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# MODELS OF ALTERNATING RENEWAL PROCESS AT DISCRETE TIME

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Abstract. We study a class of models used with success in the modelling of climatological sequences. These models are based on the notion of renewal. At first, we examine the probabilistic aspects of these models to afterwards study the estimation of their parameters and their asymptotical properties, in particular the consistence and the normality. We will discuss for applications, two particular classes of alternating renewal processes at discrete time. The first class is defined by laws of sojourn time that are translated negative binomial laws and the second class, suggested by Green is deduced from alternating renewal process in continuous time with sojourn time laws which are exponential laws with parameters $\alpha^0$ and $\alpha^1$ respectively.

**Introduction.** In this paper we study a class of alternating renewal processes in discrete time with values in $\{0,1\}$. This kind of processes has numerous applications, namely in climatology where they are used with success in the modelling of climatological sequences according to the wetness or the dryness features

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of days (Lebreton [10], Buishand [4], Green [7]). These processes are entirely described by their sojourn times laws in the states 0 and 1. Firstly, let us recall the probabilistic aspects of the model such as the stationarity of the second order, the marginal laws of finite dimension and the partial sum laws. Concerning the statistical aspect, we suggest a number of estimations of the model parameters and their asymptotic properties, particularly the consistency and the normality under the hypothesis that the means of the sojourn time laws are finite. Recently, Mitov and al. [11] and Mitov [12] have established some results having a very close relation with the topic of our paper.

For illustration, we also examine two particular classes of alternating renewal processes in discrete time. The first is defined by the sojourn time of the translated negative binomial laws (cf. [4, 10]), the second suggested by Green [7], is deduced from alternating renewal process in continuous time and with sojourn time laws which are exponential laws with the parameters $\alpha^0$ and $\alpha^1$.

**1. Definition and probabilistic characteristics.** The considered models are based on the hypothesis that each change state of the phenomenon depends on the past only through the state of that instant. This is indeed true since for every $n > 1$ and $(x_0, \dots, x_n) \in \{0,1\}^{n+1}$

$$P(X_n = x_n / X_{n-1} = x_{n-1}, \dots, X_0 = x_0) \\ = P(X_n = x_n / X_{n-1} = x_k, \dots, X_k = x_k, X_{k-1} = x_{k-1})$$

where

$$k = \begin{cases} 1 & \text{if } x_0 = \cdots = x_{n-1} \\ \sup\limits_{1 \le t \le n-1} \{x_{t-1} \neq x_t\} & \text{else.} \end{cases}$$

Let $T_n^i$ be the $n^{th}$ entry date in the state $i$, and $D_n^i$ be the $n^{th}$ sojourn time in the state $i$, for $i \in \{0,1\}$. We consider $\left(T_n^i; n \ge 1\right)$ and $\left(D_n^i; n \ge 1\right)$ for $i \in \{0,1\}$, the sequences of the entry dates and the sojourn times in the state $i$ defined by:

$$T_1^i = \inf\{t \ge 0 : x_t = i\}; \ D_1^i = \inf\left\{t \ge T_1^i : x_t = 1-i\right\} - T_1^i;$$

$$T_n^i = \inf\left\{t \ge T_{n-1}^i + D_{n-1}^i : x_t = i\right\};$$

$$D_n^i = \inf\left\{t \ge T_n^i : x_t = 1-i\right\} - T_n^i; \quad n \ge 2.$$

We notice that on the set $\{X_0 = i\}$, $i \in \{0,1\}$, we have

$$0 = T_1^i < T_1^{1-i} < T_2^i < T_2^{1-i} < \cdots < T_n^i < T_n^{1-i} < T_{n+1}^i < \cdots$$

$$D_n^{1-i} = T_n^i - T_{n-1}^{1-i} \quad \text{and} \quad T_n^{1-i} = T_n^i + D_n^i, \quad n \geq 1.$$

As $\hat{p}^0{=}(\hat{p}_n^0 : n{\geq}1)$, $\hat{p}^1{=}(\hat{p}_n^1 : n{\geq}1)$, $p^0{=}(p_n^0 : n{\geq}1)$, $p^1{=}(p_n^1 : n{\geq}1)$ four probability laws on $\mathbb{N}^*$ are obtained. Let $p$ be a real number in $[0, 1]$.

According to Gregoire [8], we define

**Definition 1.** *A time series $(X_t; t \in \mathbb{N})$ is said to be an alternating renewal process with parameters $\left(p, \hat{p}^0, \hat{p}^1, p^0, p^1\right)$ if $P(X_0 = 1) = p$ and for $i \in \{0, 1\}$, conditioned on $X_0 = i$, the sojourn times $D_n^i; (n \geq 1)$ (resp. $D_n^{1-i}; (n \geq 1)$) are independent having as law $\hat{p}^i$ for $n = 1$, and having the same law $p^i$ for $n \geq 2$ (resp. having as law $p^{1-i}$ for $n \geq 1$), the sequences $\left(D_n^i; n \geq 1\right)$ and $\left(D_n^{1-i}; n \geq 1\right)$ are also mutually independent.*

The $p^0$ and $p^1$ laws are respectively called sojourn time laws in the states 0 and 1. As for $\hat{p}^0$ and $\hat{p}^1$ they are called waiting laws.

