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FIRST ORDER CHARACTERIZATIONS OF PSEUDOCONVEX FUNCTIONS

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Abstract. First order characterizations of pseudoconvex functions are investigated in terms of generalized directional derivatives. A connection with the invexity is analysed. Well-known first order characterizations of the solution sets of pseudolinear programs are generalized to the case of pseudoconvex programs. The concepts of pseudoconvexity and invexity do not depend on a single definition of the generalized directional derivative.

1. Introduction. Three characterizations of pseudoconvex functions are considered in this paper. The first is new. It is well-known that each pseudoconvex function is invex. Then the following question arises: what is the type of

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the function η from the definition of invexity, when the invex function is pseudoconvex. This question is considered in Section 3, and a first order necessary and sufficient condition for pseudoconvexity of a function is given there. It is shown that the class of strongly pseudoconvex functions, considered by Weir [25], coincides with pseudoconvex ones.

The main result of Section 3 is applied to characterize the solution set of a nonlinear programming problem in Section 4. The base results of Jeyakumar and Yang in the paper [13] are generalized there to the case, when the function is pseudoconvex.

The second and third characterizations are considered in Sections 5, 6. In many cases the pseudoconvex functions are quasiconvex. It is interesting when a quasiconvex function is pseudoconvex. Is there any quasiconvex and invex function, which is not pseudoconvex? These questions in the case of differentiable functions defined on the space $\mathbb{R}^n$ are discussed in the papers [6, 9, 16, 11]. Crouzeix and Ferland [6] gave a first order necessary and sufficient condition for pseudoconvexity of a quasiconvex function. Giorgi [9] showed that the intersection of the sets of quasiconvex differentiable functions and invex differentiable ones is the set of pseudoconvex functions. Komlósi [16] analysed the connection of this problem with the pseudolinear functions. Giorgi and Thierfeldar [11] proved in a very short way the sufficient condition of pseudoconvexity. In their papers Komlósi [15] considered the case of Dini directional derivatives, Tanaka [24] obtained that a quasiconvex and invex function, which is locally Lipschitz and regular in the sense of Clarke [5], is pseudoconvex with respect to the Clarke's generalized directional derivative. Aussel [3] analysed the second characterization with the help of the abstract subdifferential, introduced by Aussel, Corvellec and Lassonde [2]. In the most of these papers is used the property that if $f(y) < f(x)$, then the derivative of f at the point x in the direction $y - x$ is nonpositive. In Section 6 of our paper this property is replaced by weaker. As a consequence of the main result of Sections 5, 6 quasiconvex functions cannot be characterized in a similar way like pseudoconvex ones.

A similar characterization of the pseudomonotone generalized directional derivatives to the characterization of Section 3 is considered in Section 7.

Our analysis does not depend on a single definition of the directional derivative. The cases, when the generalized directional derivatives are the Pschenichnyi quasiderivative, the Clarke's, Dini, Dini-Hadamard (sometimes called contingent), Michel-Penot derivatives, are considered especially.

2. Preliminaries. Throughout this work $\mathbf{E}$ is a given real topological vector space, $X \subset \mathbf{E}$ is an open set, and $S \subset X$ is a convex set. The (topological) dual space of $\mathbf{E}$ is denoted by $\mathbf{E}^*$, and the canonical pairing between $\mathbf{E}$ and $\mathbf{E}^*$ by $\langle \cdot, \cdot \rangle$. The set of the reals we denote by $\mathbb{R}$, and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ is its extension with the two infinite elements $-\infty$ and $+\infty$. The algebraic operations and limits with infinite elements are defined as usually in the convex analysis.

Consider a given function $f : X \to \mathbb{R}$. Suppose that $h(x, u)$ is a generalized directional derivative of f at the point x in the direction u. The function $h(x, u)$ may be considered as a bifunction $h : X \times \mathbf{E} \to \overline{\mathbb{R}}$.

Recall the following concepts.

The point $x \in X$ is said to be *stationary* with respect to h if $h(x, u) \geq 0$ for all $u \in \mathbf{E}$.

