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ON PROJECTIVE PLANE OF ORDER 13 WITH A FROBENIUS GROUP OF ORDER 39 AS A COLLINEATION GROUP

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Abstract. One of the most outstanding problems in combinatorial mathematics and geometry is the problem of existence of finite projective planes whose order is not a prime power.

It is well known that a Desarguesian plane of order p, p prime number, has not a proper subplane, also every known non Desarguesian plane (of order p^r, p – odd prime number and $r \geq 2$) has a subplane of order 2.

It was shown by E. Ademaj [1] that: If $\mathcal{P}$ is a projective plane of order 11 on which operates a group $G = \langle \rho, \tau / \rho^7 = \tau^9 = 1, \rho^\tau = \rho^2 \rangle$ of order 63, then G cannot fix a subplane of order 2.

The next number is $p = 13$.

In this paper we shall look projective plane of order 13 with a Frobenius group of order F3.13 . Using the method of tactical decomposition, we shall construct the orbit structure of Frobenius group as a collineation group of projective plane of order 13.

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We begin with two definitions

Definition 1. *A projective plane of order $n \geq 2$ is a set $\mathcal{P}$ with n^2+n+1 elements called points with n^2+n+1 subsets of $\mathcal{P}$ called lines, such that:*

1. *each line contains (is incident with) $n+1$ points,*
2. *any two (distinct) lines intersect in exactly one point.*

Definition 2. *A permutation α of the points $\mathcal{P}$ which maps lines of $\mathcal{P}$ onto lines of $\mathcal{P}$ is called a collineation (automorphims).*

The purpose of this paper is to prove the following result:

Theorem. *It exists orbit structure of a Frobenius group of order 39 as a collineation group of projective plane of order 13.*

Proof. Let $\mathcal{P}$ be a projective plane of order 13 which possesses a Frobenius group

$$G = \langle \rho, \mu / \rho^{13} = \mu^3 = 1, \rho^\mu = \rho^3 \rangle$$

of order 39 as a collineation group. Then $\mathcal{P}$ has $13^2 + 13 + 1 = 183$ points and lines. We take a partition of the point set $\mathcal{P}$.

$$P = \{P_\infty, P_1, P_2, \dots, P_{14}\},$$

where $P_i = \{i_0, i_1, i_2, \dots, i_{12}\}$, $i = 1, 2, \dots, 14$ and bloch set $B = \{B_\infty, B_1 B_2, \dots, B_{14}\}$ so that the partition

$$P = \{P_\infty, P_1, P_2, \dots, P_{14}, B_\infty, B_1 B_2, \dots, B_{14}\},$$

is a tactical decomposition of $\mathcal{P}$.

According to the definition of tactical decomposition, let r to a collineation of order 13 of Aut $\mathcal{P}$. The we can set:

$$(*) \qquad \rho = (\infty)(1_0, 1_1, \dots, 1_{12})(2_0, 2_1, \dots, 2_{12}) \dots (14_0, 14_1, \dots, 14_{12})$$

where ∞ is a fixed point and $1_0, 1_1, \dots, 1_{12}, 2_0, 2_1, \dots, 2_{12}, \dots, 14_0, 14_1, \dots, 14_{12}$ are all other points of $\mathcal{P}$.

Since the number of orbits of blocs and points is the same, for the fixed bloc we can set

$$v = \{\infty, 1_0, 1_1, \dots, 1_{12}\}$$

Since through every point of $\mathcal{P}$ passes exactly $14 = 13 + 1$ lines, for the other lines through ∞ without loos of generality we can set

$$P = \{\infty, 2_0, 3_0, 4_0, 5_0, 6_0, 7_0, 8_0, 9_0, 10_0, 11_0, 12_0, 13_0, 14_0\}$$

with the action of $\langle\rho\rangle$ on line p, we get the other lines through ∞:

$$p^{\rho} = \{\infty, 2_1, 3_1, \ldots, 14_1\}$$
$$p^{\rho^2} = \{\infty, 2_2, 3_2, \ldots, 14_2\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$p^{\rho^{12}} = \{\infty, 2_{12}, 3_{12}, \ldots, 14_{12}\}$$

We have constructed one nontrivial orbit of lines of $\mathcal{P}$ (the lines through ∞). There are exactly 13 futher nontrivial $\langle\rho\rangle$- orbits of lines of $\mathcal{P}$.