About the second order stationarity of the alternating renewal process, we can further point out (cf. Grégoire [8]):

**Theorem 1.** *Let $(X_t; t \in \mathbb{N})$ be an alternating renewal process with the parameters $\left(p, \hat{p}^0, \hat{p}^1, p^0, p^1\right)$. If the laws $p^0$ and $p^1$ are aperiodic and admit moments of the first order $m_0$ and $m_1$, then $(X_t)$ is a stationary process only if*

$$p = \frac{m_1}{m_0 + m_1} \quad \text{and} \quad \hat{p}_k^i = \frac{1}{m_i} \sum_{l \geq 0} p_{k+l}^i, \qquad \forall k \geq 1 \quad \text{and} \quad i \in \{0, 1\}.$$

Thus, a stationary alternating renewal process is uniquely determined by the sojourn time laws $p^0$ and $p^1$. We define $RA\left(p^0, p^1\right)$ the class of the stationary alternating renewal processes with the parameters $p^0$ and $p^1$.

The covariance function $\gamma_X(.)$ of the process $(X_t) \in RA\left(p^0, p^1\right)$ is entirely determined by the transition probabilities $P(X_h = 1/X_0 = 1)$ and these can be determined by simple recurrence.

**Finite dimension marginal laws and partial sum laws.** We define $Y_n^0$ and $Y_n^1$ to be the number of $0-$sequences and $1-$sequences, starting and ending over the period $\{0, 1, \ldots, n\}$. If $N_n^{ij}$ indicates the number of transitions from state $i$ to state $j$ between instant 0 and instant $n$, then

$Y_n^0 = \sup\left\{N_n^{10} + x_n - 1, 0\right\}$ and $Y_n^1 = \sup\left\{N_n^{01} - x_n, 0\right\}$,

$\Delta_1^0, \ldots, \Delta_k^0, \ldots$ (resp. $\Delta_1^1, \ldots, \Delta_k^1, \ldots$) are the durations of successive sequences of 0 (resp. of 1) appearing after the instant 0,

$\Delta_n = \inf\left\{n + 1, \Delta_1^0.\mathbf{1}_{\{X=0\}} + \Delta_1^1.\mathbf{1}_{\{X=1\}}\right\}$ is the spent time on the period $\{0, 1, \ldots, n\}$ in the initial state from the instant 0 (until the first possible instant for the change of state),

$\Gamma_n$ is the spent time in the final state from the instant
$\tau_n = \inf\{t \geq 0 : X_t = X_{t+1} = \dots . = X_n\}$.
Clearly, $\Gamma_n$ can be written thus :

$$\Gamma_n = n + 1 - \Delta_n - \sum_{k=1}^{Y_n^0} \Delta_k^0 - \sum_{l=1}^{Y_n^1} \Delta_l^1$$

We accept conventionaly that when $Y_n^0$ (resp. $Y_n^1$) is zero, the corresponding sum is also zero.

The following result can be easily checked:

**Proposition 1.** *Let $(X_t; t \in \mathbf{N})$ be a stationary alternating renewal process with parameters $p^0$ and $p^1$ with respective means $m_0$ and $m_1$. The joint law of the variables $X_0, \dots, X_n$ is written*

$$P_n(x_0, \dots, x_n) = \begin{cases} p^{x_0}(1-p)^{1-x_0} \cdot \sum\limits_{k \geq n+1} \left(\hat{p}_k^1\right)^{x_0} \cdot \left(\hat{p}_k^0\right)^{1-x_0} & \text{if } \delta = n+1, \\ \\ p^{x_0}(1-p)^{1-x_0} \cdot \left(\hat{p}_\delta^1\right)^{x_0} \cdot \left(\hat{p}_\delta^0\right)^{1-x_0} \prod\limits_{k=1}^{y^0} p^0_{\delta_k^0} \cdot \prod\limits_{l=1}^{y^1} p^1_{\delta_l^1} & \\ \times \sum\limits_{m \geq k} \left(p_m^1\right)^{x_n} \cdot \left(p_m^0\right)^{1-x_n} & \text{if } \delta \leq n, \end{cases}$$

*for $(x_0, \dots, x_n) \in \{0,1\}^{n+1}$, and $\delta, \delta_1^0, \dots, \delta_k^0, \dots, \delta_1^1, \dots, \delta_l^1, \dots, \gamma, \gamma^0, \gamma^1$ indicating the respective values of $\Delta_n$, $\Delta_1^0, \dots, \Delta_k^0, \dots, \Delta_1^1, \dots, \Delta_l^1, \dots, \Gamma_n, Y_n^0, Y_n^1$ for the observation $(x_0, \dots, x_n)$, with the convention that when $y^0 = 0$ (resp. $y^1 = 0$), then the corresponding product equals* 1.

It appears that the statistics $\left(X_0, \Delta_n, Y_n^i, \Delta_1^i, \dots, \Delta_{Y_n^i}^i, i \in \{0,1\}\right)$ is sufficient.