The function $f : X \to \mathbb{R}$ is said to be *quasiconvex* on S if

$$f(x + t(y - x)) \leq \max \{f(x), f(y)\}, \quad \text{whenever} \quad x, y \in S \quad \text{and} \quad 0 \leq t \leq 1.$$

The following cone is connected to f at each fixed point $x \in S$:

$$\mathcal{N}(x) = \{\xi \in \mathbf{E}^* \mid y \in S,\ f(y) \leq f(x) \quad \text{imply} \quad \langle \xi, y - x \rangle \leq 0\}.$$

Actually, this is the normal cone to the sublevel set $L_{f(x)} = \{y \in S \mid f(y) \leq f(x)\}$ at x. Since f is quasiconvex, then $L_{f(x)}$ is convex.

The notion of pseudoconvexity for differentiable functions is introduced by Mangasarian in 1965. Without differentiability assumptions in our case it might look so: the function f, is said to be *pseudoconvex* (*strictly pseudoconvex*) with respect to h on S if

$$x,\ y \in S,\ f(y) < f(x)\ (x, y \in S,\ x \neq y,\ f(y) \leq f(x)) \quad \text{imply} \quad h(x, y - x) < 0.$$

The notion of invexity for differentiable functions is introduced by Hanson [12]. The function $f : X \to \mathbb{R}$ is said to be *invex* with respect to h on S if there exists a mapping $\eta : S \times S \to \mathbf{E}$ such that

$$f(y) - f(x) \geq h(x, \eta) \quad \text{for all} \quad x, y \in S. \tag{1}$$

The notion of subdifferential issues from the convex analysis, and may be applied to our directional derivative h. Each continuous linear functional ξ over $\mathbf{E}$ satisfying $\langle \xi, u \rangle \leq h(x, u)$ for all $u \in \mathbf{E}$ is said to be a *subgradient* of f with respect to h at x. The set of all subgradients $\partial f(x)$ at x is called the *subdifferential* of f at x. $\partial f(x)$ is (possibly empty) closed convex subset of $\mathbf{E}$.

3. A characterization of pseudoconvex functions. In this section we assume that the derivative $h(x,u)$ satisfies for all distinct $x,\ y \in S$, $u \in \mathbf{E}$ the properties:

1. *$h(x,u) < \infty$.*

2. *If f is pseudoconvex on S, then $f(y) = f(x)$ implies $h(x, y-x) \le 0$.*

The following two theorems gives necessary and sufficient conditions for pseudoconvexity and strict pseudoconvexity.

Theorem 3.1. *Suppose that the derivative h satisfies Properties 1, 2. Then $f : X \to \mathbb{R}$ is pseudoconvex on S if and only if there exists a positive function $p : S \times S \to \mathbb{R}$ such that*

$$f(y) - f(x) \ge p(x,y)\, h(x, y-x) \quad \text{for all} \quad x,\ y \in S. \tag{2}$$

P r o o f. The sufficiency is obvious. We shall prove the necessity. Let f be pseudoconvex. If $h(x, y-x) = -\infty$, then the inequality (2) is obvious. Assume that $h(x, y-x) > -\infty$. We construct explicitly the function p in the following way.

$$p(x,y) = \begin{cases} \dfrac{f(y)-f(x)}{h(x,y-x)}, & \text{if} \quad f(y) < f(x) \quad \text{or} \quad h(x,y-x) > 0, \\ 1, & \text{otherwise.} \end{cases}$$

The function $p(x,y)$ is well defined, strictly positive, and it satisfies inequality (2). Indeed, if $f(y) < f(x)$, then $h(x, y-x) < 0$ by pseudoconvexity. If $h(x, y-x) > 0$, then $f(y) \ge f(x)$ because of the pseudoconvexity again. Let $f(y) = f(x)$ be possible. According to Property 2, $h(x, y-x) \le 0$, which is a contradiction. Hence $f(y) - f(x) > 0$. □

