MULTICRITERIA EVALUATION AND OPTIMIZATION OF HIERARCHICAL SYSTEMS

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Abstract. It is shown that any multicriteria problem can be represented by a hierarchical system. Separate properties of the object are evaluated at the lower level of the system, using a criteria vector, and a composition mechanism is used to evaluate the object as a whole at the upper level. The paper proposes a method to solve complex multicriteria problems of evaluation and optimization. It is based on nested scalar convolutions of vector-valued criteria and allows simple structural and parametrical synthesis of multicriteria hierarchical systems.

Keywords: alternative choice, multicriteriality, hierarchical systems, composition of criteria, nested scalar convolutions

Introduction

In decision-making theory [1,2], there are two different approaches to evaluating objects (alternatives) subject to choice. One of them is to evaluate an object as a whole and to choose an alternative by comparing objects as gestalts (holistic images of objects without detailing their properties). The second approach is to detail and evaluate certain vectors of characteristics (properties) of objects and to make a decision after comparing these properties. The scheme of decision making can be represented by the formula [3]

$$\{\{\chi\},\Phi\} \to \chi^*,$$

where $\{\chi\}$ is a set of objects (alternatives); Φ is a choice function, i.e., a rule that establishes preferability in the set of alternatives; and χ^* are chosen alternatives (one or more).

The holistic approach implies choosing χ^* using the choice function Φ. The vector approach requires decomposing (expanding) the function Φ into a set (vector) of some choice functions φ. By decomposition of the choice function Φ is meant [1] its equivalent representation by a certain set of other choice functions φ whose composition is the initial choice function Φ.

Both approaches have their advantages and shortcomings. Choice after comparing objects may substantially differ from choice after comparing the vector characteristics of objects [2]. This is because information on vectors sometimes gives an insufficiently adequate description of objects, even with the most careful choice of characteristics of objects. Some portion of the information on objects is lost when the objects are described by a set of characteristics. Another portion of the information, which is not directly concerned with the objects being compared, is introduced into the model. The choice of an appropriate set of properties (characteristics) of an object is subjective to a certain extent. Moreover, there is an assumption [2] that human thinking is not has been evolved to adapt for a natural (from a formal standpoint) changeover from preferences on a set of objects to preferences on a set of their characteristics.

Nevertheless, modern decision theory is inclined to using the vector approach since it is objective and comprehensive and allows employing formalized methods. The concreteness and clearness of an approach are also taken into account since it is easier to collect indisputable facts and reach a consensus on a specific issue [4].

It is assumed that it is much easier for a decision-maker to reveal a preferable alternative for a certain property of an object. For example, in choosing the best design of an aircraft, it is easier to compare design A over an design B in comfort, or reliability, or weight-lifting ability, than to compare the whole designs A and B [3]. Selecting properties of alternatives is a decomposition that leads to a hierarchical structure of properties. The properties of the first hierarchical level can be subdivided into sets of next specific properties, etc. The division depth is determined by tending to reach the properties that are convenient to be compared to each other. Indeed, in the example with an aircraft, it is easier to judge its comfort rather than the aircraft as a whole; however, such a qualitative property is also not always convenient for comparison and has to be decomposed for convenient and objective comparison of properties. Therefore, the property of comfort, in turn, is subjected to hierarchical decomposition into levels: (a) cabin noisiness, (b) floor vibrations, (c) seat spacing, etc. These characteristics can be evaluated and are objective.

Properties for which objective numerical characteristics exist are called criteria. More strictly, quantitative characteristics of properties of an object, whose numerical values are a quality measure of an object of evaluation with respect to the given property, are called criteria. Deriving a set of criteria is the final result of hierarchical decomposition. The number of levels depends on the decomposition depth required. Complexity is that decomposition depth can be different for different initial properties, and heterogeneous sets of criteria should be normalized at each hierarchy level.

Though attractive, the approach comparing individual properties involves the serious problem of inverse passage to the required comparison alternatives in general. This problem assumes a composition of criteria based on hierarchy levels, which is difficult, especially for a significant decomposition depth of properties. In an elementary and most popular case (two-level hierarchy), the composition problem can be solved traditionally, by deriving a single scalar convolution of criteria. However, other approaches are required for three (and more)-level hierarchy.

The aforesaid makes it reasonable to state that any multicriteria problem can be represented as a hierarchical system, where at the lower level, the object is evaluated in separate characteristics using a vector of criteria, and at the upper level, a composition is used to evaluate the object as a whole. The key problem is composition of criteria at hierarchy levels.