As for the partial sum laws, we apply to the alternating case, Elliot's [6] and Cox's [5] methods presented in the field of ordinary renewal process. We thus are led (Buishand [4]) to a computing algorithm for the probability law of the variable $S_n^1 = \sum\limits_{t=0}^{n} X_t$ :

**Proposition 2.** *Let $(X_t; t \in \mathbf{N})$ be a stationary alternating renewal process with the parameters $p^0$ and $p^1$ having for respective means $m_0$ and $m_1$. We have*

$$P\left(S_n^1 = m\right) = \frac{1}{m_0 + m_1}\left[m_0.Q_n^0(m) + m_1.Q_n^1(m)\right], \tag{1}$$

*where* $Q_n^i(m) = P\left(S_n^1 = m \mid X_0 = i\right)$, *for* $i \in \{0,1\}$ *and* $0 \leq m \leq n$.
*In particular, the probability that the process remains in the state* $1$ *over the period* $\{0,1,\ldots,n\}$, *is*

$$P\left(S_n^1 = n+1\right) = \frac{1}{m_0+m_1}\sum_{l\geq 1} l.p_{n+l}^1. \tag{2}$$

The relation (1) is obvious and the law of the $S_n^1$ would be entirely determined once the probabilities $Q_n^i(m)$ are known.

According to Lebreton [10], the probabilities $Q_n^i(m)$ are provided by the recurrence as follows:

for $n=0$: $\quad Q_0^i(0) = 1-\delta_{i1}$ and $Q_0^i(1) = \delta_{i1}$,

for $n \geq 1$: $\quad Q_n^0(n+1) = Q_n^1(0) = 0$, $Q_n^0(0) = \sum\limits_{l\geq n+1} \hat{p}_l^0$,

$$Q_n^1(n+1) = \sum_{l\geq n+1} \hat{p}_l^1,\ Q_n^0(n) = \frac{1}{m_0}\sum_{l\geq n} p_l^1 \text{ and } Q_n^1(1) = \frac{1}{m_1}\sum_{l\geq n} p_l^0,$$

for $n \geq 2$, $1 \leq m \leq n-1$ :

$$Q_n^0(m) = \sum_{l=1}^{n-m} \hat{p}_l^0.R_{n-l}^0(m) \text{ and } Q_n^1(m) = \sum\nolimits_{l=0}^{m-1} \hat{p}_l^1.R_{n-l}^1(m-l),$$

where

$$R_n^i(m) = P\left(S_n^1 = m \mid X_0 = i, X_1 = 1-i\right),$$

for $i \in \{0,1\}$ and $m = 0,1,\ldots,n$.
Once more, we determine the probabilities $R_n^i(m)$, in the following lemma:

**Lemma 1.** *We have*
*for* $n=0$ :

$$R_0^i(1) = \delta_{i1} \qquad \textit{and} \qquad R_0^i(0) = 1-\delta_{i1}.$$

*for* $n \geq 1$ :

$$R_n^0(0) = R_n^0(n+1) = R_n^1(0) = R_n^1(n+1) = 0,$$

$$R_n^0(n) = m_1.\hat{p}_n^1, \quad R_n^1(n) = m_0.\hat{p}_n^0$$

*and*

$$R_n^0(1) = \begin{cases} 1 & \textit{if } \ n=1 \\ m_0.p_1^1.\hat{p}_{n-1}^0 & \textit{if } \ n\geq 2 \end{cases}, \qquad R_n^1(1) = \begin{cases} 1 & \textit{if } \ n=1 \\ m_1.p_1^0.\hat{p}_{n-1}^1 & \textit{if } \ n\geq 2 \end{cases}$$

*for* $n \geq 2, \;\; 2 \leq m \leq n:$

$$R_n^1(m) = \sum_{l=1}^{n-m} p_l^0.R_{n-l}^0(m-1) \quad \textit{and} \quad R_n^0(m) = \sum_{l=1}^{m} p_l^1.R_{n-l}^1(m-l+1)\,.$$

P r o o f. The results of first two cases can be easily obtained and for $n \geq 2$ and $2 \leq m \leq n$, we can write

$$\sum_{l=1}^{m} P\left(X_1 = 1, \ldots, X_l = 1, X_{l+1} = 0, \sum_{j=l}^{n} X_j = m-l+1 \mid X_0 = 0, X_1 = 1\right)$$

$$= \sum_{l=1}^{m} P\left(X_1 = 1, \ldots, X_l = 1, X_{l+1} = 0 \mid X_0 = 0, X_1 = 1\right)$$

$$\times P\left(\sum_{j=l}^{n} X_j = m-l+1 \mid X_l = 1, X_{l+1} = 0\right),$$

that is

$$R_n^0(m) = \sum_{l=1}^{m} p_l^1.R_{n-l}^1(m-l+1)\,.$$

The proof is similar for $R_n^1(m)$. $\square$

**Persistences.** We define the persistence at the $n^{th}$ day of the state $i$, $i \in \{0,1\}$, as being the probability that the sojourn in the state $i$ lasts strictly more than $n$ days, knowing that it has lasted $n$ days. We write it thus

$$q_n^i = P\left(X_{n+1} = i \mid X_0 = 1-i, X_1 = i, \ldots, X_n = i\right).$$

The following relation is easily set:

$$q_n^i = 1 - \frac{p_n^i}{\sum\limits_{k \geq n} p_k^i}. \tag{3}$$

The sequences $\left(q_n^i : n \geq 1\right)$ of the persistences are important characteristics of a model (specially in climatology). We are interested in some properties such as the monotony and the behaviour when $n$ tends to $+\infty$.