Theorem 3.2. *Assume that the derivative h satisfies Property 1. Then $f : X \to \mathbb{R}$ is strictly pseudoconvex on S if and only if there exists a positive function $p : S \times S \to \mathbb{R}$ such that*

$$f(y) - f(x) > p(x,y)\, h(x, y-x) \quad \text{for all} \quad x, y \in S.$$

P r o o f. The proofs of this and previous theorems are similar. A function which satisfies the inequality is

$$p(x,y) = \begin{cases} 2\dfrac{f(y)-f(x)}{h(x,y-x)}, & \text{if} \quad f(y) < f(x), \\ \dfrac{1}{2}\dfrac{f(y)-f(x)}{h(x,y-x)}, & \text{if} \quad h(x,y-x) > 0, \\ 1, & \text{otherwise.} \end{cases}$$

□

We shall consider some examples of generalized directional derivatives h.

Example 3.1. *Suppose that f is Frèchet differentiable. Let $h(x,u)$ be the Frèchet directional derivative of f at x in the direction u. This is the classical case, and our pseudoconvex functions coincide with the pseudoconvex ones, introduced by Mangasarian* [19]. *All the properties are fulfilled.*

A Frèchet differentiable function, which satisfies inequality (2) *in the finite-dimensional case is called by T. Weir* [25] *strongly pseudoconvex. But as we see this class of functions coincides with the differentiable pseudoconvex ones.*

Example 3.2. *Let $h(x,u)$ be the upper or lower Dini directional derivative. When f is radially lower semicontinuous, Property* 2 *is fulfilled, since each radially lower semicontinuous pseudoconvex function is quasiconvex* [10, Theorem 3.5], *and for radially lower semicontinuous functions quasiconvexity is equivalent to the following implication $f(y) \le f(x)$ imply $h(x,y-x) \le 0$ if h is the upper or lower Dini derivative* [8, Theorem 4]. *When the function is locally Lipschitz, then the Dini derivatives are finite-valued.*

Remark 3.1. Suppose that $f(x)$ is a given pseudoconvex function, and $h(x,u)$ is its generalized directional derivative such that there exists a positive function p, satisfying (2). Let $g(x,u)$ be another smaller generalized derivative of f, that is $h(x,u) \ge g(x,u)$ for all $x \in S$, $u \in \mathbf{E}$. It is obvious that g fulfills (2) with the same function p. Using this fact we may construct other examples of generalized derivatives, which are smaller than the Dini derivatives and satisfies Theorems 3.1, 3.2: lower Dini-Hadamard, lower Clarke's, lower Michel-Penot directional derivatives.

4. Characterizations of the solution sets. Consider the global minimization problem

$$(P) \qquad \min\ f(x), \quad \text{subject to} \quad x \in S.$$

As an application of Theorem 3.1 we give first-order characterizations of the solution set of the program (P) in terms of any its minimizer. Characterizations of solutions sets are useful for understanding the behavior of solution methods for programs that have multiple optimal solutions.

Denote by $\overline{S}$ the solution set $\arg\min\,\{f(x) \mid x \in S\}$, and let it be nonempty. Suppose that the following property is satisfied in this section.

3 (Fermat rule). *If $f(\overline{x}) = \min\,\{f(x) \mid x \in S\}$, then $h(\overline{x}, x-\overline{x}) \geq 0$ for all $x \in S$.*

The following lemma is a trivial generalization of a well-known property of the Frèchet differentiable functions [19, Theorem 9.3.3].

Lemma 4.1. *Let $f : X \to \mathbb{R}$ be pseudoconvex on S, and Property 3 be satisfied. Then $\overline{S}$ coincides with the set $S^* = \{z \in S \mid h(z, x-z) \geq 0$ for all $x \in S\}$.*

P r o o f. The inclusion $\overline{S} \subset S^*$ follows from Property 3, and the opposite inclusion is a consequence of the definition of pseudoconvexity. □

The following theorem and corollary are given in the work of Jeyakumar and Yang [13, Theorem 3.1 and Corollary 3.1] in the case when f is a pseudolinear Frèchet differentiable function.