Analysis of the problem state

The case where a multicriteria problem can be represented by a two-level hierarchical system is developed most thoroughly in decision theory. The composition problem is usually solved either using a main criterion (criteria constraints) method or by using a single scalar convolution of a vector –valued criterion [3]. The latter method is used more often, the numerical value of convolution being the quality evaluation of the given object (alternative) as a whole.

It is convenient to use scalar convolution in a traditional form when the number of partial criteria is not too great (usually, $s \leq 10$). Then each criterion plays a self-depended role and all of them are comparable in importance. However, there exist complex multicriteria problems with a large number (say, several tens) of partial criteria. Then the value of each criterion separately has a weak effect on the solution of the multicriteria problem. It is expedient to group them into headings (groups, clusters) where scalar convolutions are considered as new, higher-weight criteria. These aggregated criteria, in turn, are subjected to scalar convolution and then are compared with higher weights during the solution of the multicriteria problem. Thus, as the dimension of the criteria space increases, the initially two-level hierarchical system of criteria is transformed into multilevel one and requires a mechanism to compose criteria into hierarchy levels.

As an example, let us consider alternate evaluation of research projects in biological studies in space [5]. To evaluate the efficiency of such projects, 28 partial criteria are used. Consultations with experts has allowed grouping these criteria into four headings (groups): (i) general criteria, (ii) scientific development criteria, (iii) economic criteria and (iv) social criteria (see Table 1).

Table 1

Criterion No.	Project quality criteria	Points (10-point scale)
	General Criteria	
1	Compliance of the project with the Space Program of Ukraine	10.00
2	Integration of the project into international programs of biological investigations in space	9.00
3	Probability that the approach will lead to the desired results	7.50
4	Completeness of feasibility of the project under given conditions of space experiment	8.30
	Scientific Development Criteria	
5	Compliance of the job structure and investigation methods with project tasks	9.75
6	Furtherance of gathering knowledge on the influence of space flight factors on fundamental physiological processes	8.25

7	Furtherance of gathering knowledge on the adaptation of biological objects for space flight conditions	8.25
8	Novelty of investigations	8.00
9	Originality and innovation of the purposes formulated	8.75
10	Influence of investigation on scientific concepts and methods in space biology and medicine	9.75
11	Probability that investigation will allow new break-through projects	8.50
12	Refuting the paradigms available	5.30
13	Share of worldwide investigations in the project	8.00
14	Enhancing the prestige of Ukraine in the world	8.50
15	International support of the project	9.70
16	Coverage in the scientific literature	9.70
17	Using the results in academic activity	7.70
18	Popularization and propagation of knowledge	7.75
Economic Criteria		
19	Probability of introducing the technologies into Ukrainian economy	9.33
20	Adequate number of specialists to bring the research to practical implementation	10.00
21	Attracting investments	9.33
22	Reducing production expenses	9.33
23	Increasing sales	8.67
24	Adequacy of financing the tasks planned	9.33
Social Criteria		
25	Increasing the number of worksites	10.00
26	Increasing the level of staff qualification	8.00
27	Promoting development of small and medium-scale business	7.67
28	Influence on the activity of social and youth organizations	10.00

The right-hand column of the table contains the results of expert evaluation of the Biosorbent space research project, which has been included in the program of onboard experiments at the International Space Station. These data are the lower level of a three-level hierarchical system of criteria for evaluating the project as a whole.

Totsenko considered in [6] the general case of developing a decision-making support technology when an alternative should be chosen from a set of inhomogeneous alternatives for which it is impossible to formulate a unified set of quantitative evaluation criteria. In this case, the problem can be solved by methods based on hierarchical objective evaluation of alternatives without criteria analysis. Given qualitative properties, the problem of composition can be solved using binary relations, for example, by the hierarchy analysis [7]. But the problem is facilitated substantially if quantitative (or reducible to them) criteria that permit operations in a normalized criteria space are employed to evaluate alternatives. The theory of multicriteria evaluation and optimization is applicable to such problems. The present paper addresses such a class of problems.

Formulation of the problem

The state of a hierarchical system for a given alternative is defined by the following parameters:

$I=\{1,2,...,n\}$ is a set of elementary subsystems evaluated using lower-level hierarchy criteria;

$\{y_i\}_{i\in I}$, $y=\{y_1,...,y_n\}$ are the estimates of elementary subsystems based on scalar criteria and a vector-valued criterion of the lower hierarchy level. The efficiency of each highest level depends on the estimates according to the lowest-level hierarchy criteria.