**Examples**

1. Alternating renewal process for sojourn time laws with translated negative binomial.

Let the sojourn time laws $p^i$ are translated negative binomial laws with the parameters $\left(\mu^i, r^i\right), i \in \{0,1\}$ (i.e.

$$p_k^i = \frac{\left(r^i\right)_{k-1}}{(k-1)!} \cdot \left(1 + \frac{\mu^i}{r^i}\right)^{-r} \cdot \left(\frac{\mu^i}{\mu^i + r^i}\right)^{k-1}; k \geq 1,$$

where $\left(r^i\right)_k = \left(r^i + k - 1\right) . \left(r^i + k - 2\right) \ldots \left(r^i + 1\right) .r^i$ if $k \geq 1$ and $\left(r^i\right)_0 = 1$, $\mu^0$, $r^0$ and $\mu^1$, $r^1$ are positive real numbers). The $p^i$ law have as mean $m_i = \mu^i + 1$ and variance $\sigma_i^2 = \mu^i . \left(1 + \dfrac{\mu^i}{r^i}\right)$. Thus, we show (Buishand [4]) that the sequence of persistences $\left(q_n^i : n \geq 1\right)$ of the state $i$ $(i = 0, 1)$, is monotonously increasing (resp. monotonously decreasing) with limit $\dfrac{\mu^i}{\mu^i + r^i}$ if $r^i < 1$ (resp. $r^i > 1$), and is constant in $r^i = 1$ and valued $\dfrac{\mu^i}{1 + \mu^i}$.

2. The time series $(X_t; t \in \mathbb{N})$ is deduced here from a process $(V_t; t \in \mathbb{R}^+)$ in continuous time with values in $\{0, 1\}$. Let us put:

$$X_t = \begin{cases} 1, & \text{if } \ V_s = 1 : s \in [t, t+1[\,, \\ 0, & \text{elsewhere,} \end{cases} \tag{4}$$

for $t = 0, 1, 2, \ldots$.

According to Green [7], we make the hypothesis that $(V_t; t \in \mathbb{R}^+)$ is a stationary alternating renewal process in continuous time corresponding to the sojourn time laws in the states 0 and 1 which are exponential laws of the respective parameters $\alpha^0$ and $\alpha^1$.

**Proposition 3.** *The binary time series $(X_t; t \geq 0)$ defined by (4) is a stationary alternating renewal process with the parameter sojourn time laws $\left(p^0, p^1\right)$, where the law $p^1$ is the geometrical law with the parameter $1 - e^{-\alpha^1}$ and the law $p^0$ is defined by*

$$p_n^0 = C_1 \lambda_1^n \left(1 - \lambda_1\right) + C_2 \lambda_2^n \left(1 - \lambda_2\right); \qquad n \geq 1,$$

*where*

$$\lambda_i = \frac{1}{2} \left\{1 - b + (-1)^{i-1} \sqrt{(1-b)^2 + 4(b-a)}\right\},$$

$$C_i = \frac{(-1)^{i-1}}{\sqrt{(1-b)^2 + 4(b-a)}} \left[ 1 - b - \lambda_i + \frac{b-a}{1-e^{-\alpha^1}} \right]; \qquad i = \overline{1,2}$$

*and*

$$a = \frac{\alpha^0 e^{-\alpha^1}}{\alpha^0 + \alpha^1} \left( 1 - e^{-(\alpha^0 + \alpha^1)} \right), \qquad b = e^{-\alpha^1} \left( 1 - e^{-\alpha^0} \right).$$

P r o o f. We can easily deduce from the hypothesis concerning $(V_s; s \in \mathbb{R}^+)$ that $(X_t; t \in \mathbb{N})$ is a stationary and alternating renewal process. According to Green's results the sequences of persistences in the states 1 and 0 for the process $(X_t; t \in \mathbb{N})$ are respectively constant equal to $e^{-\alpha^1}$ and given by

$$q_1^0 = 1 - b + \frac{b-a}{1-e^{-\alpha^1}}, \qquad q_n^0 = 1 - b + \frac{b-a}{q_{n-1}^0}, \quad n \geq 2. \tag{5}$$

Thus, we can immediately deduce that the sojourn time law in the state 1 is geometrical with parameter $1 - e^{-\alpha^1}$. To precisely define the sojourn time law in the state 0, we observe that

$$p_1^0 = 1 - q_1^0 \text{ and } p_n^0 = q_1^0 \ldots q_{n-1}^0 \left( 1 - q_n^0 \right) : n \geq 2,$$

or again

$$p_n^0 = \Delta_{n-1} - \Delta_n : n \geq 1 \tag{6}$$

if $\Delta_0 = 1$ and $\Delta_n = q_1^0 \ldots q_n^0; n \geq 1$.
Also according to (5), we have

$$\begin{cases} \Delta_0 = 1, \quad \Delta_1 = 1 - b + \dfrac{b-a}{1-e^{-\alpha^1}}, \\ \Delta_n = (1-b) .\Delta_{n-1} + (b-a)\Delta_{n-2}; \quad n \geq 2. \end{cases} \tag{7}$$

The equation (7) is a two term linear recurrence equation, and with constant coefficients. The solution is provided by

$$\Delta_n = C_1 \lambda_1^n + C_2 \lambda_2^n; \quad n \geq 0,$$

where $\lambda_1$ and $\lambda_2$ are the roots of the equation

$$\lambda^2 - (1-b)\lambda - (b-a) = 0$$

and $C_1$ and $C_2$ are the solutions of the system

$$\begin{cases} C_1 + C_2 = 1 \\ C_1 \lambda_1 + C_2 \lambda_2 = 1 - b + \dfrac{b-a}{1-e^{-\alpha^1}} \end{cases}.$$