Theorem 4.1. *Assume that $f : X \to \mathbb{R}$ is pseudoconvex on S, and Properties 1, 2, 3 are satisfied. Let $\overline{x}$ be any fixed point of $\overline{S}$. Then $\overline{S} = \tilde{S} \subset \hat{S}$, where $\tilde{S} = \{z \in S \mid h(z, \overline{x}-z) = 0\}$, and $\hat{S} = \{z \in S \mid h(\overline{x}, z-\overline{x}) = 0\}$.*

P r o o f. To show the inclusion $\overline{S} \subset \tilde{S}$ suppose that z is an arbitrary point of $\overline{S}$. By Theorem 3.1 there exists $p > 0$ such that

$$0 = f(\overline{x}) - f(z) \geq p\, h(z, \overline{x}-z).$$

It follows from Lemma 4.1 that $h(z, \overline{x}-z) = 0$. Therefore $z \in \tilde{S}$. The proof of the statement $\overline{S} \subset \hat{S}$ uses the same arguments.

To prove the inclusion $\tilde{S} \subset \overline{S}$ suppose that z is an arbitrary point of $\tilde{S}$, but $z \notin \overline{S}$. Therefore $f(\overline{x}) < f(z)$. By pseudoconvexity $h(z, \overline{x}-z) < 0$, which contradicts the assumption $z \in \tilde{S}$. □

Corollary 4.1. *Let all the hypotheses of Theorem* 4.1 *be satisfied. Then* $\overline{S} = \tilde{S}_1$, *where* $\tilde{S}_1 = \{z \in S \mid h(z, \overline{x} - z) \geq 0\}$.

Proof. The inclusion $\overline{S} \subset \tilde{S}_1$ is a corollary of the theorem. The opposite inclusion follows from the definition of pseudoconvexity.

5. Criterion for pseudoconvexity of a quasiconvex function. The viewpoint of the following section is to establish whether a quasiconvex function can be characterized by a similar inequality of the type (1) like pseudoconvex functions.

Let from now on $S \equiv X$. In this section we suppose that the derivative h satisfies for all $x \in X$ Property 3 and the following other properties:

4. *The set* $\partial f(x)$ *is nonempty.*

5. $h(x, u)$ *considered as a function of* u *is the support function of* $\partial f(x)$, *and* $h(x, u) = \max\{\langle \xi, u \rangle \mid \xi \in \partial f(x)\}$ *for all* $u \in \mathbf{E}$.

6. *If* f *is quasiconvex, then* $\partial f(x) \subset \mathcal{N}(x)$.

In the considered case Property 3 implies that each local minimizer is a stationary point. As a consequence of Property 5, $h(x, u)$ is positively homogeneous, subadditive function of u, and $h(x, 0) = 0$. It follows from Property 4 that $h(x, u) > -\infty$ for all $x \in X$, $u \in H$, since for all $x \in X$ there exists $\xi \in \partial f(x)$. Therefore $h(x, u) \geq \langle \xi, u \rangle > -\infty$ for all $u \in \mathbf{E}$.

Properties 4, 5, 6 are stronger than Properties 1, 2. To prove this fact, we need of the following statements.

Proposition 5.1. *Each pseudoconvex function* $f : X \to \mathbb{R}$, *which satisfies Property* 4, *is quasiconvex.*

Proof. Assume in the contrary that there exist x, $y \in X$ and $z = x + t(y - x)$, $t \in (0, 1)$ with $f(z) > \max\{f(x), f(y)\}$. By Property 4, there exists $\xi \in \partial f(z)$. Hence, by pseudoconvexity

$$t\langle \xi, x - y \rangle = \langle \xi, x - z \rangle \leq h(z, x - z) < 0$$

$$(1 - t)\langle \xi, y - x \rangle = \langle \xi, y - z \rangle \leq h(z, y - z) < 0.$$

These inequalities contradict each other. □

Lemma 5.1. *Let the function $f : X \to \mathbb{R}$ satisfy Properties 4, 5. Then Property 6 is equivalent to the following implication:*

(3) *If f is quasiconvex, then $x,\ y \in X,\ f(y) \le f(x)$ imply $h(x, y-x) \le 0$.*

P r o o f. Suppose that Property 6 is satisfied and $f(y) \le f(x)$. By Property 5 there exists $\xi \in \partial f(x)$ such that $h(x, y-x) = \langle \xi, y-x \rangle$. It follows from Property 6 that $\xi \in \mathcal{N}(x)$. Therefore $\langle \xi, y-x \rangle \le 0$.

