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# DISCRIMINANT SETS OF FAMILIES OF HYPERBOLIC POLYNOMIALS OF DEGREE 4 AND 5*

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*Communicated by J.-P. Françoise*

*To the memory of my mother*

Abstract. A real polynomial of one real variable is *hyperbolic* (resp. *strictly hyperbolic*) if it has only real roots (resp. if its roots are real and distinct). We prove that there are 116 possible non-degenerate configurations between the roots of a degree 5 strictly hyperbolic polynomial and of its derivatives (i.e. configurations without equalities between roots). The standard Rolle theorem allows 286 such configurations. To obtain the result we study the *hyperbolicity domain* of the family $P(x;a,b,c) = x^5 - x^3 + ax^2 + bx + c$ (i.e. the set of values of $a,b,c \in \mathbf{R}$ for which the polynomial is hyperbolic) and its stratification defined by the discriminant sets $\mathrm{Res}(P^{(i)}, P^{(j)}) = 0$, $0 \le i < j \le 4$.

## 1. Introduction.

### 1.1. Statement of the problem.

**Definition 1.** *A real polynomial of degree $n$ of one real variable is called* hyperbolic *if it has only real roots (multiple roots are allowed). The derivatives*

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*of all orders* $\leq n-1$ *of a hyperbolic polynomial are also hyperbolic. A polynomial is* strictly hyperbolic *if its roots are real and distinct (hence, this is the case of all its derivatives of orders* $\leq n-1$ *as well). If the coefficients of a polynomial depend on parameters, then the set of the values of the parameters for which the polynomial has only real roots is called its* hyperbolicity domain.

Denote by $x_j^i$ the roots of the $i$-th derivative of a hyperbolic polynomial of degree $n$, $x_1^i \leq \ldots \leq x_{n-i}^i$; we set $x_j = x_j^0$. The standard Rolle theorem implies that if the polynomial is strictly hyperbolic, then for $k > i$, $j = 1, \ldots, n-k$ one has

$$x_j^i < x_j^k < x_{j+k-i}^i \tag{1}$$

In the present paper we are interested in the question which *non-degenerate configurations* of the roots of a hyperbolic polynomial and of its derivatives consistent with these inequalities are possible to take place (i.e. configurations without equalities between roots $x_{j_1}^{i_1} = x_{j_2}^{i_2}$, for any $(i_1, j_1) \neq (i_2, j_2)$). For $n = 1, 2$ or $3$ all such configurations are realized by hyperbolic polynomials. For $n = 1$ and $2$ there is a single possible configuration, respectively $x_1$ and $x_1 < x_1^1 < x_2$. For $n = 3$ there are two possible such configurations:

$$x_1 < x_1^1 < x_1^2 < x_2 < x_2^1 < x_3 \quad \text{and} \quad x_1 < x_1^1 < x_2 < x_1^2 < x_2^1 < x_3$$

realized by the polynomial $x^3 - x + q$ respectively for $q \in (0, 2\sqrt{3}/9)$ and $q \in (-2\sqrt{3}/9, 0)$. For $n = 4$ there are 12 non-degenerate configurations consistent with (1) two of which cannot be realized by hyperbolic polynomials, see Section 3.

In the present paper we consider a generic family of monic polynomials of degree 5 and their derivatives of all orders. Our aim is to describe the possible non-degenerate configurations of the roots of a hyperbolic polynomial of degree 5 and of its derivatives. We prove the following

**Theorem 2.** *There are* 116 *possible non-degenerate configurations of the roots of a hyperbolic polynomial of degree* 5 *and of its derivatives up to order* 4.

On the other hand, there are 286 non-degenerate configurations consistent with conditions (1) (for arbitrary $n$ this number is $\binom{n+1}{2}!\dfrac{1!2!\ldots(n-1)!}{1!3!\ldots(2n-1)!}$, see [14] or [15]). The different cases where non-degenerate configurations consistent with (1) are not realized by hyperbolic polynomials are discussed in the form of Observations. The absence of some of the configurations is closely connected with the presence of *overdetermined strata* in any generic family of monic hyperbolic polynomials, see Section 4.

**1.2. Historical remarks.** The property of being hyperbolic has been considered in the case of several variables as well. Some properties of hyperbolic polynomials and criteria of hyperbolicity have been studied at the beginning of the twentieth century, see e.g. [13], ch. 5-6. In the 60's and 70's the interest to hyperbolic polynomials (mainly in the case of several variables) was stimulated by the works of I. G. Petrovsky and L. Hörmander contributing to the theory of linear partial differential equations with constant coefficients.

Also in the case of one variable some new results appeared, see e.g. [11]. In the 80's V. I. Arnold and his students wrote several papers on hyperbolic polynomials motivated by their application to potential theory, see [2], [3], [5], [6] and [7]. Some of these results appeared in parallel in the thesis and papers of I. Meguerditchian, see [9] and [10].

The case $n = 4$ is mentioned in [1].

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**2. Preliminaries.** Consider the generic family of polynomials $P(x, a) = x^n + a_1x^{n-1} + \ldots + a_n$, $x, a_i \in \mathbf{R}$. Denote by $\Pi^*$ its hyperbolicity domain. After the shift $x \mapsto x - a_1/n$ the study of $\Pi^*$ is reduced to the case $a_1 = 0$.

**Lemma 3.** *In the case $a_1 = 0$ the polynomial $P$ is hyperbolic only if $a_2 \leq 0$. If $a_1 = a_2 = 0$, then $P$ is hyperbolic only for $a_2 = \ldots = a_n = 0$.*

Indeed, all derivatives of $P$ must be hyperbolic, in particular $P^{(n-2)} = (n!/2)x^{n-2} + (n-2)!a_2$, therefore $a_2 \leq 0$. Let $a_1 = a_2 = 0$. As $P^{(n-3)} = (n!/6)x^{n-3} + (n-3)!a_3$ must be hyperbolic, one has $a_3 = 0$ etc. □

For $t \in \mathbf{R}$ set $\tilde{a}_j = e^{jt}a_j$. One has $P(e^tx, \tilde{a}) = e^{nt}P(x, a)$. Therefore to study $\Pi^*$ it suffices to study $\Pi = \Pi^* \cap \{a_1 = 0, a_2 = -1\}$. This is what we do. To find out the possible non-degenerate configurations of the roots of $P$ and of all its derivatives we consider the *discriminant sets* $D(i, j) := \{(a_3, \ldots, a_n) \in \Pi | \mathrm{Res}(P^{(i)}, P^{(j)}) = 0\}$.

The paper is structured as follows. We consider the case $n = 4$ in Section 3. In Section 4 we define a stratification of $\Pi$ and we introduce the notion of an *overdetermined stratum*.

The rest of the paper deals with the case $n = 5$, i.e. with the family $P = x^5 - x^3 + ax^2 + bx + c$. In Section 5 we describe the discriminant sets and we give pictures of them together with $D(0,1)$. We describe the non-degenerate configurations of the roots of $P^{(i)}$ and $P^{(j)}$ (not of all derivatives simultaneously) for $(a,b,c)$ from each open part of $\Pi$ into which it is divided by the set $D(i,j)$.

In Section 6 we prove Theorem 16 from Section 4 which states that for $n=5$ there are only three overdetermined strata, all of dimension 0.

In Section 7 we use the results of Section 5 to prove Theorem 2 which amounts to counting the number of open parts of $\Pi$ into which all sets $D(i,j)$ together divide $\Pi$. Having the information of Section 5 the reader should be able to describe the configuration of the roots of $P$ and its derivatives in each of these open parts.

**3. The case $n = 4$.** Consider the family of monic polynomials $P = x^4 - x^2 + ax + b$, $x, a, b \in \mathbf{R}$. The hyperbolicity domain $\Pi$ of $P$ is the interior of the curvilinear triangle $D'E'A'$, see Fig. 1. The configurations of the roots of $P$, $P'$, $P''$ and $P'''$ (denoted respectively by 0, f, s, t) are indicated on the figure by *configuration vectors* on which coinciding roots are put in square brackets. E.g. the configuration vector corresponding to the point $A'$ $([0f0], s, [ft], s, [0f0])$ means that $x_1 = x_2 = x_1^1 < x_1^2 < x_2^1 = x_1^3 < x_2^2 < x_3 = x_4 = x_3^1$.

The discriminant sets are:

| | |
|---|---|
| $D(0,1) : 4b(4b-1)^2 + a^2 - 27a^4/4 - 36a^2b = 0$ | $D(0,2) : \pm a/\sqrt{6} + b = 5/36$ |
| $D(1,2) : a = \pm 4/3\sqrt{6}$ | $D(0,3) : b = 0$ |
| $D(1,3) : a = 0$ | $D(2,3) = \emptyset$ |

The set $D(0,1)$ has a self-intersection point at $A'$ and cusps at $D'$ and $E'$; these cusp points belong to $D(1,2)$. The set $D(0,2)$ has a self-intersection point at the point $B'$; the latter and $A'$ belong to $D(1,3)$. The line $D(0,3)$ is tangent to $D(0,1)$ at $C'$ where $\{C', A'\} = D(0,1) \cap D(1,3)$. All discriminant sets are invariant w.r.t. the involution $(a,b) \mapsto (-a,b)$.

There are 10 open domains in $\Pi$ defined by the 10 non-degenerate configurations indicated beside the figure. By the general formula from the Introduction there are 12 non-degenerate configurations consistent with (1). The absence of the two configurations $(0,f,0,s,t,f,0,s,f,0)$ and $(0,f,s,0,f,t,s,0,f,0)$ is connected with the fact that $D(1,3)$ passes through the intersection point of the two lines of which $D(0,2)$ consists.