We shall find for $\lambda_1, \lambda_2, C_1$ and $C_2$ the values already provided in the Proposition 3. Using the relations (6) we obtain the form announced for the $p^0$ law. $\square$

**Remark:** Let us note that $b > a$ (when $\alpha^0$ and $\alpha^1$ are positive) and $0 < -\lambda_2 < \lambda_1 < 1$. We then deduce that the sequence $\left(q_n^0; n \geq 1\right)$ of the persistences in the state 0 converges to the constant $\lambda_1$. Moreover and according to Hardy and Wright's results [9], the partial sequences $\left(q_{2n}^0; n \geq 1\right)$ and $\left(q_{2n+1}^0; n \geq 0\right)$ are respectively monotonously increasing and monotonously decreasing and $q_{2n}^0 \leq q_{2n+1}^0, n \geq 1$. Thus, the persistence in state 0 varies about the limit value $\lambda_1$.

**Corollary 1.** *We have*

$$p = P\left(X_t = 1\right) = \frac{\alpha^0 e^{-\alpha^1}}{\alpha^0 + \alpha^1} e^{-\alpha^1}, \quad t \in \mathbb{N}.$$

*The mean and the variance of the sojourn time law in state* 1 *(resp.* 0*) are given by:* $m_1 = \dfrac{1}{1 - e^{-\alpha^1}}$*;* $\sigma_1^2 = \dfrac{e^{-\alpha^1}}{\left(1 - e^{-\alpha^1}\right)^2}$ *(resp.* $m_0 = \dfrac{\alpha^1 + \alpha^0\left(1 - e^{-\alpha^1}\right)}{\alpha^0 e^{-\alpha^1}\left(1 - e^{-\alpha^1}\right)}$*;* $\sigma_0^2 = C_1.\dfrac{1 + \lambda_1}{\left(1 - \lambda_1\right)^2} + C_2.\dfrac{1 + \lambda_2}{\left(1 - \lambda_2\right)^2} - m_0^2$*, where* $\lambda_1$*,* $\lambda_2$*,* $C_1$ *and* $C_2$ *are the constants given in Proposition* 3*).*

In this case, we can write the function of covariance $\gamma_X(.)$ of the process $(X_t; t \in \mathbb{N})$ as:

$$\gamma_X(h) = \begin{cases} \dfrac{\alpha^0 \alpha^1}{\left(\alpha^0 + \alpha^1\right)^2} e^{-(h+2)\alpha^1 - h\alpha^0}, & \text{if} \quad h > 0 \\ \dfrac{\alpha^0 e^{-\alpha^1}}{\left(\alpha^0 + \alpha^1\right)^2}\left(\alpha^0 + \alpha^1 - \alpha^0 e^{-\alpha^1}\right), & \text{if} \quad h = 0. \end{cases}$$

**2. Statistical study.** Let us suppose we observe the trajectory of a stationary alternating renewal process with sojourn time laws $p^0$ and $p^1$ having means $m_0$ and $m_1$ and variances $\sigma_0^2$ and $\sigma_1^2$.

Concerning the unbiased estimator $\dfrac{S_n^1}{n+1}$ of $p = \dfrac{m_1}{m_0 + m_1}$, we have:

**Proposition 4.** *The statistics* $\dfrac{S_n^1}{n+1}$ *is a weakly consistent estimator of*

$\dfrac{m_1}{m_0+m_1}$, *and the sequence* $\left\{\sqrt{n}\left(\dfrac{S_n^1}{n+1}-\dfrac{m_1}{m_0+m_1}\right)\right\}_{n\geq 1}$ *converges in distribution to zero-mean gaussian variable with variance* $\sigma^2=\dfrac{m_1^2\sigma_0^2+m_0^2\sigma_1^2}{(m_0+m_1)^2}$.

P r o o f. We can write

$$S_n^1=\sum_{l=1}^{Y_n^1}\Delta_l^1+\Delta_n.\mathbf{1}_{\{X_0=1\}}+T_n.\mathbf{1}_{\{X_n=1\}},$$

and as $\Delta_l^1=D_{l+1}^1;(1\leq l\leq q-1)$, and $Y_n^1=q-1$ if

$$\{X_0=1\}\cap\left\{\sum_{t=1}^{n}(1-X_t).X_{t-1}+\sum_{t=1}^{n}X_t(1-X_{t-1})=2q\right\},$$

then conditioned on this event we have

$$S_n^1=\begin{cases} D_1^1+\ldots+D_q^1, & \text{if} \quad T_q^0\leq n<T_{q+1}^1; q=1,2,\ldots\\ D_1^1+\ldots+D_q^1+n-T_{q+1}^1, & \text{if} \quad T_{q+1}^1\leq n<T_{q+1}^0; q=0,1\ldots\end{cases}$$

Let then $\varepsilon$ be a positive random variable which, conditioned on $X_0=1$, follows the law $p_\varepsilon$ so that $\hat{p}^1*p_\varepsilon=p^1$, and is independent of $D_{j+1}^1$ and $D_j^0$; $j\geq 1$. Let us define $\tilde{S}_n^1=S_n^1+\varepsilon$.