Assume that implication (3) is fulfilled. To prove Property 6 suppose that $\xi \in \partial f(x)$ and $f(y) \le f(x)$. Hence $\langle \xi, y-x \rangle \le h(x, y-x) \le 0$, and $\xi \in \mathcal{N}(x)$. $\square$

Property 1 follows from Properties 4, 5. By Proposition 5.1 and Lemma 5.1 we conclude that Property 2 is a consequence of Properties 4, 5, 6.

We shall use the following lemma, which is given without proof in the book of Nesterov [21], when the space is $\mathbb{R}^n$.

Lemma 5.2. *Let $f : X \to \mathbb{R}$ be upper semicontinuous and quasiconvex. Then*

$$x,\ y \in X,\ f(y) < f(x) \quad \textit{imply} \quad \langle \xi, y-x \rangle < 0 \quad \textit{for all} \quad \xi \in \mathcal{N}(x),\ \xi \ne 0.$$

P r o o f. Assume in the contrary that there exist $x,\ y \in X,\ f(y) < f(x)$ and $\xi \in \mathcal{N}(x),\ \xi \ne 0$ satisfying $\langle \xi, y-x \rangle \ge 0$. By the definition of the cone $\mathcal{N}(x)$, we obtain $\langle \xi, y-x \rangle = 0$. Since f is upper semicontinuous, then the strict sublevel set $SL_{f(x)} = \{z \in X \mid f(z) < f(x)\}$ is open. Therefore there exists a neighborhood U_y of y such that $U_y \subset SL_{f(x)}$. There exists a neighborhood U of the origin of the space such that $U_y = y + U$. By the definition of the cone $\mathcal{N}(x)$, $\langle \xi, y+z-x \rangle \le 0$ for all $z \in U$. Hence $\langle \xi, z \rangle \le 0$ for all $z \in U$. There exists a balanced neighborhood V of the origin, that is $\lambda z \in V$ for all $z \in V$ and $\lambda \in [-1, 1]$, satisfying $V \subset U$. Since $-z \in V$ for all $z \in V$, then $\langle \xi, z \rangle = 0$ for all $z \in V$. Using the continuity of the linear operations, it follows from $0.z_1 = 0$ for all $z_1 \in \mathbf{E}$ that there exist $t \in \mathbb{R}$ and $z \in V$, which fulfill the equality $z_1 = tz$. Consequently, $\langle \xi, z_1 \rangle = 0$ for all $z_1 \in \mathbf{E}$. This is impossible according to the assumption $\xi \ne 0$. $\square$

The following theorem is a necessary and sufficient condition for pseudoconvexity of a quasiconvex function [3, 6, 9, 11, 15].

Theorem 5.1. *Assume that the derivative h satisfies Properties* 3, 4, 5 *and* 6. *Let $f : X \to \mathbb{R}$ be quasiconvex and upper semicontinuous. Then f is pseudoconvex on X if and only if the set of global minimizers coincides with the set of stationary points.*

P r o o f. *Necessity.* By Property 3 each global minimizer is a stationary point. Suppose in the contrary that there exist a stationary point x, which is not a global minimizer. Therefore there exist $y \in X$ such that $f(y) < f(x)$. By pseudoconvexity, $h(x, y - x) < 0$. This is impossible, since x is stationary.