**4. Overdetermined strata.** Denote by $\mathrm{Pol}_n^{\mathbf{C}}$ (resp. by $\mathrm{Pol}_n$) the space of all monic polynomials of degree $n$ with complex coefficients (resp. of

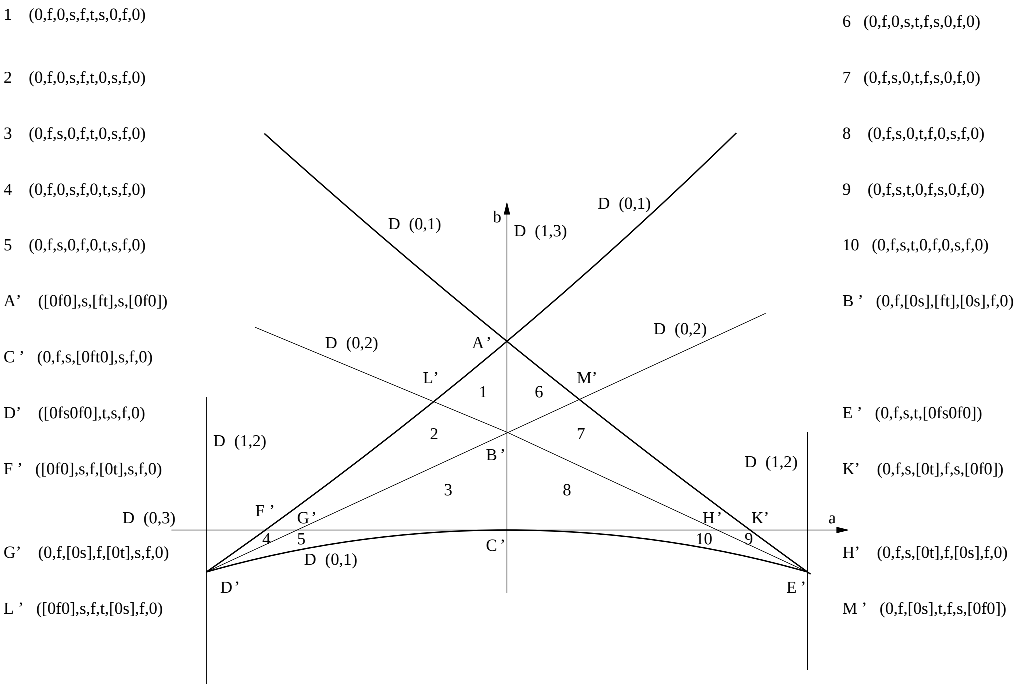


Fig. 1. The case n=4

all monic hyperbolic polynomials of degree $n$). The set of all possible $n$-tuples $(P_n, \ldots, P_1)$ where $P_i \in \ Pol_i^{\mathbf{C}}$ (resp. $P_i \in \mathrm{Pol}_i$) can be identified with the space $\tilde{X}$ of their roots (we denote the roots of $P_i$ by $x_j^i$, in the case of $\mathrm{Pol}_n$ one has $x_1^i \leq \ldots \leq x_{n-i}^i$).

Stratify $\tilde{X}$. A *stratum* of codimension $k$ in $\tilde{X}$ is a subset of $\tilde{X}$ defined by $k$ independent equalities of the form $x_{j_1}^{i_1} = x_{j_2}^{i_2}$, $(j_1, i_1) \neq (j_2, i_2)$.

For $P \in \mathrm{Pol}_n^{\mathbf{C}}$ or $P \in \mathrm{Pol}_n$ set $P_{n-i} = ((n-i)!/n!)P^{(i)}$. One can embed $\mathrm{Pol}_n$ into $\tilde{X}$ by sending $P \in \mathrm{Pol}_n$ into $(P_n, \ldots, P_1)$; denote the embedding by $\pi$. For a stratum $S$ of $\tilde{X}$ denote by $S'$ its intersection with $\pi(\mathrm{Pol}_n)$.

**Definition 4.** *The stratum $S$ is said to be* overdetermined *if its codimension in $\tilde{X}$ (called its* order – $\mathrm{ord}\, S$*) is greater than the codimension of $S'$ in $\pi(\mathrm{Pol}_n)$ (called its* codimension – $\mathrm{codim}\, S$*). Call* surplus *of $S$ the difference* $\mathrm{surp}\, S = \mathrm{ord}\, S - \mathrm{codim}\, S$.

**Definition 5.** *A* multiplicity vector *is a vector whose components are the multiplicities of the roots of a hyperbolic polynomial. The roots are listed in increasing order. Example: the multiplicity vector* $[1, 2, 2]$ *means that one has* $x_1 < x_2 = x_3 < x_4 = x_5$.

**Example 6.** *For even* $n$ *(resp. for odd* $n$*) the polynomial* $P_e = (x^2 - 2/n)^{n/2}$ *(resp.* $P_o = x(x^2 - 2/(n-1))^{(n-1)/2}$*) defines an overdetermined stratum. Indeed, one has*

$$P_e'' = (n(n-1)x^2 - 2)(x^2 - 2/n)^{n/2-2}, \quad P_e^{(n-2)} = (n!/2)(x^2 - 2/n(n-1)),$$

*hence,* $P_e^{(n-2)}$ *divides* $P_e''$*. On the other hand, the polynomial* $P_e$ *is completely defined by the conditions its multiplicity vector to be* $[n/2, n/2]$ *and its first three coefficients to be* $1$, $0$ *and* $-1$*; these conditions do not formally imply that* $P_e^{(n-2)}$ *divides* $P_e''$*. It is also true that* $P_e^{(n-1)} = n!x$ *divides all derivatives of* $P_e$ *of odd order.*

*For* $P_o$ *one has*

$$P_o' = (nx^2 - 2/(n-1))(x^2 - 2/(n-1))^{(n-3)/2}, \quad P_o^{(n-2)} = (n!/2)(x^2 - 2/n(n-1)),$$

*hence,* $P_o^{(n-2)}$ *divides* $P_o'$*. However,* $P_o$ *is completely defined by the conditions its multiplicity vector to be* $[(n-1)/2, 1, (n-1)/2]$*,* $P_o$ *to be divisible by* $P_o^{(n-1)} = n!x$ *and the first three coefficients of* $P_o$ *to be* $1$, $0$ *and* $-1$*; these conditions do not involve* $P_o^{(n-2)}$ *or* $P_o'$*.*

*Observe that if* $n$ *is odd, then up to a constant factor and rescaling of the* $x$*-axis one has* $P_e = \int_0^x P_o(t)dt$ *where* $P_e$ *is defined for* $n + 1$*.*

**Remarks 7.** 1) In this example one has

$$\begin{array}{lll} \operatorname{ord} P_e \geq 3n/2 - 1 & \operatorname{codim} P_e = n - 2 & \operatorname{surp} P_e \geq n/2 + 1 \\ \operatorname{ord} P_o \geq (3n-3)/2 & \operatorname{codim} P_o = n - 2 & \operatorname{surp} P_o \geq (n+1)/2 \end{array}$$

because in the definition of the stratum of $P_e$ in $\tilde{X}$ at least the following equalities are involved: $x_1 = \ldots = x_{n/2}$, $x_{n/2+1} = \ldots = x_n$; $x_1^{n-1} = x_{n/2-i+1}^{2i-1}$ for $i = 1, 2, \ldots, n/2-1$; $x_1^{n-2} = x_{n/2-1}^2$, $x_2^{n-2} = x_{n/2}^2$ and eventually some more of them; for the stratum of $P_o$ we have the equalities $x_1 = \ldots = x_{(n-1)/2}$, $x_{(n+3)/2} = \ldots = x_n$; $x_1^{n-1} = x_{(n+1)/2-i}^{2i}$ for $i = 0, 1, \ldots, (n-3)/2$; $x_1^{n-2} = x_{(n-1)/2}^1$, $x_2^{n-2} = x_{(n+1)/2}^1$ and eventually others.

2) By definition, the polynomial $(n/2)^{n/2}((n/2)!/n!)(P_e)^{(n/2)}(x\sqrt{2/n})$ is the Legendre polynomial of degree $n/2$.

To estimate more exactly the orders of the strata defined by $P_o$ and $P_e$ one can use the following

**Proposition 8.** *For all even* $n$ *and* $0 < s < n/2$ *the polynomial* $P_e^{(n/2+s)}$ *divides the polynomial* $P_e^{(n/2-s)}$.

P r o o f. The polynomial $P_e^{(n/2)}(x\sqrt{2/n})$ (denoted by $y$ for short) satisfies the differential equation

$$(x^2-1)y'' + 2xy' - n(n+1)y = 0 \tag{2}$$

(recall that $(n/2)^{n/2}((n/2)!/n!)y$ is a Legendre polynomial). Denote (for $k = 1, \ldots, n/2$) by $y^{(-k)}$ the primitive of $y^{(-k+1)}$ divisible by $(x^2-1)^k$. By (2), one has $((x^2-1)y')' = n(n+1)y$, hence, $(x^2-1)y' = n(n+1)y^{(-1)}$ and $y'$ divides $y^{(-1)}$. This means that $P_e^{(n/2+1)}$ divides $P_e^{(n/2-1)}$.

Suppose that for $i \leq s$ one has $(x^2-1)^i y^{(i)} = \alpha_i y^{(-i)}$, $\alpha_i \in \mathbf{R}^*$. Differentiating equation (2) $s$ times one gets

$$(x^2-1)y^{(s+2)} + (2s+2)y^{(s+1)} - (n(n+1) - s(s+1))y^{(s)} = 0, \text{ i.e.}$$

$$((x^2-1)^{s+1}y^{(s+1)})' = (x^2-1)^s \beta_s y^{(s)}$$

where $\beta_s = (n(n+1) - s(s+1))$. Hence, $((x^2-1)^{s+1}y^{(s+1)})' = \alpha_s\beta_s y^{(-s)}$. Integrating both sides one gets $(x^2-1)^{s+1}y^{(s+1)} = \alpha_s\beta_s y^{(-s-1)}$. This means that $P_e^{(n/2+s+1)}$ divides $P_e^{(n/2-s-1)}$. □

**Remarks 9.** 1) The presence of overdetermined strata of dimension 1 in the family of hyperbolic polynomials $P(x,a) = x^n + a_2x^{n-2} + a_3x^{n-3} + \ldots + a_n$ can be explained in part by the fact that outside 0, $n-1$ sheats of discriminant sets $\mathrm{Res}(P^{(i)}, P^{(j)}) = 0$ never intersect transversally because they are all invariant under the one-parameter group of quasi-homogeneous transformations $a_j \mapsto e^{jt}a_j$, $t \in \mathbf{R}$.