According to the central limit theorem (Takács [14], Renyi [13]), the sequence $\left\{\sqrt{n}\left(\dfrac{\tilde{S}_n^1}{n+1}-\dfrac{m_1}{m_0+m_1}\right)\right\}_{n\geq 1}$ conditioned on $X_0=1$, converges in distribution to zero mean gaussian variable with variance $\sigma^2$. As $\sqrt{n}\dfrac{\tilde{S}_n^1-S_n^1}{n+1}=\sqrt{n}\dfrac{\varepsilon}{n+1}$ converges to 0 almost surely when $n\to+\infty$, then the same result is true for the sequence $\left\{\sqrt{n}\left(\dfrac{S_n^1}{n+1}-\dfrac{m_1}{m_0+m_1}\right)\right\}_{n\geq 1}$. In an analogous manner, we show that the same tendency occurs conditioned on $X_0=0$. Consequently, the result of the proposition is demonstrated. □

Buishand has precised the asymptotic behavior of the variance $\sigma^2$ of $S_n^1$ when the moments of order 3 of the sojourn time laws exist. We have

$$\begin{aligned} var\left(S_n^1\right) &= (n+1).\frac{m_1^2\sigma_0^2+m_0^2\sigma_1^2}{(m_0+m_1)^3}+\frac{\left(m_1\sigma_0^2-m_0\sigma_1^2\right)^2}{2(m_0+m_1)^4}-\frac{m_1^2\mu_{0,3}+m_0^2\mu_{1,3}}{3(m_0+m_1)^3}\\ &\quad+\frac{2m_1m_0+m_0^2m_1^2}{6(m_0+m_1)^2}+o(1), \end{aligned}$$

where $\mu_{i,3}$ is the centered moment of the order 3 of the law $p^i; i = 0, 1$.

It follows in particular that $\frac{S_n^1}{n+1}$ converges in mean-square to $\frac{m_1}{m_0+m_1}$.

**Estimation of the moments of sojourn time laws.** Following this introductory result, we now propose to estimate the means $m_0$ and $m_1$ of the sojourn time laws $p^0$ and $p^1$.

**Proposition 5.** *The statistics* $\frac{S_n^1}{(n+1)\,.\theta_n}$ *(resp.* $\frac{n+1-S_n^1}{(n+1)\,.\theta_n}$*),* $\theta_n \in \left\{\frac{N_n^{01}}{n}, \frac{N_n^{10}}{n}, \frac{Y_n^1}{n}, \frac{Y_n^0}{n}\right\}$ *are estimators of* $m_1$ *(resp.* $m_0$*). These estimators are weakly consistent.*

P r o o f. We only have to prove that the variable $\theta_n$ is a weakly consistent estimator of $\frac{1}{m_0+m_1}$.

Firstly, it is obvious that the variables $\frac{N_n^{01}}{n} = \frac{1}{n}\sum_{i=1}^{n}(1-X_{t-1})\,.X_t$ and $\frac{N_n^{10}}{n} = \frac{1}{n}\sum_{i=1}^{n}X_{t-1}.\,(1-X_t)$, are estimators of $\frac{1}{m_0+m_1}$, and similarly for the variables $\frac{Y_n^0}{n}$ and $\frac{Y_n^1}{n}$. In addition, these estimators are convergent in probability. For example, the convergence of the estimator $\frac{Y_n^0}{n}$ is a consequence of Takács's theorem (Renyi [13]). The conditions of validity for this theorem are found in the renewal hypothesis, conditioned on $X_0 = 1$ where $X_0 = 0$, this added to the fact that

$$T^1_{Y_n^0+1} \leq n < T^1_{Y_n^0+2}$$

exception made on the event $\{X_0 = \ldots = X_n = 0\}$ (the probability of which tends to 0). The convergence in probability of the variable $Y_n^1$ can be demonstrated in the same way and the convergence of the variable $N_n^{1-i,i}$ follows from the fact that the variable $Y_n^i - N_n^{1-i,i}$ is bounded. □

The expression of the likelihood function leads to estimations of the $p^0$ and $p^1$ parameters from the approximate log-likelihood function

$$\sum_{k=1}^{Y_n^0}\log\left(p^0_{\Delta_k^0}\right) + \sum_{l=1}^{Y_n^1}\log\left(p^1_{\Delta_l^1}\right)$$

which suggests to get back to a problem of two estimations: the estimation of the law parameter $p^0$ and the estimation of the law parameter $p^1$.