Sufficiency. Suppose that $x,\ y \in X$ and $f(y) < f(x)$. Since x is not a global minimizer, according to the hypotheses of the theorem, x is not stationary. This means that $0 \notin \partial f(x)$. By Lemma 5.2, $\langle \xi, y - x \rangle < 0$ for all $\xi \in \mathcal{N}(x),\ \xi \neq 0$. It follows from Properties 4, 6 that $\langle \xi, y - x \rangle < 0$ for all $\xi \in \partial f(x)$. Using Property 5 we obtain that $h(x, y - x) < 0$. □

Proposition 5.2. *Suppose that the derivative h satisfies Properties* 3, 4, 5 *and* 6. *Then each pseudoconvex function $f : X \to \mathbb{R}$ is invex.*

P r o o f. The claim follows directly from Theorem 3.1. □

As a consequence of Propositions 5.1 and 5.2 we make a conclusion that the set of pseudoconvex functions is a subset of the intersection between the quasiconvex functions and the invex ones. The converse is valid, too [9, 24].

Theorem 5.2. *Assume that the derivative h satisfies Properties* 3, 4, 5 *and* 6. *If the upper semicontinuous function $f : X \to \mathbb{R}$ is quasiconvex and invex, then it is pseudoconvex.*

P r o o f. Otherwise, by Theorem 5.1, there exist a stationary point $x \in X$, which is not a global minimizer. Therefore there exists $y \in X$ with $f(y) < f(x)$. According to the invexity, there exists $\eta \in \mathbf{E}$ such that $0 > f(y) - f(x) \geq h(x, \eta) \geq 0$. Now, we obtained a contradiction. □

Example 5.1. *Let f be locally Lipschitz, and $h(x, u)$ be the upper Clarke's generalized derivative of f at x in the direction u. $\partial f(x)$ coincides with the generalized gradient of Clarke. All the properties are fulfilled* [5]*, except for Property* 6. *If f is regular, then Property* 6 *is also satisfied* [5, Corollary 1 from Theorem 2.4.7].

Example 5.2. *Let f be quasidifferentiable in the sense of Pschenichnyi* [23]*, that is there exists the usual directional derivative $f'(x, u)$ for all $x \in X,\ u \in$ $\mathbf{E}$, it is finite, $f'(x, \cdot)$ is convex, and there exists a nonempty closed convex set*

$\partial f(x)$ such that $f'(x,u) = \max\{\langle \xi, u\rangle \mid \xi \in \partial f(x)\}$ for all $x \in X$. This class of functions contains the functions, which are regular in the sense of Clarke. In this case $h(x,u) \equiv f'(x,u)$. All the properties are fulfilled.

Example 5.3. *Let $f : X \to \mathbb{R}$ be locally Lipschitz and its upper Dini derivative $f^{\uparrow}_{D}(x,u) = \limsup_{t\to+0} t^{-1}(f(x+tu) - f(x))$ be upper semicontinuous (as a function of x) for any fixed direction u. Then $f^{\uparrow}_{D}(x,u) = f^{\uparrow}_{Cl}(x,u)$ for all x and u* [7, Proposition 2.1.9]. *Therefore the upper Dini derivative satisfies Properties* 3, 4, 5. *Let's verify that Property* 6 *is fulfilled, too. Suppose that f is quasiconvex, $\xi \in \partial f(x)$, and $f(y) \le f(x)$. Hence $\langle \xi, y - x\rangle \le f^{\uparrow}_{D}(x, y - x) \le 0$, since f is quasiconvex.*

The locally Lipschitz function $f : X \to \mathbb{R}$ is said to be pseudo-regular *if $f^{\uparrow}_{D}(x,u) = f^{\uparrow}_{Cl}(x,u)$ for all $x \in X$ and $u \in \mathbf{E}$. It is known that a pseudo-regular function is not necessarily regular* (*see* [22] *for an example*).

In the considered case the upper Dini-Hadamard derivative coincides with the the upper Dini derivative, and it satisfies all these properties.

Example 5.4. *Let $h(x,u)$ be the Michel-Penot derivative* [20]. *When the function is locally Lipschitz, Properties* 3, 4, 5 *are fulfilled. If the function is* semiregular [4]*, that is there exists the usual directional derivative, and it coincides with the Michel-Penot derivative, then it satisfies Property* 6*, too.*

Remark 5.1. A quasiconvex function $f : X \to \mathbb{R}$, which is not pseudoconvex on X cannot be characterized by a similar inequality of the type (1), as the pseudoconvex functions can. Indeed, suppose that there exist a mapping $\eta : X \times X \to \mathbf{E}$ of some type, which satisfies (1). Thanks to Theorem 5.2, f is pseudoconvex. This conclusion contradicts the assumptions that f is not pseudoconvex.