2) If at some point three discriminant sets intersect along a variety of dimension higher than the expected one, then this variety does not necessarily define an overdetermined stratum. Example: the three discriminant sets define locally the conditions $x_a^b = x_c^d$, $x_c^d = x_g^h$ and $x_a^b = x_g^h$; the third equality being a corollary of the first two, the intersection with the third discriminant set does not decrease the dimension of the intersection of the first two.

**Proposition 10.** *There are no overdetermined strata for* $n < 4$. *For* $n = 4$ *the points* $A'$ *and* $B'$ *(see Fig. 1) define the only overdetermined strata.*

Indeed, for $n < 4$ the proposition is to be checked directly. For $n = 4$ at $A'$ the condition $x_1^3 = x_2^1$ is not a corollary of $x_1 = x_2$, $x_3 = x_4$. At $B'$ the condition $x_1^3 = x_2^1$ is not a corollary of $x_2 = x_1^2$, $x_3 = x_2^2$. Only at $A'$, $B'$, $C'$ $D'$ and $E'$ do three discriminants or sheats of discriminants meet at one point and all intersections by two are transversal. The points $C'$, $D'$ and $E'$ do not define overdetermined strata, see part 2) of Remarks 9. □

For $n = 5$ the exhaustive answer to the question which strata are overdetermined is given by Theorem 16, see Subsection 5.1.

**Definition 11.** *Call* Gegenbauer's polynomial of degree $n$ *the hyperbolic polynomial of the form* $x^n - x^{n-2} + \ldots$ *which is divisible by its second derivative (one can show easily that for* $n \geq 3$ *such a polynomial exists and is unique; moreover, it is even or odd together with* $n$*). Hence, for any* $n$ *Gegenbauer's polynomial defines an overdetermined stratum of dimension* 0 *in the family* $P(x,a)|_{a_2=-1}$ *(defined in 1) of Remarks 9) because it is completely defined by the condition to be divisible by its second derivative and one gets the additional condition that* $P^{(n-1)} = n!x$ *divides all its derivatives which are odd polynomials.*

**Remark 12.** Denote by $Q$ the polynomial $P_e$ defined for $n \in 2\mathbf{N}^*$. Proposition 8 implies that up to rescaling of the $x$-axis and up to a non-zero constant factor Gegenbauer's polynomial of degree $n/2$ equals $Q^{(n/2-1)}$. Indeed, $Q^{(n/2+1)} = (Q^{(n/2-1)})''$ divides $Q^{(n/2-1)}$.

**Example 13.** *For* $n = 3$ *(resp.* $n = 4$, $n = 5$*) Gegenbauer's polynomial equals* $x^3 - x$ *(resp.* $x^4 - x^2 + 5/36 = (x^2 - 1/6)(x^2 - 5/6)$, $P_G = x^5 - x^3 + 21x/100 = x(x^2 - 3/10)(x^2 - 7/10)$*).*

One can obtain examples of overdetermined strata of higher dimension (and in families of polynomials of higher degree) by integrating already existing examples. In the case of $\mathrm{Pol}_n$ one has to check that the polynomial obtained by integrating is hyperbolic for some values of the constant of integration.

**Example 14.** *Consider the family of polynomials* $Q(x,a,b,c,d) = x^6/6 - x^4/4 + ax^3 + bx^2 + cx + d$. *One can transform this family (up to a constant factor) into* $P(x,a)|_{a_2=-1}$ *(*$P$ *is defined in* 1) *of Remarks* 9*), by rescaling the variable* $x$. *The family* $Q$ *contains an overdetermined stratum of dimension* 1 *(hence,* $P$ *contains such a stratum of dimension* 2*). Indeed, the polynomial* $R(x,d) = x^6/6 - x^4/4 + 21x^2/200 + d$ *is hyperbolic for* $d \in [-81/6000, -49/6000]$. *(To see this it suffices to evaluate* $R$ *at its critical points which are the zeros of* $P_G$ *where* $P_G$ *is defined in Example* 13.*) This polynomial is defined by the condition that* $R'''$

*divides $R'$ (these are in fact three conditions). They imply that $R^{(5)}$ divides $R^{(3)}$. Hence, this defines an overdetermined stratum of dimension 1 of the family $Q$.*

*The example can be given also in the context of polynomials with complex coefficients in which case the question of hyperbolicity of $R(x,d)$ is not raised.*

## 5. Description of the discriminant sets for $n = 5$.

**5.1. Generalities and notation.** In what follows we consider the family $P$ of polynomials where

$$P = x^5 - x^3 + ax^2 + bx + c \quad P' = 5x^4 - 3x^2 + 2ax + b \quad P'' = 20x^3 - 6x + 2a$$
$$P''' = 60x^2 - 6 \qquad P^{(4)} = 120x$$

For some of the couples $(i,j) \neq (0,1)$ the set $D(i,j)$ divides $\Pi$ into parts whose open interiors are the subsets of $\Pi$ where the configuration of the roots of $P^{(i)}$ and $P^{(j)}$ is non-degenerate and fixed.

The set $\Pi$ and the discriminant sets are invariant under the involution

$$\sigma : (a,b,c) \mapsto (-a,b,-c) \tag{3}$$

**Notation 15.** *We denote the roots of $P$ by $x_1 \leq \ldots \leq x_5$, the ones of $P'$ (resp. $P''$, $P'''$, $P^{(4)}$) by $f_1 \leq \ldots \leq f_4$ (resp. $s_1 \leq s_2 \leq s_3$, $t_1 \leq t_2$, $l$); we choose these letters to match "first", "second", "third" and "last". If it is necessary, we use also the notation $x_j^i$ for the roots of $P^{(i)}$.*

The following theorem is proved in Section 6.

**Theorem 16.** 1) *For $n = 5$ the only overdetermined strata of $\Pi$ are the points $\Sigma$, $\Phi$ and $F$ defined as follows:*

$$\Sigma \ : \ x_2 = t_1, \ x_4 = t_2, \ x_3 = s_2 = l$$
$$\Phi \ : \ x_2 = s_1, \ x_3 = s_2, \ x_4 = s_3, \ l = s_2$$
$$F \ : \ x_1 = x_2, \ x_4 = x_5, \ f_2 = t_1, \ f_3 = t_2, \ l = s_2 = x_3$$

*(the point $\Phi$ defines Gegenbauer's polynomial of degree 5).*

2) *One has*

$$\operatorname{ord}\Sigma = 4 \quad \operatorname{codim}\Sigma = 3 \quad \operatorname{surp}\Sigma = 1$$
$$\operatorname{ord}\Phi = 4 \quad \operatorname{codim}\Phi = 3 \quad \operatorname{surp}\Phi = 1$$
$$\operatorname{ord}F = 6 \quad \operatorname{codim}F = 3 \quad \operatorname{surp}F = 3$$

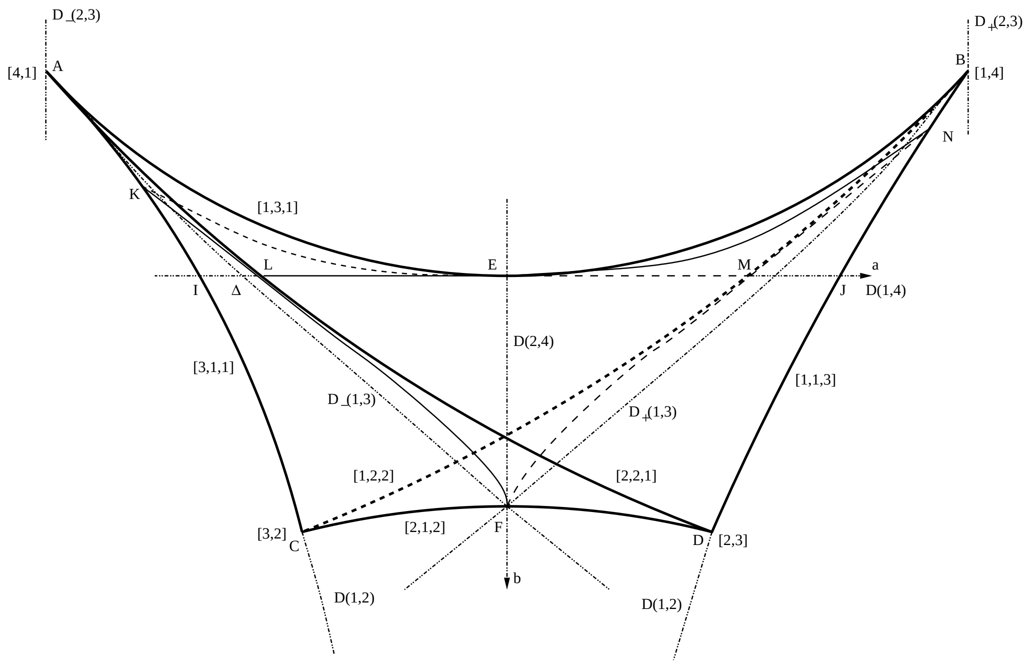


Fig. 2. The sets $D(0,1)$, $D(0,3)$, $D(0,4)$, $D(1,2)$, $D(1,3)$, $D(1,4)$, $D(2,3)$ and $D(2,4)$

