \frac{\eta}{\eta-z}\right] > \frac{1}{2} + k\left|\frac{zf'(z)}{f(z)-f(\eta)} + \frac{z}{\eta-z}\right|, \quad z,\eta \in \mathcal{U}. \tag{2.2}$$

Proof. Assume $z \neq \eta$, and write $F(z,\zeta,\eta)$ as

$$\begin{aligned} F(z,\zeta,\eta) &= \frac{2zf'(z)}{f(z)-f(\eta)} - \frac{z+\eta}{z-\eta} - \left[\frac{2\zeta f'(z)}{f(z)-f(\eta)} - \frac{2\zeta}{z-\eta}\right] \\ &= 2\left[\frac{zf'(z)}{f(z)-f(\eta)} + \frac{\eta}{\eta-z}\right] - \left[1 + \frac{2\zeta f'(z)}{f(z)-f(\eta)} + \frac{2\zeta}{\eta-z}\right]. \end{aligned}$$

So we get $\operatorname{Re} F(z,\zeta,\eta) \geq 0$ if and only if

$$\operatorname{Re}\left[\frac{zf'(z)}{f(z)-f(\eta)} + \frac{\eta}{\eta-z}\right] \geq \frac{1}{2} + \operatorname{Re}\left[\frac{\zeta f'(z)}{f(z)-f(\eta)} + \frac{\zeta}{\eta-z}\right].$$

The condition (2.2) will follow upon choosing $\zeta = ke^{i\alpha}z$ such that

$$\operatorname{Re}\left[\frac{\zeta f'(z)}{f(z)-f(\eta)} + \frac{\zeta}{\eta - z}\right] = k\left|\frac{zf'(z)}{f(z)-f(\eta)} + \frac{z}{\eta - z}\right|.$$

Assume now that (2.2) holds. Clearly (2.2) implies (2.1) if $k|z| \geq |\zeta|$. Applying once more the minimum principle for harmonic functions we see that this implies that $\operatorname{Re} F(z,\zeta,\eta) \geq 0$ for all $z, \eta \in \mathcal{U}$, $|\zeta| \leq k$ hence $f \in k\text{-}\mathcal{UCV}$. Taking the limit as $\eta \to z$ in (2.2), we see that this inequality turns into

$$\operatorname{Re}\left[1 + \frac{1}{2}\frac{zf''(z)}{f'(z)}\right] > \frac{1}{2} + \frac{k}{2}\left|\frac{zf''(z)}{f'(z)}\right|,$$

which is equivalent to (1.6). Hence, the result holds also in the case $\eta = z$. $\square$

Since the classes k-$\mathcal{UCV}$ and k-$\mathcal{ST}$ are connected by the Alexander relation we may also obtain the two-variable representation of the class k-$\mathcal{ST}$ as follows.

Corollary 2.3. *Let* $0 \leq k < \infty$. *The function* $f \in \mathcal{H}$ *belongs to* k-$\mathcal{ST}$ *if and only if*

$$\operatorname{Re}\left[\frac{(z-\zeta)f'(z)}{f(z)} + \frac{\zeta}{z}\right] > 0, \quad z \in \mathcal{U}, \quad |\zeta| \leq k.$$

The alternative characterizations of k-$\mathcal{UCV}$ can be used to derive some new properties which we state in the next corollary.

Corollary 2.4. *Let* $0 \leq k < \infty$ *and* $f \in k$-$\mathcal{UCV}$. *Then*

$$\operatorname{Re}\frac{(z-\zeta)f'(z)}{f(z)-f(\zeta)} > \frac{1}{2}, \quad z \in \mathcal{U},\ |\zeta| \leq k \tag{2.3}$$

$$\operatorname{Re}\left[\frac{(z-\zeta)f'(z)}{f(z)} + \frac{\zeta}{z}\right] > \frac{1}{2}, \quad z \in \mathcal{U},\ |\zeta| \leq k \tag{2.4}$$

and

$$k\left|\frac{zf'(z)}{f(z)} - 1\right| < \operatorname{Re}\frac{zf'(z)}{f(z)} - \frac{1}{2}, \quad z \in \mathcal{U}. \tag{2.5}$$

P r o o f. The inequality (2.3) follows from (2.1) by choosing $\zeta \in \mathcal{U}$ and taking $\eta = \zeta$ and (2.4) follows from (2.1) by taking $\eta = 0$. Finally (2.5) follows from (2.2) by taking $\eta = 0$. □

Remark 2.5. Considering the results of Corollary 2.4 as results concerning the order of starlikeness of the corresponding classes we observe that the inequality (2.3) can be understood to say that any k-uniformly convex function is uniformly starlike of order 1/2. This result resembles the result for the classical classes of starlike and convex functions. Further the inequality (2.4) together with Corollary 2.3 implies that each k-uniformly convex function is k-starlike of order 1/2.