6. ∂-pseudoconvex functions. The properties assumed in Section 5 that the directional derivative must satisfy are too strong. In this section we weaken Property 5 into the following.

7. *$h(x,u)$ is positively homogeneous function of u for all $x \in X$.*

8. $h(x,0) \le 0$ *for all $x \in X$.*

In this case another notion of pseudoconvexity may be used to satisfy Theorems 5.1 and 5.2. The function $f : X \to \mathbb{R}$ is called *∂-pseudoconvex* if

$$x,\ y \in X,\ f(y) < f(x) \quad \text{imply} \quad \langle \xi, y - x\rangle < 0 \quad \text{for all} \quad \xi \in \partial f(x).$$

Each pseudoconvex function is ∂-pseudoconvex. When Properties 4, 5 are satisfied, these two notions coincide. But when Property 5 is replaced into Properties 7, 8, Property 6 is weaker than the implication (3). In this case a ∂-pseudoconvex function is not necessarily quasiconvex.

Example 6.1. *Consider the function*

$$f(x) = \begin{cases} x^2, & \textit{if} \quad x \neq 0, \\ 1, & \textit{if} \quad x = 0. \end{cases}$$

It is ∂-pseudoconvex, if we take $h(x, u)$ to be the lower Dini-Hadamard derivative, but $f(x)$ is not quasiconvex.

If Property 4 is fulfilled, then Proposition 5.1 can be generalized to the following.

Proposition 6.1. *Each ∂-pseudoconvex function $f : X \to \mathbb{R}$, which satisfies Property* 4, *is quasiconvex.*

Proof. It repeats the arguments of the proof of Proposition 5.1. □

We must change Propositions 5.2 as follows.

Proposition 6.2. *Let Properties* 7, 8 *be fulfilled. Then each ∂-pseudoconvex function $f : X \to \mathbb{R}$ is invex.*

Proof. Assume that there exists a function f, which is ∂-pseudoconvex, but it isn't invex. Then there exist $x,\ y \in X$ such that

$$f(y) - f(x) < h(x, u) \quad \text{for all} \quad u \in \mathbf{E} \tag{4}$$

Since $h(x, 0) \leq 0$, we conclude that $f(y) < f(x)$. If there exists some $\eta \in \mathbf{E}$ such that $h(x, \eta) < 0$, then the inequality (4) will not be satisfied for all directions of the type $t\eta$, where $t > 0$ is sufficiently large. Therefore $\langle 0, u\rangle = 0 \leq h(x, u)$ for all $u \in \mathbf{E}$. According to the definition of the subdifferential $0 \in \partial f(x)$. But the last inclusion contradicts the inequality $f(y) < f(x)$, because f is ∂-pseudoconvex. □

We must change Theorems 5.1, 5.2 as follows. Their proofs remain the same.

Theorem 6.1. *Assume that the derivative* h *satisfies Properties* 3 *and* 6. *Let* $f : X \to \mathbb{R}$ *be quasiconvex and upper semicontinuous. Then* f *is* ∂*-pseudoconvex on* X *if and only if the set of global minimizers coincides with the set of stationary points.*

Theorem 6.2. *Assume that the derivative* h *satisfies Properties* 3 *and* 6. *If the upper semicontinuous function* $f : X \to \mathbb{R}$ *is quasiconvex and invex, then it is* ∂*-pseudoconvex.*

Example 6.2. *The following example shows that the word "global" in Theorem* 6.1 *cannot be replaced by the word "local", as it is in the differentiable case* [6, Theorem 2.2]. *Consider the function*

$$f(x) = \begin{cases} 0, & \text{if} \quad x < 0, \\ 1, & \text{if} \quad x \geq 0. \end{cases}$$

It is quasiconvex and upper semicontinuous on $\mathbb{R}$. *Let* h *be the lower Dini-Hadamard derivative. Then this function satisfies Properties* 3 *and* 6. *The set of local minimizers and the set of stationary points coincide with* $(-\infty, 0) \cup (0, \infty)$, *but* f *is not* ∂*-pseudoconvex on* $\mathbb{R}$.