**5.2. The sets $D(0,1)$, $D(0,4)$, $D(1,2)$, $D(1,3)$, $D(1,4)$, $D(2,3)$ and $D(2,4)$.** In what follows we use in general the same notation for the points and curves in $Oabc \simeq \mathbf{R}^3$ and for their projections in the $ab$-plane. On Fig. 2 we show **the set** $D(0,1)\cap\Pi$ (the boundary of $\Pi$) – this is the curvilinear tetrahedron $ABCD$. Its interior (denoted by $\tilde{\Pi}$) represents all strictly hyperbolic polynomials from the family $P$. Its faces (i.e. two-dimensional strata) represent hyperbolic polynomials with multiplicity vectors as follows:

$$ABC \quad [1,2,1,1] \qquad ACD \quad [2,1,1,1] \qquad BCD \quad [1,1,1,2] \qquad ABD \quad [1,1,2,1]$$

The multiplicity vectors of the six edges and of the four vertices are noted on the figure. The reader can find the proof of the properties of $\Pi$ in [6] and [7]. One can deduce from the results of [6] part 1) of the following proposition, parts 2) and 3) being well-known classical results:

**Proposition 17.** 1) *The hyperbolicity domain* $\Pi$ *is the set of points contained between and on the graphs of two Lipschitz functions of the variables* $a, b$ *defined on the projection of* $\Pi$ *on the* $(a, b)$*-plane. The first (resp. the second) graph consists of the closures of the faces* $ACD$ *and* $ABD$ *(resp.* $ABC$ *and* $BCD$*). The values of the two functions coincide only on the union of the projections in the* $(a, b)$*-plane of the four edges* $AB$*,* $AC$*,* $CD$ *and* $BD$*.*

2) *Each of the four faces of* $\Pi$ *is locally concave toward the interior of* $\Pi$*.*

3) *The discriminant sets* $D(i, j)$ *are ruled surfaces.*

The positive $a$-axis (resp. $b$-axis) is the half-line $EM$ (resp. $EF$). The positive $c$-axis is the vertical half-line at $E$ directed downward. The tetrahedron $ABCD$ is invariant under the involution (3), its projection on the $ab$-plane is symmetric w.r.t. $EF$.

**Proposition 18.** *The projections on the ab-plane of the six edges of* $\Pi$ *are real algebraic simple Jordan curves without inflection points. Their concavity is as shown on Fig.* 2*.*

The proposition is proved in Subsection 5.5.

**The sets** $D(1, 4) = \{b = 0\}$ **and** $D(2, 4) = \{a = 0\}$ are vertical planes, i.e. planes parallel to the $c$-axis, they are represented by the dash-dot-dot lines $IJ$ and $EF$.

**Remark 19.** In the part of $\tilde{\Pi}$ whose projection on the $ab$-plane is inside $AKILEA$ (resp. inside $BNJMEB$ or $ICFDJMELI$) one has $f_3 < l < f_4$ (resp. $f_1 < l < f_2$ or $f_2 < l < f_3$). To the left (resp. right) of $EF$ one has $s_2 < l < s_3$ (resp. $s_1 < l < s_2$). This is easy to deduce from the presence in the closures of some of these projections of the points $A$, $B$ and $C$ whose multiplicity vectors are $[4, 1]$, $[1, 4]$ and $[3, 2]$.

**The set** $D(0, 4) = \{c = 0\}$ is a horizontal plane. The set $D(0, 4) \cap \Pi$ is the curvilinear plane figure $FMNEKLF$. It is tangent to the faces $ADB$ and $ABC$ along the line segments $LE$ and $EM$. (These line segments belong to the line $b = c = 0$, they define hyperbolic polynomials for which 0 is a double root of $P$, so they belong to $D(0, 4) \cap \Pi$).

**Remarks 20.** 1) The tangency along $LE$ and $EM$ can be proved like this. On $LE$ one has $x_3 = x_4 = l$. By continuous deformation one can obtain one of the two conditions $x_3 < x_4 = l$ or $x_3 = l < x_4$ the polynomial remaining hyperbolic and becoming strictly hyperbolic (the reader will easily prove the existence of these deformations). Hence, the deformations lead to different half-planes of $\{c = 0\}$ (defined by the line $\{b = c = 0\}$) and at the same time inside $\Pi$. A similar remark holds for $EM$ on which one has $x_2 = x_3 = l$ etc.

2) In the part $AKLEA$ (resp. $KLFCMEK$, $LFDNEL$, $BEMNB$) of $\tilde{\Pi}$ one has $x_4 < l < x_5$ (resp. $x_3 < l < x_4$, $x_2 < l < x_3$, $x_1 < l < x_2$) – observe that each of the closures of these parts contains exactly one of the vertices $A$, $B$, $C$, $D$ with known multiplicity vectors.

3) The segments $LE$ and $EM$ of tangency between $D(0,4)$ and $D(0,1)$ divide $D(0,4) \cap \Pi$ into three parts in which this intersection is defined by three different conditions: $x_4 = l$ in $KLEK$, $x_3 = l$ in $LFMEL$ and $x_2 = l$ in $EMN$.

**The set** $D(1,3)$ consists of the two vertical planes

$$D_{\pm}(1,3) : b \pm a\sqrt{2/5} = 1/4.$$

Hence, they pass through the point $F$. Indeed, $F$ defines the polynomial $P_F = x(x^2 - 1/2)^2$ (with multiplicity vector $[2,1,2]$ and set of roots invariant under the involution $x \mapsto -x$). For this polynomial one has $x_1 = x_2$, $x_4 = x_5$, $f_2 = t_1$, $f_3 = t_2$ and $x_3 = s_2 = l$. Recall that the point $F$ defines an overdetermined stratum, see Theorem 16.

**Proposition 21.** *The plane $D_-(1,3)$ intersects the face $ABD$ only at $A$. It is tangent to the edges $AD$ and $AC$ at $A$ and it intersects the edge $CD$ only at $F$. One has $D_-(1,3) \cap AD = \{A\}$, $D_-(1,3) \cap AC = \{A\}$. The curve $\theta = AF = ACD \cap D_-(1,3)$ is a simple Jordan curve projecting on the ab-plane into a segment. Analogous statements hold for the plane $D_+(1,3)$.*

Proof. The tangency of $D_-(1,3)$ to $AB$ and $AC$ at $A$ can be derived from the fact that $D(0,1)$ has a swallowtail singularity at $A$. The plane $D_-(1,3)$ intersects the face $ABD$ and the edge $AD$ only at $A$ because their projections on the $ab$-plane are tangent at $A$ and the projection of the edge $AD$ has the concavity as shown on Fig. 2, see Proposition 18. The plane $D_-(1,3)$ intersects the edge $AC$ also only at $A$ (use again the tangency at $A$ and the concavity).

The plane $D_-(1,3)$ intersects the edge $CD$ only at $F$. Indeed, if not, then due to the concavity of its projection in the $ab$-plane the edge $CD$ would intersect $D_-(1,3)$ exactly twice and the points $C$ and $D$ would be in one and the same half-space w.r.t. $D_-(1,3)$. However, one has $f_2 < t_1$ at $C$ and $f_2 > t_1$ at $D$ (to be checked directly; see also Remarks 22 below).

The curve $\theta$ is a simple Jordan curve. Indeed, the face $ACD$ is the graph of a Lipschitz function defined on the projection of the face $ACD$ in the $ab$-plane, see Proposition 17. To obtain $\theta$ one intersects this graph by the vertical plane $D_-(1,3)$ and the intersection projects in the $ab$-plane into a segment. □

Use the notation $(fftftf)$ in the sense that $f_1 < f_2 < t_1 < f_3 < t_2 < f_4$.

**Remarks 22.** 1) In the part of $\tilde{\Pi}$ which projects on the $ab$-plane into $AFCA$ (resp. $AFBEA$, $BFDB$) one has $(fftftf)$ (resp. $(ftfftf)$, $(ftftff)$). This can be deduced from the presence of the points $A$, $B$, $C$, $D$ and $E$ in the closures of some of these parts. Hence, there is one non-degenerate configuration of the roots only of $P'$ and $P'''$ consistent with (1) which is not realized by a hyperbolic polynomial of degree 5 – this is

$$(fttff) \quad \text{i.e.} \quad f_1 < f_2 < t_1 < t_2 < f_3 < f_4 \tag{4}$$

Geometrically this is illustrated as follows: the intersecting planes $D_{\pm}(1,3)$ divide the space into four sectors the lower of which does not intersect with $\tilde{\Pi}$, see Fig. 2.

In the case of degree 4 there are two non-degenerate configurations of the roots of the polynomial with *all* its derivatives which are missing but if one considers only couples of derivatives, then all configurations are realized.

2) It is proved in [8] that all non-degenerate configurations of the roots of $P$ and $P^{(i)}$ (for any degree $n$ and for any $i \geq 1$) which are consistent with (1) can be realized by hyperbolic polynomials. Hence, this result is optimal in the sense that if one wants to have all non-degenerate configurations of the roots of $P$ and two or more of its derivatives, or just of the roots of two or more derivatives, then there are counterexamples (e.g. choose the missing configuration $(ffttff)$ or complete it to a configuration of the roots of $P$, $P'$ and $P'''$ etc.).

**Observation 23.** *There are* 40 *non-degenerate configurations consistent with* (1) *and* (4) *(hence, none of them is realized by a hyperbolic polynomial).*

Proof. Suppose that $t_1 < x_3 < t_2$. One must have $t_1 < x_3, s_2, l < t_2$ which gives 6 possible permutations of $x_3, s_2, l$ within the interval $(t_1, t_2)$. One has also $f_1 < x_2, s_1 < f_2$ and $f_3 < x_4, s_3 < f_4$; one has two possible permutations of $x_2, s_1$ within $(f_1, f_2)$ and two permutations of $x_4, s_3$ within $(f_3, f_4)$. Combining all possibilities independently one gets $6 \times 2 \times 2 = 24$ non-degenerate configurations.

Let now $f_2 < x_3 < t_1$. One has two possible permutations of $x_2, s_1$ within $(f_1, f_2)$, two possible permutations of $x_4, s_3$ within $(f_3, f_4)$ and two possible permutations of $s_2, l$ within $(t_1, t_2)$ which gives $2 \times 2 \times 2 = 8$ non-degenerate configurations.