Finally, we obtain the following

Corollary 2.6. *Let $0 \le k < \infty$. If $f \in k$-$\mathcal{UCV}$ then f is starlike of order $(2k+1)/(2k+2)$.*

P r o o f. If f is k-uniformly starlike and $zf'(z)/f(z) = u + iv$ then, in view of (2.5), we have

$$k^2(u-1)^2 + k^2v^2 < \left(u - \frac{1}{2}\right)^2, \; u > 1/2, \tag{2.6}$$

that yields the desired result. Indeed (2.6) states that $zf'(z)/f(z)$ lies inside the convex domain contained in the right half plane and bounded by the conic section which intersects the real axis at the point $u_0 = (2k+1)/(2k+2)$. □

3. The class $\mathcal{UM}(\alpha, k)$. In this section we shall introduce the class $\mathcal{UM}(\alpha, k)$ which corresponds to the class $\mathcal{M}(\alpha)$ in the case of the classical classes of convex and starlike functions. The class $\mathcal{UM}(\alpha, k)$ provides similar 'passage' between classes with prefix 'k', namely between the class k-$\mathcal{UCV}$ and k-$\mathcal{ST}$.

In the sequel we will use the notation

$$J(\alpha, f, z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right) \quad (z \in \mathcal{U}).$$

Definition 3.1. *Let $\alpha \in [0,1]$ and $k \in [0, \infty)$. We say that the function $f \in \mathcal{S}$ belongs to the class $\mathcal{UM}(\alpha, k)$ if*

$$\operatorname{Re} J(\alpha, f, z) > k\,|J(\alpha, f, z) - 1| \tag{3.1}$$

for $z \in \mathcal{U}$, $\zeta \in \mathbf{C}$ *and* $|\zeta| \leq k$.

Remark 3.2. Let

$$J(\alpha, f, z) = u + i\,v$$

then the condition (3.1) can be written as

$$u^2 > k^2(u-1)^2 + k^2 v^2,\ u > o, \tag{3.2}$$

that describes a family of conic domains Ω_k which kind depends only on the parameter k. Further we denote by p_k the functions that satisfy the conditions $p_k(\mathcal{U}) = \Omega_k$, $p_k(0) = 1$ and Re $p_k(z) > 0$ in $\mathcal{U}$. The family of domains Ω_k and the properties of the related functions p_k were characterized in details in [4] and [5]. We recall that the domains Ω_k are convex and symmetric with respect to the real axis.

Remark 3.3. The class $\mathcal{UM}(\alpha, k)$ contains certain classes of functions, considered by various authors. We present below a survey of them.

$$\mathcal{UM}(\alpha, 0) = \mathcal{M}(\alpha),$$

$$\mathcal{UM}(0, 0) = \mathcal{ST},$$

$$\mathcal{UM}(1, 0) = \mathcal{CV},$$

$$\mathcal{UM}(0, k) = k\text{-}\mathcal{ST},$$

$$\mathcal{UM}(1, k) = k\text{-}\mathcal{UCV}.$$

Theorem 3.4. *The function $h(z) = z/(1 - Az)$ is an element of $\mathcal{UM}(\alpha, k)$ if and only if*

$$|A| \leq \frac{\alpha}{k(\alpha+1)+\alpha}. \tag{3.3}$$

P r o o f. It is easy to calculate that

$$\text{Re } J(\alpha, h, z) = \text{Re}\left[\frac{1+\alpha A z}{1 - Az}\right]$$

and

$$k\ |J(\alpha, h, z) - 1| = k\left|\frac{(\alpha+1)Az}{1-Az}\right|.$$

Thus, in this case (3.1) becomes

$$\operatorname{Re}\left[\frac{1+\alpha Az}{1-Az}\right] > k\left|\frac{(\alpha+1)Az}{1-Az}\right|$$

or equivalently

$$\operatorname{Re}(1+\alpha Az)(1-A\overline{z}) > k|(\alpha+1)Az||1-Az|. \tag{3.4}$$

Now, it is enough to prove the condition (3.4) for $z = e^{it}, t \in [0, 2\pi]$. A brief computation leads to the thesis. □

Theorem 3.5. $\mathcal{UM}(\alpha,k) \subset \mathcal{UM}(0,k) = k\text{-}\mathcal{ST}$.