Then we consider the problem of sequential estimation of the mean $m_i$ and the variance $\sigma_i^2$ of the sojourn time law $p^i$ in the state $i : i \in \{0,1\}$. Regarding the observation to the random instant $Y_n^i$ of the set $\Delta_1^i, \ldots, \Delta_k^i, \ldots$ of the random variables independent and identically distributed according to the $p^i$ law and taking into account that $\frac{Y_n^i}{n}$ is an estimator which converges in probability to $\frac{1}{m_0+m_1}$, we deduce from the Anscombe's theorem [2] the following result:

**Lemma 2.** *Let us suppose that $p^i$ law admits moments of order up to 4. Then the sequences* $\left\{\frac{1}{\sqrt{Y_n^i}} \sum_{k=1}^{Y_n^i} \frac{\Delta_k^i - m_i}{\sigma_i}\right\}_{n\geq 1}$ *and* $\left\{\frac{1}{\sqrt{Y_n^i}} \sum_{k=1}^{Y_n^i} \frac{\left(\Delta_k^i\right)^2 - E\left(\Delta_1^i\right)^2}{\sqrt{var\left(\Delta_1^i\right)^2}}\right\}_{n\geq 1}$ *converge in distribution to a normal random variable which is centered and reduced.*

Then we have:

**Corollary 2.** *If the law $p^i$ admits moments of order up to 4, the estimators* $\overline{m}_{i,n} = \frac{1}{\sqrt{Y_n^i}} \sum_{k=1}^{Y_n^i} \Delta_k^i$ *and* $\overline{\sigma}_{i,n}^2 = \frac{1}{\sqrt{Y_n^i}} \sum_{k=1}^{Y_n^i} \left(\Delta_k^i - \overline{m}_{i,n}\right)^2$ *of $m_i$ and $\sigma_i^2$ are weakly consistent, and the sequence $\{\sqrt{n}\left(\overline{m}_n - m\right); n \geq 1\}$ converges in distribution to a gaussian random variable which is centered and having as variance $\sigma_i^2 . \left(m_0 + m_1\right)$.*

Proof. The results of convergence in probability of $\frac{Y_n^i}{n}$ and $S_n^1$ assure the convergence in probability of $\frac{1}{Y_n^i} \sum_{k=1}^{Y_n^i} \Delta_k^i$ to $m_i$ and $\frac{1}{Y_n^i} \sum_{k=1}^{Y_n^i} \left(\Delta_k^i\right)^2$ to $E\left(\Delta_1^i\right)^2 = m_i^2 + \sigma_i^2$. Then $\frac{1}{Y_n^i} \sum_{k=1}^{Y_n^i} \left(\Delta_k^i - \overline{m}_{i,n}\right)^2$ converges to $\sigma^2$. As $\sqrt{\frac{n}{Y_n^i}}$ converges in probability to $\sqrt{m_0 + m_1}$, we obtain the convergence in distribution of the sequence $\{\sqrt{n}\left(\overline{m}_n - m\right); n \geq 1\}$. $\square$

**Examples**

**1.** If the sojourn time laws $p^i; i \in \{0,1\}$ are translated negative binomial laws with parameters $\left(\mu^i, r^i\right)$, then this law has as mean $m_i + 1$ and as variance $\sigma_i^2 = \mu_i . \left(\frac{\mu_i}{r_i} + 1\right)$, $i \in \{0,1\}$. The results concerning the estimators of $p =$

$\dfrac{\mu_i+1}{\mu_0+\mu_1+2}$, $m_i$, $\sigma_i^2, i \in \{0,1\}$, are applied here again; it is easy to notice that the conditions validating these results are satisfied.

Now let us figure the estimation problem with the parameters $\left(\mu^i, r^i\right)$, $i = 0, 1$. We shall get back once more to an estimation problem consisting in the estimation of the parameters $\left(\mu^0, r^0\right)$ of the $p^0$ law and the estimation of the parameters $\left(\mu^1, r^1\right)$ of the $p^1$ law. For ease of presentation, the index $i$ identifying the law, its parameters and the corresponding observations will be omitted in the following.

The log-likelihood function which corresponds to the observation $(\delta_1, \ldots, \delta_n)$ of an independent sample $(\Delta_1, \ldots, \Delta_n)$ of a translated negative binomial law with the parameter $(\mu, r)$ is:

$$l\left(\delta_1, \ldots, \delta_n; \mu, r\right) = -\sum_{j=1}^{+\infty} n_j.r.\log\left(1+\frac{\mu}{r}\right) + \sum_{j=1}^{+\infty} n_j.\left(j-1\right).\log\left(\frac{\mu}{\mu+r}\right)$$

$$+\sum_{j=3}^{+\infty} n_j.\left\{\log\left(r\right)+\log\left(r+1\right)+\ldots+\log\left(r+j-2\right)-\log\left(\left(j-2\right)!\right)\right\},$$

where $n_j$ is the number of $\delta_k : 1 \leq k \leq n$, equal to $j$ and $n = \sum\limits_{j=1}^{+\infty} n_j$.

To estimate $\mu$ and $r$ by the maximum likelihood method, we look then for $(\mu, r)$ solution of

$$\begin{cases} \left(-\sum\limits_{j=1}^{+\infty} n_j\right).\dfrac{r}{\mu+r} + \sum\limits_{j=1}^{+\infty} n_j.\left(j-1\right).\dfrac{r}{\mu\left(\mu+r\right)} = 0 \\ -\sum\limits_{j=1}^{+\infty} n_j.\left[\log\left(1+\dfrac{\mu}{r}\right) - \dfrac{\mu}{\mu+r} + \dfrac{j-1}{\mu+r}\right] + \sum\limits_{j=3}^{+\infty} n_j.\sum\limits_{l=0}^{j-2}\dfrac{1}{r+l} = 0. \end{cases}$$