The collineation ρ regarded as the permutation of the points of $\mathcal{P}$ is given by $(*)$. We can see that restriction of ρ at $Z_{13} = \{0, 1, \ldots, 12\}$ is $\rho/Z_{13} = (0, 1, 2, \ldots, 12)$ that is $\rho/Z_{13} : x \to x + 1(\operatorname{mod} 13)$. The assumption is that on $\langle\rho\rangle$ acts the collineation μ of order 3. That is μ normalizes $\langle\rho\rangle$ and replaces $\langle\rho\rangle$-orbits (of points) again into the $\langle\rho\rangle$ orbits and also it permutes the ordinal numbers (indeces) $0, 1, 2, \ldots, 12$. We can take of restriction ρ/Z_{13}.

From

$$\rho_1 = \rho/Z_{13} = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12)$$

$$\rho_{13} = (0, 3, 6, 9, 12, 2, 5, 8, 11, 1, 4, 7, 10)$$

it follovs that

$$\mu_1 = \mu/Z_{13} = (0)(1, 3, 9)(2, 6, 5)(4, 12, 10)(7, 8, 11).$$

One can see that $\langle\mu_1\rangle$ to the conjugation (also conjugation with help of ρ_1^i, $i \in Z_{13}$) is the only subgroup of order 3 (in Σ_{13}) normalizing $\langle\rho_1\rangle = \langle(0, 1, 2, \ldots, 12)\rangle$. Besides we can easily verify that

$$\mu_1 : x \to kx \qquad (x \in Z_{13},\ \text{compution} \quad \operatorname{mod} 13),\ k \in \{3, 9\}.$$

In the above presentation of μ_1 we have chosen μ_1 so that it fixes index 0. In fact, μ_1 which is of order 3, fixes one index from Z_{13}, which we take to be zero. Other wise, it can be any other index by the conjugation $\mu_1 \to \mu_1^{\rho_1^i}$.

The presentation of the permutation of the big (orbital) numbers $1, 2, \ldots, 14$ — under μ, without loss of generality, can be deseribed by

$$\mu = (\infty)(1)(2)(3, 4, 5)(6, 7, 8)(9, 10, 11)(12, 13, 14)$$

The collineation μ (as it normalizes ρ) fixes globaly the sets $\{\infty\}, \{1\}, \{2\}$, of the points and $\{v\}$, $\{p\}$, $\{l\}$ of the lines of $\mathcal{P}$. The other lines p^{ρ^i}, $i = 0, 1, 2, \ldots, 12$ leavs invariant, as m permutes the orbital numbers.

Hence we have two possibilities of the action of $G = \langle \rho, \mu \rangle$ on a projective plane $\mathcal{P}$ of order 13.

Case I: G acts faithfully on a subplane of order 3.

Case II: Does not fix a subplane of order 3.

We must contruct the representatives other remaining 13 nontrivial orbits of lines of $\mathcal{P}$.

At first we difine the Haming number:

$$H = (|\rho| - 1)\lambda = (|13| - 1)1 = 12$$

We have the following presentation of number 12:

(a) $12 = 4 \cdot 3$

(b) $12 = 3 \cdot 2 + 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1$

(c) $12 = 3 \cdot 2 + 3 \cdot 2$

(d) $12 = 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1$.

Thus, we shall look the case (a) for invariant line, but other lines we get of from type (b).

Since the next line must be μ-invariant, we have the following presentation for the μ-invariant line l:

I: $l = \{1_0, 2, 2, 2, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

II: $l = \{1, 3, 3, 4, 4, 5, 5, 6, 6, 7, 7, 8, 8, 2\}$

In case I, we construct other 12 representatives (of $\langle \rho \rangle$-orbits lines) which passes through 10.