Proof. Denote by $p(z) = zf'(z)/f(z)$ and let p_k be the functions mentioned in Remark 3.2. Then the condition (2.1) is equivalent to

$$(1-\alpha)p(z) + \alpha\left(p(z) + \frac{zp'(z)}{p(z)}\right) \prec p_k(z)$$

or equivalently

$$p(z) + \alpha\frac{zp'(z)}{p(z)} \prec p_k(z).$$

The above subordination is of Briot-Bouquet type (see [1]) and so implies that $p \prec p_k$ in $\mathcal{U}$, as desired. □

Theorem 3.6. $\mathcal{UM}(\alpha,k) \subset \mathcal{UM}(\beta,k)$ *for* $0 \le \beta \le \alpha \le 1$.

Proof. Since the case $\beta = 0$ was considered in the previous theorem we assume $\beta > 0$. Suppose also that $f \in \mathcal{UM}(\alpha,k)$ and $\beta \le \alpha \le 1$. Now, denoting by $p(z) = zf'(z)/f(z)$ and taking into account Theorem 3.5 we have that

$$J(\alpha,f,z) = p(z) + \alpha\frac{zp'(z)}{p(z)} =: Q(z) \prec p_k(z),$$

and also $p \prec p_k$ in $\mathcal{U}$. In order to prove that $f \in \mathcal{UM}(\beta,k)$ we need to show that

$$p(z) + \beta\frac{zp'(z)}{p(z)} \prec p_k(z).$$

Since $\beta/\alpha \le 1$ and

$$p(z) + \beta\frac{zp'(z)}{p(z)} = \frac{\beta}{\alpha}Q(z) + \left(1-\frac{\beta}{\alpha}\right)p(z)$$

then for each fixed z the right hand side of above is an element of the line segment with endpoints at Q and p, which lie inside the convex domain Ω_k. Thus $p(z) + \beta\, zp'(z)/p(z)$ also lies in the domain Ω_k. This completes the proof. □

Taking into consideration Theorem 2.1 and Corollary 2.3 together with Definition 3.1 we may also derive another representation of $\mathcal{UM}(\alpha, k)$.

Corollary 3.7. *Let $0 \le k < \infty$. The function $f \in \mathcal{H}$ belongs to $\mathcal{UM}(\alpha, k)$ if and only if*

(3.5)
$$\mathrm{Re}\left\{(1-\alpha)\left[\frac{(z-\zeta)f'(z)}{f(z)} + \frac{\zeta}{z}\right] + \alpha\left[\frac{2(z-\zeta)f'(z)}{f(z)-f(\eta)} - \frac{(z+\eta-2\zeta)}{z-\eta}\right]\right\} > 0$$

or

$$\mathrm{Re}\left\{(1-\alpha)\left[\frac{(z-\zeta)f'(z)}{f(z)} + \frac{\zeta}{z}\right] + \alpha\left[1 + \frac{(z-\zeta)f''(z)}{f'(z)}\right]\right\} > 0. \tag{3.6}$$

where $z, \eta \in \mathcal{U}$, $|\zeta| \le k$

Taking $\eta = 0$ in (3.5) we obtain the following results.

Corollary 3.8. *Let $0 \le k < \infty$. If the function $f \in \mathcal{UM}(\alpha, k)$ then*

$$\mathrm{Re}\left[\frac{(z-\zeta)f'(z)}{f(z)} + \frac{\zeta}{z}\right] > \frac{\alpha}{\alpha+1} \quad (z \in \mathcal{U},\ |\zeta| \le k). \tag{3.7}$$

The above inequality can be read as the order of k-starlikeness of the functions from the class $\mathcal{UM}(\alpha, k)$. This means that if the function $f \in \mathcal{UM}(\alpha, k)$ then $f \in k\text{-}\mathcal{ST}_{\alpha/(\alpha+1)}$.

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