The first equation will then give

$$\hat{\mu} = \frac{1}{n}\sum_{j=1}^{+\infty} j.n_j - 1 = \frac{1}{n}\sum_{j=1}^{+\infty} j.\delta_j - 1$$

and for $\mu = \hat{\mu}$, the likelihood maximum estimator $\hat{r}$ of $r$ is the solution of the equation

$$n.\log\left(1+\frac{\mu}{r}\right)=\sum_{j=3}^{+\infty}n_j.\left(\frac{1}{r}+\frac{1}{r+1}+\ldots+\frac{1}{r+j-2}\right).$$

This equation has a single root if $s_n^2 > \overline{\delta}_n$ (Anscombe [1] for the existence, and Bonitzer [3] for the uniqueness), where $\overline{\delta}_n$, $s_n^2$ designate respectively the empirical mean and the variance of $(\delta_1,\ldots,\delta_n)$.
However, we have to use frequently iterative methods to calculate $\hat{r}$. Also, in order to suggest explicit estimators of $(\mu, r)$, we use the moments' method which provides:

$$\hat{\mu}_n=\frac{1}{n}\sum_{j=1}^{n}\Delta_j-1=\overline{\Delta}_n-1, \qquad \hat{r}_n=\frac{\hat{\mu}_n^2}{S_n^2-\hat{\mu}_n},$$

with $S_n^2=\frac{1}{n}\sum_{j=1}^{n}\left(\Delta_j-(\hat{\mu}+1)\right)^2$.

We immediatly deduce from Corollary 2 the following result:

**Corollary 3.** *The estimators $\overline{\mu}_{i,n}$ and $\overline{r}_{i,n}$ of $\mu_i$ and $r_i$ respectively $(i\in\{0,1\})$, are convergent in probability, and the sequence $\left\{\sqrt{n}\left(\overline{\mu}_{i,n}-\mu_i\right);n\geq 1\right\}$ converges in distribution to a centered gaussian random variable having variance $\mu_i\left(\mu_0+\mu_1+2\right).\left(1+\frac{\mu_i}{r_i}\right);i\in\{0,1\}$.*

**2.** Estimation of some characteristics of the model (4). The results concerning the estimators of $p=\dfrac{\alpha^0.e^{-\alpha^1}}{\alpha^0+\alpha^1}$, $m_i,\sigma_i^2;i\in\{0,1\}$ are again applicable here.

We now consider the estimation problem with the parameters $\alpha^0$ and $\alpha^1$. The expressions of $m_1$ and $m_2$ depending of $\alpha^0$ and $\alpha^1$ lead to the estimators

$$\overline{\alpha}_n^1=-\log\left(1-\frac{1}{\overline{m}_{1,n}}\right) \quad\text{and}\quad \overline{\alpha}_n^0=\frac{\overline{\alpha}_n^1.\overline{m}_{1,n}}{\overline{m}_{0,n}.e^{-\overline{\alpha}_n^1}-1}$$

of $\alpha^1$ and $\alpha^0$ respectively, where $m_{i,n}=\dfrac{1}{Y_n^i}\displaystyle\sum_{k=1}^{Y_n^i}\Delta_k^i;\quad i\in\{0,1\}$.

We immediatly deduce from corollary 1 the following result:

**Corollary 4.** *The estimators $\overline{\alpha}_n^1$ and $\overline{\alpha}_n^0$ of $\alpha^0$ and $\alpha^1$ are convergent in probability.*

We can build the estimation of $\alpha^0$ and $\alpha^1$ from the estimation of the transition probabilities

$$p_{11} = P\left(X_t = 1/X_0 = 1\right) \quad \text{and} \quad p_{01} = P\left(X_t = 1/X_0 = 0\right),$$

which are respectively given by

$$p_{11} = e^{-\alpha^1} \quad \text{and} \quad p_{01} = \frac{\alpha^0.e^{-\alpha^1}\left(1 - e^{-\alpha^1}\right)}{\alpha^1 + \alpha^0\left(1 - e^{-\alpha^1}\right)}.$$

The estimators of $p_{11}$ and $p_{01}$ are given by:

$$\hat{p}_{11}^{(n)} = \frac{N_n^{11}}{N_n^{01} + N_n^{11}} = \frac{U_n}{S_n^1 - x_n}; \qquad \hat{p}_{01}^{(n)} = \frac{N_n^{01}}{N_n^{00} + N_n^{01}} = \frac{S_n^1 - U_n - x_0}{n - S_n^1 + x_n};$$

where $U_n = \sum_{t=1}^{n} X_t.X_{t-1}$, are the estimators for the maximum of approximate likelihood in the case of first order markov chains. Here, of course the process is not markovian but, taking into account that the $p^1$ law is geometrical (let us recall $0 < -\lambda_2 < \lambda_1 < 1$), it seems reasonable to use these estimators. We are thus led to the estimators

$$\hat{\alpha}_n^1 = -\log \hat{p}_{11}^n \quad \text{and} \quad \hat{\alpha}_n^0 = -\frac{\hat{p}_{01}^{(n)}.\log \hat{p}_{11}^{(n)}}{\left(\hat{p}_{11}^n - \hat{p}_{01}^n\right)\left(1 - \hat{p}_{11}^{(n)}\right)};$$

for $\alpha^0$ and $\alpha^1$.

It follows immediately the result below:

**Corollary 5.** *The estimators $\hat{\alpha}_n^1$ and $\hat{\alpha}_n^0$ of $\alpha^0$ and $\alpha^1$ are convergent in probability.*

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