The additional conditions that define the hierarchical structure are as follows:

$J=\{1,2,...,m\}$ is the set of hierarchical levels;

$\{I_j\}_{j\in J}$ is the distribution of subsystems into levels, $I_j=\{1,2,...,n_j\}$;

$\{\lambda_j\}_{j\in J}$ are priority vectors.

It is required to find an analytical estimate $\varphi*$ and a qualitative efficiency evaluation of the hierarchical structure and to choose the best alternative from available ones.

Solution technique

To analytically evaluate the hierarchical structures for efficiency, we propose to apply the method of nested scalar convolutions [8]. A composition is carried out by a nested doll ("matreshka") principle: scalar convolutions of weighed components of vector criteria of the lowest level are components of vector criteria of the highest level. The scalar convolution of criteria obtained at the uppermost level automatically becomes the expression for the efficiency evaluation for the whole hierarchical system.

The algorithm of nested scalar convolutions can be represented as sequential weighed scalar convolutions of vector criteria at each hierarchy level in view of priority vectors based on the compromise (trade-off) scheme selected

$$\{(\varphi^{(j-1)}, \lambda^{(j-1)}) \to \varphi^{(j)}\}_{j\in J}\ ,\ \varphi^{(1)} \equiv y\ ,$$

and efficiency evaluation of the whole hierarchical system can be expressed by determining scalar convolution of the upper hierarchy level:

$$\varphi* = \varphi^{(m)}\ .$$

In choosing solutions, the number of alternatives is $n_a \geq 1$. Each alternative is characterized by a hierarchical structure. For $n_a = 1$, the problem posed is transformed into the evaluation of the given hierarchical structure. If $n_a > 1$, then each structure is evaluated as a given one and the alternative whose hierarchical structure has been evaluated best, is chosen. Therefore. in case of discrete multicriteria optimization, the base problem is to evaluate a given hierarchical structure. However, this method can only be used if the number of alternatives n_a is relatively small, when simple enumeration does not involve significant computational difficulties. For large sets of alternatives, other optimization methods should be applied, for example those stated in [9].

Compromise scheme

As a base trade-off scheme for the method of nested scalar convolutions, we propose to use the nonlinear scheme described in [10]. It was established that without loss of generality, a premise for its application is that all of the partial criteria are subject to minimization and are bounded:

$$y_i \leq A_i, A = \{A_i\}_{i=1}^{n},\ i \in [1, n],$$

where *A* is the vector of constraints.

The scalar convolution

$$\varphi(\lambda, y) = \sum_{i=1}^{n} \lambda_i [A_i - y_i]^{-1}$$

or

$$\varphi(\lambda, y_0) = \sum_{i=1}^{n} \lambda_i [1 - y_{0i}]^{-1},$$

if quantitative criteria are normalized by the formula $y_0 = y/A$, is a simple informative model of utility function of the decision maker at the lower level of hierarchy for criteria being minimized according to the concept of nonlinear trade-off scheme.

Qualitative (but reducible to quantitative) criteria are usually determined by experts using scale points. A questionnaire with partial criteria is given to experts. Criteria are associated with a continuous scale divided, for example, into ten intervals. Zero on the scale is indicative of no weight, 10 corresponds to the maximum weight. An expert should estimate the relative influence of each partial criterion on the general estimate under given conditions and to associate it with the corresponding point on the scale, characterized by a number *f*. It is admissible to select points between numbers or to assign some criteria to one point on the scale.

An analysis of decision-making processes has shown that evaluating objects on a 10-points scale, experts are guided by gradations of a so-called fundamental scale represented in a general form in Table 2 and described in [7], where qualitative gradations of properties of objects are associated with the corresponding quantitative estimates *f*. It is possible to say that in terms of fuzzy-set theory [11], the fundamental scale appears as a universal membership function for a passage from a number to the corresponding qualitative gradation and back. A passage from a linguistic variable (satisfactory quality, excellent quality, etc) to corresponding quantitative estimates *f* according to point scale (5.5, 7.0), i.e. passage from fuzzy qualitative gradations to numbers and back, is carried out.

Table 2

Quality category	Ranges of fundamental scale for *f*	Ranges of normalized inverse fundamental scale for y_0, φ_0
Unacceptable	0-3	1.0-0.7
Low	3-5	0.7-0.5
Satisfactory	5-6	0.5-0.4
Good	6-8	0.4-0.2
High	8-10	0.2-0.0

Estimates *f* are determined according to a direct 10-point fundamental scale for the criteria being maximized. The technique applied in the paper for multicriteria evaluation according to a nonlinear trade-off scheme is developed for normalized minimized criteria y_0 whose estimates are obtained from *f* by the formula [12]

$$y_0=1-0,1\cdot f,\ y_0\in[0;1].$$

This is reflected in Table 2 by an inverse normalized scale. This scale is used to measure normalized scalar convolutions of criteria φ_0 as well.