We define number (game product):

$$(k_1, l) = |r|\lambda = 13 \cdot 1 = 13$$

Since, Haming number $H = 3 \cdot 2 + 2 \cdot 1 + 2 \cdot 1 + 2 \cdot 1$, then for line k_1 we can set:

$$k_1 = \{1, a, a, a, b, b, c, c, d, d, e, f, g, h\}$$

so that $(k_1, l) = 13$. Let $a = 2$, then $(k_1, l) = 1 + 12 = 13$. Since $b, c, d, e, f, g, h \in \{12, 13, 14\}$, and on the one other and $b, c, d, e, f, g, h \notin \{3, 4, 5, 6, 7, 8, 9, 10, 11\}$, then we conclude $a \neq 2$.

Suppose $a = 3$. Then we have

$$k_1 = \{1, 3, 3, 3, b, b, c, c, d, d, e, f, g, h\}$$

and $(k_1, l) = 1 + 3 = 4$. For $b = 2$ have $(k_1, l) = 4 + 8 = 12$. Since $c, d \notin \{4, 5, 6, 7, 8, 9, 10, 11\}$ and $c, d \in \{12, 13, 14\}$, then we must have $c = 12$, $d = 13$ and have $(k_1, l) = 12 + 0 = 12$. Since $e \in \{4, 5, 6, 7, 8, 9, 10, 11, 14\}$, then have $e = 4$, and have $(k_1, l) = 12 + 1 = 13$. Since $f, g, h \notin \{5, 6, 7, 8, 9, 10, 11\}$ and $f, g, h \in \{14\}$, then we conclude $b \neq 2$. Permuting numbers $4, 5, 6$, we conclude that $b \notin \{4, 5, 6\}$.

Let $b = 7$, then

$$k_1 = \{1, 3, 3, 3, 7, 7, c, c, d, d, e, f, g, h\}$$

and $(k_1, l) = 4 + 2 = 6$. For $c = 2$ we have $(k_1, l) = 6 + 8 = 14$ it means $c \neq 2$. Permuting numbers $4, 5, 6$ we conclude that $c \notin \{4, 5, 6\}$. If $c = 8$, then

$$k_1 = \{1, 3, 3, 3, 7, 7, 8, 8, d, d, e, f, g, h\}$$

and $(k_1, l) = 6 + 2 = 8$. Since, and $d \neq 2$, $d \notin \{4, 5, 6\}$ then we take $d = 9$, and we have

$$k_1 = \{1, 3, 3, 3, 7, 7, 8, 8, 9, 9, e, f, g, h\}$$

where $(k_1, l) = 8 + 2 = 10$. For $e = 2$ have $(k_1, l) = 10 + 4 = 14$ it means $e \neq 2$. Since $e \in \{4, 5, 6, 10, 11, 12, 13, 14\}$ then let $e = 4$, and we have

$$k_1 = \{1, 3, 3, 3, 7, 7, 8, 8, 9, 9, 4, f, g, h\}$$

where $(k_1, l) = 10 + 1 = 11$. Permuting numbers $5, 6, 10, 11, 12, 13, 14$ we may conclude $f, g \notin \{5, 6\}$ and $f, g \in \{10, 11, 12, 13, 14\}$. If $f = 10$ and $g = 11$, then we have

$$k_1 = \{1, 3, 3, 3, 7, 7, 8, 8, 9, 9, 4, 10, 11, h\}$$

where $(k_1, l) = 11 + 2 = 13$. Since, $h \notin \{5, 6\}$ and $h \in \{12, 13, 14\}$ suppose $h = 12$. Then, we have the following solution for line k_1

$$k_1 = \{1, 3, 3, 3, 7, 7, 8, 8, 9, 9, 4, 10, 11, 12\}.$$

We act with $\langle \mu \rangle$ on line k_1, then

$$k_1^{\mu} = \{1, 4, 4, 4, 5, 8, 8, 6, 6, 10, 10, 11, 9, 13\}$$
$$k_1^{\mu^2} = \{1, 5, 5, 5, 3, 6, 6, 7, 7, 11, 11, 9, 10, 14\}$$

Since, the conditions are filled $(k_1, l) = (k_1, k_1^{\mu}) = (k_1, k_1^{\mu^2}) = 13$ and $(k_1^{\mu}, k_1^{\mu^2}) = 13$ then we conclude that these are three representatives lines (of $\langle \rho \rangle$ – nontrivial orbits) which passes through 1_0. Now, we shall construct the line k_2:

$$k_2 = \{1, a, a, a, b, b, c, c, d, d, e, f, g, h\}$$

so that $(k_2^{\mu^i}, l) = 13$, $(k_2^{\mu^i}, k_2^{\mu^j}) = 13$, where $i \neq j$ and $i, j \in \{0, 1, 2\}$.