If $t_2 < x_3 < f_3$, then in the same way one gets 8 more non-degenerate configurations; all in all, we have $24 + 8 + 8 = 40$ of them. □

**Observation 24.** *There are* 102 *non-degenerate configurations consistent with* (1) *and* $(ftfftf)$*. Only* 66 *of them are realized by hyperbolic polynomials, see Lemmas* 40*,* 42 *and* 43*.*

**Observation 25.** *It follows from the general formula, see the Introduction, that for each of the cases* $(fftftf)$ *and* $(ftftff)$ *there are* 72 *non-degenerate configurations consistent with* (1)*. Indeed, it is clear that the numbers of configurations for both cases are equal because* $\Pi$ *is invariant under the involution* (3)*; on the other hand one has* $72 = (286 - 102 - 40)/2$*, see the general formula from the Introduction and Observations* 23 *and* 24*. Out of these* 72 *configurations only* 25 *are realized by hyperbolic polynomials, see Lemma* 41*.*

Proof of Observation 24. Set $J_l = (f_2, x_3)$, $J_r = (x_3, f_3)$, $J = (t_1, t_2)$. If $x_2 \in (f_1, s_1)$ or $x_2 \in (s_1, t_1)$, and if $x_4 \in (t_2, s_3)$ or $x_4 \in (s_3, f_4)$, then either $s_2 \in J_l$ or $s_2 \in J_r$, the interval $J$ is divided by the points $f_2$, $x_3$, $f_3$, $s_2$ into 5 parts to each of which $l$ can belong. This gives $2 \times 2 \times 2 \times 5 = 40$ non-degenerate configurations consistent with (1) and (4).

If $x_2 \in (t_1, f_2)$ and $x_4 \in (t_2, s_3)$ or $x_4 \in (s_3, f_4)$, then $s_2 \in J_l$ or $s_2 \in J_r$, and the interval $J$ is divided into 6 parts by the points $x_2$, $f_2$, $x_3$, $f_3$, $s_2$. This gives another $2 \times 2 \times 6 = 24$ non-degenerate configurations and one gets in the same way another 24 of them if $x_2 \in (f_1, s_1)$ or $x_2 \in (s_1, t_1)$ and $x_4 \in (f_3, t_2)$.

If $x_2 \in (t_1, f_2)$ and $x_4 \in (f_3, t_2)$, then $s_2 \in J_l$ or $s_2 \in J_r$ and the points $x_2$, $f_2$, $x_3$, $f_3$, $s_2$, $x_4$ divide $J$ into 7 parts which gives $2 \times 7 = 14$ non-degenerate configurations. Thus the total number of non-degenerate configurations consistent with (1) and (4) is $40 + 24 + 24 + 14 = 102$. □

**The set** $D(1,2)$ consists of lines parallel to the $c$-axis. It contains all strata having a triple root (i.e. the edges $AEB$, $AKC$ and $BND$).

**Remark 26.** The set $D(1,2)$ is the set of values of $a$ and $b$ for which $P'$ has a multiple root and its form can be deduced from the well-known picture of the swallowtail; its projection on the $ab$-plane contains the projections of the analytic continuations of the edges $AKC$ and $BND$ (given in dash-dot-dot line) which intersect further down (outside the figure). Hence, $D(1,2)$ does not contribute to the partitioning of $\tilde{\Pi}$ into parts with fixed configuration of the roots of $P$ and of its derivatives. The form of $D(1,2)$ can be deduced from Fig. 1 turned upside down, the lines $D'M'$ and $E'L'$ from Fig. 1 become the lines $BF$ and $AF$ from Fig. 2. (One has to rescale the axes but this leaves the picture essentially the same.)

**The set** $D(2,3)$ consists of the two planes $a=\pm\sqrt{2/5}$. They are vertical and contain the vertices $A$ and $B$ (because there is a quadruple root of $P$ there, hence, a root of $P''$ and of $P'''$). Thus $D(2,3)\cap\Pi=\{A,B\}$.

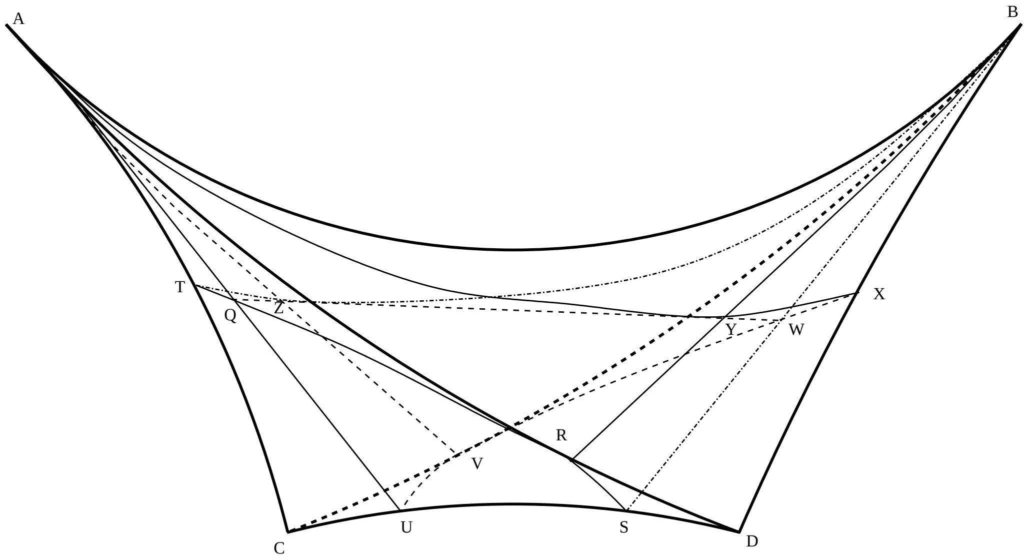


Fig. 3. The set $D(0,3)$

**5.3. The set $D(0,3)$.** The set $D(0,3)$ is the union of two planes

$$D_{\pm}(0,3): c\pm b/\sqrt{10}+a/10=\pm 9/100\sqrt{10}$$

(one substitutes $x$ in $P(x)=0$ by each of the two roots $\pm 1/\sqrt{10}$ of $P''$). Its intersection with $\Pi$ consists of the two curvilinear triangles $D_{+}(0,3)\cap\Pi = SWBZTQRS$ and $D_{-}(0,3)\cap\Pi = UQAYXWVU$ (see Fig. 3), tangent respectively to the faces $ABD$ and $ABC$ along the line segments $RB$ and $VA$. The latter belong to the set $D(0,1)\cap D(0,3)$ and to the planes $D_{\pm}(1,3)$.

**Remark 27.** The tangency along $RB$ and $VA$ can be proved like this (by analogy with Remark 20): on $RB$ one has $x_3=x_4=t_2=1/\sqrt{10}$. One can define two deformations as a result of which the polynomial $P$ becomes strictly hyperbolic and either with $x_3<x_4=t_2$ or with $x_3=t_2<x_4$. Such polynomials define points from $D(0,3)$ lying on different half-planes of $SWBZTQRS$ (defined

by the line $BR$) and inside $\Pi$. Hence, $SWBZTQRS$ is tangent to $ABD$ along $RB$. The two parts of $SWBZTQRS$ defined by the segment of tangency $RB$ are defined by two different conditions: $x_3 = t_2$ for $RSWBYR$ and $x_4 = t_2$ for $TQRYBZT$. (One has $x_2 = x_3 = t_1 = -1/\sqrt{10}$ on $VA$, $x_3 = t_1$ on $VUQAZV$ and $x_2 = t_1$ on $XWVZAYX$.)

The curvilinear triangles $SWBZTQRS$ and $UQAYXWVU$ intersect along the line segment $QZYW$ which is tangent to the faces $ABC$ and $ABD$ respectively at the points $Z$ and $Y$. The triangles divide $\Pi$ into the following 8 parts characterized by configurations between the roots of $P$ and $P'''$ (denoted by $x$ and $t$):

| | | | |
|---|---|---|---|
| $ATQZ$ | $(xxxtxtx)$ | $AQRYZ$ | $(xxtxxtx)$ |
| $TQUCVZ$ | $(xxxttxx)$ | $QRSUVWYZ$ | $(xxtxtxx)$ |
| $RSDXWY$ | $(xxttxxx)$ | $WXBY$ | $(xtxtxxx)$ |
| $VWBYZ$ | $(xtxxtxx)$ | $AZYB$ | $(xtxxxtx)$ |

(the notation $(xxxtxtx)$ means $x_1 < x_2 < x_3 < t_1 < x_4 < t_2 < x_5$).

The points defining the curvilinear triangle $SWBZTQRS$ are characterized as follows:

| | | | |
|---|---|---|---|
| $S$ | $x_1 = x_2,\ x_4 = x_5,\ x_3 = t_2$ | $W$ | $x_4 = x_5,\ x_2 = t_1,\ x_3 = t_2$ |
| $Z$ | $x_2 = x_3 = t_1,\ x_4 = t_2$ | $T$ | $x_1 = x_2 = x_3,\ x_4 = t_2$ |
| $Q$ | $x_1 = x_2,\ x_3 = t_1,\ x_4 = t_2$ | $R$ | $x_1 = x_2,\ x_3 = x_4 = t_2$ |

The equalities characterizing the points defining $UQAYXWVU$ are obtained from these ones by symmetry induced by the involution $x \mapsto -x$.