Allowance for priorities

The simplex

$$\Gamma_\lambda = \{\lambda \,|\, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1\} \quad (1)$$

is the domain of definition of priority coefficients $\lambda \in \Gamma_\lambda$, where λ_i =const are formal parameters with double physical meaning. On the one hand, these are priority coefficients that express the preference of a decision-maker according to certain criteria. On the other hand, these are coefficients of an informative regression model constructed based on the concept of nonlinear trade-off scheme. The coefficients *λ* can be determined at each hierarchy level through optimization on a simplex using the dual approach described in [10] or by the formula

$$\lambda_{ik}^{(j-1)} = \frac{f_{ik}}{\sum_{i=1}^{n_k^{(j)}} f_{ik}}, k \in I_j,$$

where $\lambda_{ik}^{(j-1)}$ is the *i*-th component of the priority vector at the $(j-1)$-th hierarchy level in evaluating the efficiency of the *k*-th subsystem on the *j*-th level; f_{ik} is the significance parameter of the *i*-th subsystem of the $(j-1)$-th level for the *k*-th subsystem of the *j*-th level (determined by experts using a 10-point scale); and $n_k^{(j)}$ is the number of subsystems of the $(j-1)$-th level that support the *k*-th subsystem of the *j*-th level.

In the most simple and popular case, a multicriteria problem without priorities is formulated and solved, where the decision-maker assumes that all of the significance parameters are identical for all of the subsystems. In this case, an elementary scalar convolution under a nonlinear trade-off scheme in a unified form [10] is used.

For a composition of criteria on hierarchy levels, it is expedient to calculate all of the scalar convolutions from top to bottom based on the concept of a nonlinear trade-off scheme. In this case, the efficiency of the *k*-th subsystem at the *j*-th hierarchy level as a normalized nested scalar convolution, in view of priority coefficients, can be evaluated by the formula

$$\varphi_{0k}^{(j)} = N_k^{(j)} \sum_{i=1}^{n_k^{(j)}} \lambda_{ik}^{(j-1)} [1-\varphi_{0ik}^{(j-1)}]^{-1}, \; k \in I_j, \; \varphi_0^{(1)} \equiv y_0, \qquad (2)$$

where $\varphi_{0ik}^{(j-1)}$ is the estimate of the *i*-th component of the normalized vector-valued criterion at the $(j-1)$-th hierarchy level for evaluating the *k*-th subsystem of the *j*-th level; and $N_k^{(j)}$ is a normalizing factor.

Normalizing conditions

Normalization of nested scalar convolutions at each hierarchy level is of importance for the theory presented here. In [5,8], the possibility of calculating normalizing conditions based on the principle of joint liability of criteria is considered. The value of the greatest (worst) criterion is separated out in the set of normalized criteria. It is agreed that if this criterion has attained the worst value, the remaining normalized criteria are assigned the possibility of attaining the same values, which constitute components of the normalizing vector. Such an approach is simple but works only if the criteria are really “equivalent”.

It is logical that if estimates with respect to all of the partial relative criteria $\varphi_{0ik}^{(j-1)}, i \in [1, n_k^{(j)}]$ are identical and equal to $\varphi_{0ik}^{(j-1)} \equiv \varphi_{0k}$, then their normalized scalar convolution in formula (2) should express the same analytical and qualitative estimate according to the inverted normalized fundamental scale:

$$\varphi_{0k} = \frac{N_k^{(j)}}{(1-\varphi_{0k})} \sum_{i=1}^{n_k^{(j)}} \lambda_{ik}^{(j-1)}.$$

Since by normalizing conditions (1) $\sum_{i=1}^{n_k^{(j)}} \lambda_{ik}^{(j-1)} = 1,$ the expression for the normalizing factor becomes

$$N_k^{(j)} = \varphi_{0k}(1-\varphi_{0k}). \qquad (3)$$

Let us use formula (3) to perform calibration calculations of the normalizing factor $N_k^{(j)}(\varphi_{0ik}^{(j-1)})$ for the estimates $\varphi_{0ik}^{(j-1)}, \; i \in [1, n_k^{(j)}]$. Let us compose the measure of the total quadratic error that occurs since the unknown factor $N_k^{(j)}$ is used rather than exact values of the normalizing factor at calibration points $N_k^{(j)}(\varphi_{0ik}^{(j-1)})$:

$$M = \sum_{i=1}^{n_k^{(j)}} [N_k^{(j)} - N_k^{(j)}(\varphi_{0ik}^{(j-1)})]^2.$$

Using the necessary extremum condition for the function

$$\frac{\partial M}{\partial N_k^{(j)}} = 0,$$

we get the normalizing factor

$$N_k^{(j)} = \frac{1}{n_k^{(j)}} \sum_{i=1}^{n_k^{(j)}} N_k^{(j)}(\varphi_{0ik}^{(j-1)})$$

and with allowance for (3), we get

$$N_k^{(j)} = \frac{1}{n_k^{(j)}} \sum_{i=1}^{n_k^{(j)}} \varphi_{0ik}^{(j-1)}(1-\varphi_{0ik}^{(j-1)}).$$

Conclusions

The recurrence formula (2) allows us to evaluate qualitatively and quantitatively the scalar convolutions of criteria, normalized using the inverted fundamental scale, with respect to all of the hierarchy levels up to the upper one: $\varphi_0^*=\varphi_0^{(m)}$.

The solution of the multicriteria evaluation problem for the example represented by Table 1 has resulted in the following. From the scalar convolution of partial criteria of the lower level of hierarchy, going under the heading General Criteria, we obtained an aggregated criterion of the second hierarchy level $\varphi_{01}^{(2)}=0{,}121$. Similarly, we obtained the value of the aggregated criterion $\varphi_{02}^{(2)}=0{,}143$ for Scientific Development Criteria, $\varphi_{03}^{(2)}=0{,}022$ for Economic Criteria, and $\varphi_{04}^{(2)}=0{,}100$ for Social Criteria.

The scalar convolution of the indicated aggregated criteria of the second hierarchy level has allowed obtaining the normalized estimate for the whole Biosorbent space research project as an aggregated criterion of the third hierarchy level $\varphi_0*=\varphi_0^{(3)}=0{,}094$. Comparison of the analytical estimates to the converted normalized fundamental scale (Table 2) allows concluding that all of the aggregated criteria are within the limits of the High Quality gradation. Note that in calculations of this example, all of the criteria were assumed to be of identical significance, i.e., a multicriteria problem without priorities was solved.

Evaluation of a given alternative and selection of the best one pertain to the class of problems of structural synthesis. Problems of parametrical synthesis solved by the method of nested scalar convolutions are described in [12]. Thus, any multicriteria problem can be represented as a hierarchical system; at its lower level partial properties of the object are evaluated using a vector of criteria, and at the upper level, the object is evaluated as a whole by means of composition.

Bibliography

1. I.M. Makarov, T.M. Vinogradskaya, A.A. Rubchinskij, and V.B. Sokolov, The Theory of Choice and Decision Making [in Russian], Nauka, Moscow (1982).
2. D.B. Yudin, Computational Methods in Decision Theory [in Russian], Nauka, Moscow (1989).
3. V.A. Gubanov, V.V. Zakharov, and A.N. Kovalenko, Introduction to Systems Analysis [in Russian], Izd. LGU, Leningrad (1988).
4. O.I. Larichev, “Problems of collective decision-making in small groups”, in: Nonlinear Dynamics and Control (To the 70-th Anniversary of Acad. S.V. Yemel’yanov [in Russian], Editorial URSS, Moscow (1999), pp. 91-103.
5. A.N. Voronin, “Systems analysis and multicriteria evaluation of space projects by expert methods”, Probl. Upravl. Inform., No. 1, 121-135 (2004).
6. V.G. Totsenko, Decision Support Systems and Methods [in Russian], Naukova Dumka, Kiev (2002).
7. T.L. Saaty, Multicriteria Decision Making: The Analytical Hierarchy Process, McGraw-Hill, New York (1990).
8. A.N. Voronin, “Nested scalar convolutions of a vector-valued criterion”, Probl. Upravl. Inform., No. 5, 10-21 (2003).
9. A.N. Voronin, “Adaptive approximation models in optimization”, Cybern. Syst. Analysis, 30, No. 5, 700-708 (1994).
10. A.N. Voronin, Yu. K. Ziatdinov, and A.I. Kozlov, Vector Optimization of Dynamic Systems [in Russian], Tekchnika, Kiev, (1999).
11. L.A. Zadeh, The Concept of a Linguistic Variable and its Application to Approximate Reasoning [in Russian], Mir, Moscow, (1976).
12. A.N. Voronin, “Vector optimization of hierarchical structures”, Probl. Upravl. Inform., No. 6, 26-34 (2004).

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