Using the same method we may construct other representatives line k_2:

$$k_2 = \{1, 7, 7, 7, 4, 4, 6, 6, 12, 12, 9, 13, 14, 2\}$$

and the conditions are filled $(k_2, l) = (k_2, k_1^{\mu^i}) = (k_1^{\mu^i}, k_1^{\mu^j}) = 13$, where $i \neq j$ and $i, j \in \{0, 1, 2\}$.

We act with $\langle \mu \rangle$ on line k_2 and we have

$$k_2^{\mu} = \{1, 2, 5, 5, 8, 8, 8, 7, 7, 10, 13, 13, 14, 12\}$$
$$k_2^{\mu^2} = \{1, 2, 3, 3, 6, 6, 6, 8, 8, 11, 14, 14, 12, 13\}$$

Using the same method we may construct other the representatives line k_3:

$$k_3 = \{1, 2, 5, 6, 8, 9, 9, 9, 10, 10, 12, 12, 14, 14\}$$

which fills given conditions.

We act with $\langle \mu \rangle$ on line k_3, then we have

$$k_3^{\mu} = \{1, 2, 3, 7, 6, 10, 10, 10, 11, 11, 13, 13, 12, 12\}$$
$$k_3^{\mu^2} = \{1, 2, 4, 8, 7, 11, 11, 11, 9, 9, 14, 14, 13, 13\}$$

Finally, using the same method, we obtain the following solution for line k_4:

$$k_4 = \{1, 2, 3, 3, 4, 4, 5, 7, 10, 10, 13, 14, 14, 14\}$$

and with the action of $\langle \mu \rangle$ on in this line we may set:

$$k_4^{\mu} = \{1, 2, 4, 4, 5, 5, 3, 8, 11, 11, 14, 12, 12, 12\}$$
$$k_4^{\mu^2} = \{1, 2, 5, 5, 3, 3, 4, 6, 9, 9, 12, 13, 13, 13\}$$

Thus, we obtain the following orbital structure (in case I):

$l =$	1	2	2	2	2	3	4	5	6	7	8	9	10	11
$k_1 = k_1 =$	1	3	3	3	4	7	7	8	8	9	9	10	11	12
$k_2 = k_1^{\mu} =$	1	4	4	4	5	8	8	6	6	10	10	11	9	13
$k_3 = k_1^{\mu^2} =$	1	5	5	5	3	6	6	7	7	11	11	9	10	14
$k_4 = k_2 =$	1	2	4	4	7	7	7	6	6	9	12	12	13	14
$k_5 = k_2^{\mu} =$	1	2	5	5	8	8	8	7	7	10	13	13	14	12
$k_6 = k_2^{\mu} =$	1	2	3	3	6	6	6	8	8	11	14	14	12	13
$k_7 = k_3 =$	1	2	5	6	8	9	9	9	10	10	12	12	14	14
$k_8 = k_3^{\mu} =$	1	2	3	7	6	10	10	10	11	11	13	13	12	12
$k_9 = k_3^{\mu^2} =$	1	2	4	8	7	11	11	11	9	9	14	14	13	13
$k_{10} = k_4 =$	1	2	3	3	4	4	5	7	10	10	13	14	14	14
$k_{11} = k_4^{\mu} =$	1	2	4	4	5	5	3	8	11	11	14	12	12	12
$k_{12} = k_4^{\mu^2} =$	1	2	5	5	3	3	4	6	9	9	12	13	13	13

Thus, we showed that, there exists orbit structure of the Frobenius group of order 39 as a collineation group of projective plane of order 13. We can say that in orbit structure of the lines the Haming number is 12 but the game product is 13 that in fact is a condition that lines will be compatible (it means that they fulfill conditions of projective plane).

The prof of the Theorem is complete.

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