Example 6.3. *Let* $h(x, u)$ *be the lower Dini-Hadamard directional derivative. It satisfies all the properties assumed in this section* [1]. *We must only prove Property* 6. *Really, let* $\xi \in \partial f(x)$ *and* $y \in X$ *be arbitrary such that* $f(y) \leq f(x)$. *Using the quasiconvexity of* f, *we obtain*

$$\langle \xi, y - x \rangle \leq f^{\downarrow}_{DH}(x, y - x) = \liminf_{(t,u') \to (+0, y-x)} \frac{1}{t}(f(x + tu') - f(x))$$
$$\leq \liminf_{t \to +0} \frac{1}{t}(f(x + t(y - x)) - f(x)) \leq 0.$$

Here we have denoted by $f^{\downarrow}_{DH}(x, u)$ *the lower Dini-Hadamard directional derivative of* f *at* x *in the direction* u. *Therefore* $\xi \in \mathcal{N}(x)$, *and* $\partial f(x) \subset \mathcal{N}(x)$.

It is seen that a function for which the lower Dini-Hadamard subdifferential $\partial f(x) \neq \emptyset$ *for all* x, *is lower semicontinuous.*

Example 6.4. *The lower Dini derivative* $f^{\downarrow}_{D}(x, u)$ *of an arbitrary function is an example of generalized derivative, which satisfies Properties* 3, 6, 7, 8.

7. A characterization of pseudomonotone bifunctions. In a very general treatment, a generalized directional derivative might be considered as a bifunction $h(x, u)$ with values from $\overline{\mathbb{R}}$, where x refers to a given point of a given subset X of $\mathbf{E}$ and u refers to a given direction of $\mathbf{E}$.

The pseudomonotone maps were introduced by Karamardian [14]. A study of the pseudomonotonicity for generalized directional derivatives (bifunctions) has appeared in Komlósi [17, 18] (see also references contained therein).

The bifunction $h(x, u)$ is called *pseudomonotone (strictly pseudomonotone)* on X, if for every pair of distinct points $y,\ z \in X$ we have

$$h(y, z-y) > 0 \quad \text{(respectively} \quad h(y, z-y) \geq 0) \quad \text{implies} \quad h(z, y-z) < 0.$$

The following theorem is a necessary and sufficient condition for pseudomonotonicity.

Theorem 7.1. *Let the bifunction $h(x, u)$ take only finite values. Then h is pseudomonotone (strictly pseudomonotone) on X if and only if there exists a negative function $p : X \times X \to \mathbb{R}$ such that*

$$h(z, y-z) \leq p(y, z)\ h(y, z-y) \quad \textit{for all} \quad y,\ z \in X \tag{5}$$

$$(h(z, y-z) < p(y, z)\ h(y, z-y) \quad \textit{for all distinct} \quad y,\ z \in X).$$

Proof. We shall consider only the pseudomonotone case, because the strictly pseudomonotone one is analogous. The proof is similar to the proof of Theorem 3.1. The sufficiency is obvious.

Necessity. Suppose that h is pseudomonotone. We construct explicitly the function p in the following way.

$$p(y, z) = \begin{cases} \dfrac{h(z, y-z)}{h(y, z-y)}, & \text{if } h(y, z-y) > 0 \text{ or } h(z, y-z) > 0, \\ -1, & \text{otherwise.} \end{cases}$$

The function $p(y, z)$ is well defined, strictly negative, and it satisfies inequality (5). □

Example 7.1. *An example of a generalized derivative, which is pseudomonotone or strictly pseudomonotone, is the lower Dini derivative of pseudoconvex (respectively strictly pseudoconvex) functions* [17, Theorems 4, 5]. *It is* *finite-valued, when the function is locally Lipschitz.*

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