**Remarks 28.** 1) The $b$-coordinates of the points $T$, $X$ and $Q$, $Z$, $Y$, $W$ are positive. Indeed, the point $T$ is defined by the polynomial

$$(40)^{-5/2} P_T(\sqrt{40}x) \qquad \text{where}$$

$$P_T(x) = (x+3)^3(x-2)(x-7) = x^5 - 40x^3 - 90x^2 + 135x + 378$$

and one has $P_T''' = 60(x-2)(x+2)$; the projection on the $ab$-plane of the line $QW$ is parallel to the $a$-axis (this follows from the symmetry w.r.t. the $b$-axis of the projection of $\Pi$ on the $ab$-plane). The intersection of $QW$ with $\{a = 0\}$ is the point $\Sigma$ (the middle of the segment $QZYW$; see Theorem 16) defining the polynomial

$$P_\Sigma = x^5 - x^3 + (9/100)x = x(x^2 - 1/10)(x^2 - 9/10) = x(x^2 - 9/10)P_\Sigma'''/60\ .$$

This means that the $b$-coordinates of the points $Q$, $Z$, $Y$, $W$ equal 9/100. Hence, the points $A$, $K$, $I$, $T$, $C$ (resp. $B$, $N$, $J$, $X$, $D$) are situated in this order on the edge $AC$ (resp. $BD$). Indeed, the point $K$ defines the polynomial

$$P_K = x(x - \sqrt{3/2})(x + 1/\sqrt{6})^3 = x^5 - x^3 - (4/3\sqrt{6})x^2 - x/12 \ .$$

2) Denote by $\Lambda \in AC$ the point where one has $x_1 = x_2 = x_3 < x_4 = s_3 < x_5$. Hence, $\Lambda$ lies between $T$ and $C$. Indeed, when a point runs over the arc $CA$, then the root $x_4$ of the polynomial defined by the point goes from $x_5$ to $x_3$. Before having $x_4 = t_2$ (which takes place at $T$) one must have somewhere $x_4 = s_3$ because $t_2 < s_3 < x_5$.

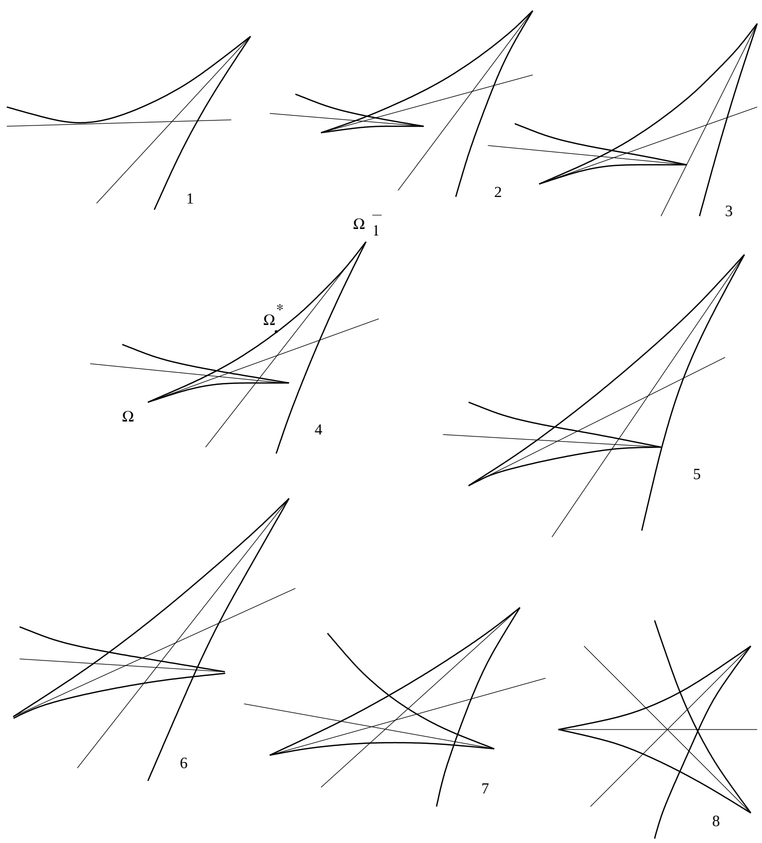


Fig. 4. The set $D(0,2)$

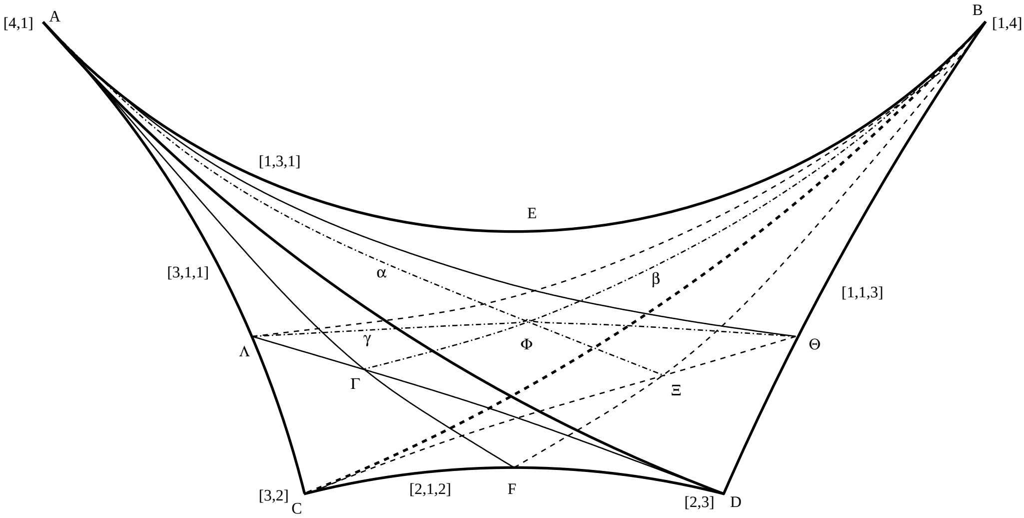


Fig. 5. The sets $D(0,2)$ and $D(0,1)$

**5.4. The set $D(0,2)$.** The set $D(0,2)$ is defined by the condition that $P(x_i) = P''(x_i) = 0$ for $i = 2, 3$ or 4. It can be easily described by its intersections with the planes $\{a = \text{const}\}$, see Fig. 4 where these intersections of $D(0,1)$ and $D(0,2)$ are shown. (We make use here of the pictures of the so-called butterfly catastrophe from [4] and [12].) On Fig. 4 the parameter $a$ takes 8 values from $-\sqrt{2/5}$ (the first value) to 0 (the last one); for $a \in [0, \sqrt{2/5}]$ the form of $D(0,2)$ can be deduced from the one for $a \in [\sqrt{2/5}, 0]$ using the involution (3).

The set $D(0,2)$ is represented together with $D(0,1)$ also on Fig. 5. It consists of three sheats – the surfaces $A\Gamma F\Xi BEA = \{x_3 = s_2\}$, $\Lambda\Gamma D\Theta B\Lambda = \{x_4 = s_3\}$ and $A\Lambda C\Xi\Theta A = \{x_2 = s_1\}$ which are parts of one and the same irreducible hypersurface. Inside the curvilinear tetrahedron $A\Lambda\Gamma\Phi$ (resp. $B\Theta\Xi\Phi$) one has $x_2 < s_1$, $x_3 < s_2$, $x_4 < s_3$ (resp. $x_2 > s_1$, $x_3 > s_2$, $x_4 > s_3$); one can deduce from here the configurations of the roots of $P$ and $P''$ in the other parts as well.

For $a \in [-\sqrt{2/5}, \sqrt{2/5}]$ the intersections of $D(0,2)$ with the planes $\{a = \text{const}\}$ consist each of three lines. Indeed, the equation $P''(x) = 20x^3 - 6x + 2a = 0$ defines three real values of $x$ which are distinct for $a \in (-\sqrt{2/5}, \sqrt{2/5})$. Substituting them in $P = x^5 - x^3 + ax^2 + bx + c = 0$ gives the equations of three lines in the $bc$-plane. For $a = \pm\sqrt{2/5}$ two of the lines coincide. These lines are the geometrical tangents at the singular points of the sets $D(0,1) \cap \{a = \text{const}\}$ (where one has $P = P' = P'' = 0$).

Set $\alpha = \{(a,b,c) \in \Pi | x_2 = s_1, x_3 = s_2\}$, $\beta = \{(a,b,c) \in \Pi | x_3 = s_2, x_4 = s_3\}$, $\gamma = \{(a,b,c) \in \Pi | x_2 = s_1, x_4 = s_3\}$. The sets $\alpha$, $\beta$, $\gamma$ consist of the self-intersection points of $D(0,2)$ (denoted by dash-dotted curves on Fig. 5). One has $\alpha = A\Phi\Xi$, $\beta = \Gamma\Phi B$, $\gamma = \Lambda\Phi\Theta$.

**Lemma 29.** *At $A$ and $B$ the set $D(0,2)$ has Whitney umbrella singularities.*

Indeed, set $\alpha = a + \sqrt{2/5}$. One deduces from $P''$ having a double root for $\alpha = 0$ that for $\alpha$ close to 0 this root equals $\sqrt{\alpha}\psi(\alpha)$ where $\psi$ is a germ of a smooth function, $\psi(0) \neq 0$. The equation $P(\sqrt{\alpha}\psi(\alpha)) = 0$ reads $\sqrt{\alpha}\psi(\alpha)(b - \alpha\psi^2(\alpha) + \alpha^2\psi^4(\alpha)) = c$. Set $\xi = \psi(\alpha)(b - \alpha\psi^2(\alpha) + \alpha^2\psi^4(\alpha))$. The last equation after taking squares of both sides becomes $\alpha\xi^2 = c^2$ which is the equation of Whitney's umbrella. $\square$

**Remarks 30.** 1) The point on 1 of Fig. 4 where the horizontal line is tangent to the set $D(0,1) \cap \{a = \text{const}\}$ is *not* a smooth point for the latter one – it is a point where $\Pi$ has a swallowtail singularity (the points $A$ and $B$ of Fig. 2 correspond to $a = \pm\sqrt{2/5}$). Note that the two planes from $D(2,3)$ are tangent to $D(0,2)$ at the points $A$ and $B$.

2) The sets $D(0,1) \cap \{a = \text{const}\}$ can be parametrized with the parameter $x$ by writing $c = -bx - ax^2 + x^3 - x^5$, $b = -2ax + 3x^2 - 5x^4$. Hence, the function $(dc/dx)/(db/dx)$ equals $-x$ on $D(0,1) \cap \{a = \text{const}\}$. This function equals $\tan\phi$ where $\phi$ is the angle between the tangent line and the positive $b$-axis. This means that $\phi$ is a strictly monotonous function of $x$. Hence, for $a \in (-\sqrt{2/5}, \sqrt{2/5})$ the three lines intersect two by two. Moreover, the intersection of $D(0,2)$ with $D(0,1)$ is transversal inside each of the four faces of $\Pi$.

3) Only for $a = 0$ do the three lines intersect at one point $\Phi$ (see 8 of Fig. 4 or look at the middle of Fig. 5). This point defines Gegenbauer's polynomial $P_G = x^5 - x^3 + 21x/100$ of degree 5 (one has $x_2 = s_1$, $x_3 = s_2$, $x_4 = s_3$), see Definition 11.

**Observation 31.** *The fact that $\Phi \in D(2,4)$ and that one has $x_2 < s_1$, $x_3 < s_2$, $x_4 < s_3$ inside $A\Lambda\Gamma\Phi$ explains why 36 non-degenerate configurations consistent with (1) are not realized by hyperbolic polynomials. (We do not discuss the question whether some of them are mentioned in previous observations or not.)*

Indeed, if $x_2 < s_1$, $x_3 < s_2$, $x_4 < s_3$, then $s_2 < l$ (i.e. $A\Lambda\Gamma\Phi \subset \{a \leq 0\}$) and the same is true if the directions of all inequalities are reversed (this can be deduced from $\Phi \in D(2,4)$). Hence, all non-degenerate configurations satisfying the inequalities $(A) : x_2 < s_1,\ x_3 < s_2,\ x_4 < s_3,\ s_2 > l$ are impossible for hyperbolic polynomials (and all configurations with the directions of all inequalities reversed).

Conditions $(A)$ and (1) imply that $t_2$ can belong to one of the three intervals (independently of the positions of $t_1$ and $l$): $(s_2, f_3)$, $(f_3, x_4)$ and $(x_4, s_3)$. Set $I_1 = (s_1, f_2)$, $I_2 = (f_2, x_3)$, $I_3 = (x_3, s_2)$. If $l \in I_3$ (resp. $l \in I_2$, $l \in I_1$), then $t_1 \in I_1$ or $I_2$ or $I_3$ (resp. $t_1 \in I_1$ or $I_2$, $t_1 \in I_1$). This gives 6 possibilities for $t_1, l$ which combined with the three possibilities for $t_2$ gives 18 non-degenerate configurations which cannot be realized by hyperbolic polynomials. Inverting the direction of the inequalities $(A)$ gives 18 more impossible non-degenerate configurations. □

**Proposition 32.** 1) *At the point $\Phi$ the set $D(0,2)$ is the transversal intersection of three smooth hypersurfaces.*

2) *The projection on the ab-plane of the set $\alpha$ (resp. $\beta$) belongs to $AFDBA$ (resp. to $ACFBA$).*

3) *The set $\{x_3 = s_2\} \subset D(0,2)$ intersects the face $ACD$ of $\Pi$ along the simple Jordan curve $A\Gamma F$ which projects on the ab-plane inside $AKICFA$ (i.e. below the line $AF$ on Fig. 2), with extremities at $A$ and $F$.*

The proposition is proved in Subsection 5.6.

**Observation 33.** 1) *The fact that the set $\alpha$ intersects $D_-(1,3) \cap \Pi$ only at $A$, see part 2) of the proposition, explains the absence of some non-degenerate configurations consistent with (1). Hence, the absence of configurations is explained not only with the presence of overdetermined strata.*

2) *The intersection of $\Pi$ with a cylinder of small radius centered at $F$ and parallel to the c-axis looks like shown on Fig. 6. (The positive a-axis (c-axis) is to the left (downward).) There are 10 non-degenerate configurations of the roots of $P$ and of its derivatives close to the one defined by the point $F$ (they correspond to the 10 components of the interior of $\Pi$, see the figure).*

*On the other hand, there are 24 non-degenerate configurations consistent with (1) that can be obtained by perturbing the configuration at $F$ (for the latter one has $x_1 = x_2$, $x_4 = x_5$, $f_2 = t_1$, $f_3 = t_2$, $x_3 = s_2 = l$; perturbing $x_1 = x_2$ or $x_4 = x_5$ (resp. $f_2 = t_1$ or $f_3 = t_2$, resp. $x_3 = s_2 = l$) gives one possibility (resp. two, resp. six); all possibilities have to be combined independently).*

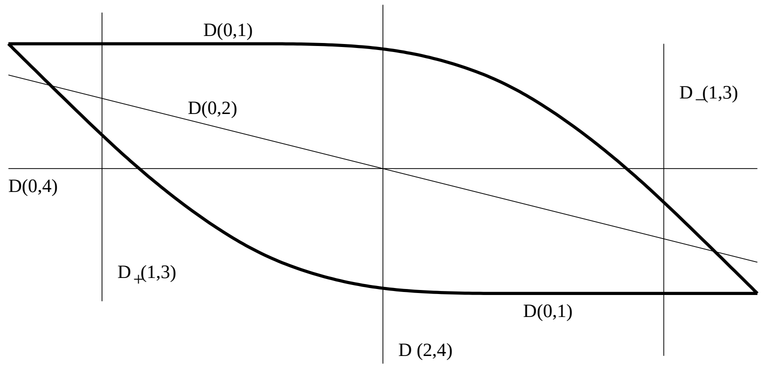


Fig. 6. The discriminant sets close to the point $F$

*Out of these* 24 *configurations six have already been mentioned in Observation* 23 *(see also* 1) *of Remarks* 22*) which leaves* 18 *configurations a priori possible to realize by hyperbolic polynomials. Nevertheless,* 8 *of them are missing.*

**Remark 34.** It follows from the proof of Proposition 32 (see (8)) that locally, at $\Phi$, the tetrahedron $A\Lambda\Gamma\Phi$ lies not only in $\{a \leq 0\}$ but also in $\{c \geq 0\}$ (in both cases equality is possible only at $\Phi$).

**Observation 35.** *There are* 24 *non-degenerate configurations consistent with* (1) *that can be obtained by perturbing the (degenerate) configuration at* $\Sigma$*. Out of them only* 20 *are encountered close to* $\Sigma$*.*

Indeed, Fig. 7 and 9 show that for $a = a_0 < 0$ close to 0 there are 10 components containing $\Sigma$ in their closures (the line $\alpha\gamma$, see Fig. 9, does not pass close to $\Sigma$ for $a_0$ small enough). By symmetry induced by the involution (3), there are 10 components for $a = a_0 > 0$ as well. On the other hand, $\Sigma$ is defined by $x_2 = t_1$, $x_4 = t_2$, $x_3 = s_2 = l$. Perturbing these equalities gives 24 different non-degenerate configurations. □

**5.5. Proof of Proposition 18.** $1^0$. Observe that each multiplicity vector of an edge has two equal components. In order to simplify technically the proof we prove the concavity separately for the edges $AD$, $BC$ and $CD$, and then for the other three edges. Recall that the multiplicity vectors of the edges $AD$,$BC$

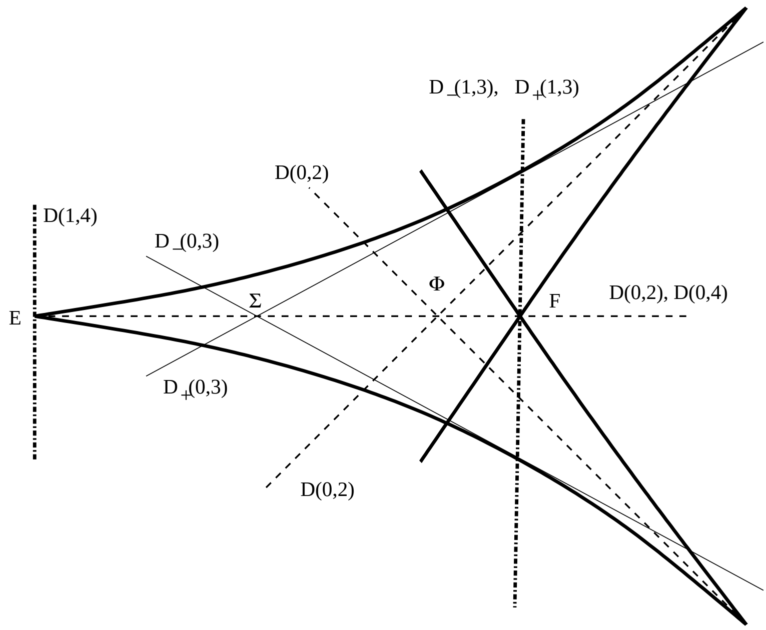


Fig. 7. The discriminant sets for $a = 0$

and $CD$ equal respectively $[2, 2, 1]$, $[1, 2, 2]$ and $[2, 1, 2]$. Hence, the edges can be considered as curves parametrized as follows (we denote the three distinct roots of $P$ by $u, v, w$):

$$(5) \qquad a = 4uvw + 2u^2v + 2uv^2 + u^2w + v^2w, \quad b = 2uv^2w + 2u^2vw + u^2v^2$$

where the parameters $u, v, w$ satisfy the equalities

$$(6) \qquad 2u + 2v + w = 0, \quad u^2 + v^2 + 2uw + 2vw + 4uv = -1$$

Set $u + v = s$, $uv = p$. Equations (6) imply that $w = -2s$, $-3s^2 + 2p = -1$, i.e. $p = (3s^2 - 1)/2$. Substituting in (5) one gets the couple of equations

$$(7) \qquad \begin{aligned} a &= -8ps + 2ps + (s^2 - 2p)(-2s) = -5s^3 + s, \\ b &= 2p(-2s)s + p^2 = (-15s^4 + 2s^2 + 1)/4 \end{aligned}$$

An inflection point is defined by the condition $d\tau/ds = 0$ where $\tau = (db/ds)/(da/ds) = s$. Hence, one has $d\tau/ds \equiv 1$ which means that there are no inflection points.

$2^0$. For the other three edges (whose multiplicity vectors up to permutation equal $[3, 1, 1]$) write down the analogs of equations (5) and (6):

$$a = 3uvw + 3uw^2 + 3vw^2 + w^3, \quad b = 3uvw^2 + uw^3 + vw^3$$

$$u + v + 3w = 0, \quad uv + 3uw + 3vw + 3w^2 = -1$$

One has $w = -s/3$, $p - 2s^2/3 = -1$ and the analog of (7) reads

$$a = 3p(-s/3) + 3s(-s/3)^2 + (-s/3)^3 = -10s^3/27 + s,$$
$$b = 3p(-s/3)^2 + s(-s/3)^3 = 5s^4/27 - s^2/3$$

One has $\tau = -2s/3$, $d\tau/ds \equiv -2/3$, i.e. there are again no inflection points.

$3^0$. The concavity of the projections of the edges $AB$, $AC$, $BD$ and $CD$ can be deduced from [10], Proposition 1.3.4, or by computing the $(a, b)$-coordinates of the points $A$, $B$, $C$, $D$, $E$, $F$, $I$ and $J$. For the edges $AD$ and $BC$ one can deduce the sense of concavity from the $(a, b)$-coordinates of the points $A$, $B$, $C$, $D$, $L$ and $M$. $\square$

**5.6. Proof of Proposition 32.** $1^0$. Prove 1). The equation defining $D(0, 2)$ can be presented in the form

$$2 \times 5^6 \operatorname{Res}(P, P'') = \begin{vmatrix} 100 & 0 & -100 & 100a & 100b & 100c & 0 & 0 \\ 0 & 100 & 0 & -100 & 100a & 100b & 100c & 0 \\ 0 & 0 & 100 & 0 & -100 & 100a & 100b & 100c \\ 10 & 0 & -3 & a & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & -3 & a & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & -3 & a & 0 & 0 \\ 0 & 0 & 0 & 10 & 0 & -3 & a & 0 \\ 0 & 0 & 0 & 0 & 10 & 0 & -3 & a \end{vmatrix} = 0$$

(because $P''/2 = 10x^3 - 3x + a$). Denote by $r_i$ the $i$-th row of the determinant. Replace $r_1$, $r_2$, $r_3$ respectively by $r_1 - 10r_4 + 7r_6$, $r_2 - 10r_5 + 7r_7$, $r_3 - 10r_6 + 7r_8$. This allows one to develop the determinant thrice w.r.t. the first column in which there is a single non-zero entry equal to 10. The newly obtained determinant of size 5 equals up to a factor $(-10)^3$

$$\begin{vmatrix} 90a & 100b - 21 & 100c + 7a & 0 & 0 \\ 0 & 90a & 100b - 21 & 100c + 7a & 0 \\ 0 & 0 & 90a & 100b - 21 & 100c + 7a \\ 10 & 0 & -3 & a & 0 \\ 0 & 10 & 0 & -3 & a \end{vmatrix} .$$

Replace $r_1$ by $r_1 - 9ar_4 - (10b - 21/10)r_5$ and $r_2$ by $r_2 - 9ar_5$. Developing twice w.r.t. the first column one gets the determinant of size 3 (up to a factor $(-10)^5$)

$$\left| \begin{array}{ccc} 100c + 34a & (3/10)(100b - 21) - 9a^2 & -a(100b - 21)/10 \\ 100b - 21 & 100c + 34a & -9a^2 \\ 90a & 100b - 21 & 100c + 7a \end{array} \right| =$$

$$= \left| \begin{array}{ccc} \varphi & 3\theta/10 - 9a^2 & -a\theta/10 \\ \theta & \varphi & -9a^2 \\ 90a & \theta & \varphi - 27a \end{array} \right| = \tag{8}$$

$$= (\varphi - 27a)(\varphi^2 - 3\theta^2/10) + \text{terms of degree } > 3$$

where $\theta = 100b - 21$, $\varphi = 100c + 34a$. Set $u = \varphi - 27a$, $v = \varphi - \sqrt{3/10}\theta$, $w = \varphi + \sqrt{3/10}\theta$. Hence, the set $D(0,2)$ at the point $a = \theta = \varphi = 0$ is the union of three hypersurfaces defined by equations of the form $u + \cdots = 0$, $v + \cdots = 0$ and $w + \cdots = 0$ (dots indicate non-linear terms). They meet transversally at 0.

This proves 1).

$2^0$. Prove 3) before proving 2). Set $P_F = x^5 - x^3 + x/4 = x(x^2 - 1/2)^2$ (the polynomial defined by the point $F$) and $P_1 = (x + 1/\sqrt{2})^2(x - 1/\sqrt{2} + 2/\sqrt{10})^2$. One has

$$P_1'(-1/\sqrt{10}) = P_1'''(-1/\sqrt{10}) = P_F'(-1/\sqrt{10}) = P_F'''(-1/\sqrt{10}) = 0 \ .$$

Consider the family of polynomials $S_\lambda = P_F + \lambda P_1$, $\lambda \leq 0$. All polynomials from this family have a double root at $-1/\sqrt{2}$ and $S_\lambda'(-1/\sqrt{10}) = S_\lambda'''(-1/\sqrt{10}) = 0$.

All polynomials from this family are hyperbolic. Indeed,

1) one has $P_1(-1/\sqrt{10}) > 0$ and $P_F(-1/\sqrt{10}) < 0$, hence, $S_\lambda(-1/\sqrt{10}) < 0$ for $\lambda \leq 0$;

2) set $\chi = 1/\sqrt{2} - 2/\sqrt{10} \in (0, 1/\sqrt{2}]$; one has $P_F(\chi) > 0$, $P_1(\chi) = 0$ (hence, $S_\lambda(\chi) > 0$) and $P_F(1/\sqrt{2}) = 0$, $P_1(1/\sqrt{2}) > 0$ (hence, $S_\lambda(1/\sqrt{2}) < 0$);

3) finally, $S_\lambda \to \pm\infty$ when $x \to \pm\infty$.

$3^0$. Recall that the curve $\theta$ was defined in Proposition 21.

*All polynomials from $\theta$ (when defined up to a constant non-zero factor), can be obtained from polynomials of the family $S_\lambda$ by means of an affine change of the independent variable $x$ and vice versa.*

Indeed, the roots of $S_\lambda$ depend continuously on $\lambda$. For each $\lambda \leq 0$ the polynomial $S_\lambda$ satisfies the conditions $x_1 = x_2$ and $x_2^1 = x_1^3$, see $2^0$. Hence, up to rescaling of the $x$-axis and a constant factor such a polynomial equals a polynomial from $\theta$. Thus the family $S_\lambda$ defines a curve $\theta' \subset \theta$. These two curves have the point $F$ in common.

In the family $S_\lambda$ the sets of ratios of the differences between the roots are different for different values of $\lambda$. (One can observe that the ratio $(x_4 - x_2)/(x_5 - x_2)$ decreases from 1 to 0 because $(x_4 - x_2)$ decreases and $(x_5 - x_2)$ increases tending to $\infty$.) For $\lambda \to -\infty$ the set of ratios tends to the set of respective ratios at $A$. The curve $\theta$ being simple and Jordan this means that $\theta' = \theta$.

$4^0$. When $\lambda$ decreases from 0 to $-\infty$, the middle root $x_3$ of $S_\lambda$ increases because one has $S'_\lambda(x_3) > 0$ and $P_1(x_3) \geq 0$ with equality only when $x_3 = 1/\sqrt{2} - 2/\sqrt{10}$.

At the same time the root $x_2^2$ of $S''_\lambda$ decreases. Indeed, $S'''_\lambda(x_2^2) < 0$ and the right root of $P''_1$ equals $\varphi = -1/\sqrt{10} + 1/\sqrt{6} - 1/\sqrt{30} < 0$; hence, for $\lambda = 0$ one has $P''_1(x_2^2) > 0$ and when $\lambda$ decreases from 0 to $-\infty$ the root $x_2^2$ decreases from 0 to $\varphi$.

Hence, for all points of the curve $AF$ one has $x_3 > x_2^2$. At the same time one has $x_3 < x_2^2$ at the vertex $C$, i.e. to reach the vertex from a point of the curve $AF$ one has to cross the set $D(0,2)$ which means that the curve $\{x_3 = x_2^2\} \cap ACD$ is projected inside $AKICF$. There remains to observe that for each $a < 0$ fixed the intersection $\{x_3 = x_2^2\} \cap ACD$ consists of at most one point, see part 2) of Remarks 30.

This proves 3).

$5^0$. To prove 2) we use again the family $S_\lambda$. We show in $6^0$ – $7^0$ that one has

$$S_\lambda(x_1^2) < S_\lambda(x_2^2) \quad \text{for all} \quad \lambda \leq 0 \;. \tag{9}$$

This implies that the set $\alpha \cap \{a \leq 0\}$ has no point in common with the plane $D_-(1,3)$ except possibly $A$. Indeed, if not, then for the polynomial $P$ defined by such an intersection point there would hold $P(x_1^2) = P(x_2^2)$, $P(x_2^1) = P(x_1^3) = 0$; by adding to the polynomial a suitable constant one can obtain in addition to these conditions the one that $x_1 = x_2$. Such a polynomial belongs to the curve $AF \subset ACD$ and up to a factor equals a polynomial from the family $S_\lambda$ after an affine change of the independent variable, see $3^0$. By (9) this is impossible.

$6^0$. Recall that $P'''_F = 60x^2 - 6 = 60(x - 1/\sqrt{10})(x + 1/\sqrt{10})$. For $x \in (-1/\sqrt{10}, 0]$ one has $0 \leq -P'''_F(x) \leq P'''_F(-x - 2/\sqrt{10})$ with equality only for $x = -1/\sqrt{10}$. One has $-P'''_1(x) = P'''_1(-x - 2/\sqrt{10})$ and the signs of $P'''_F$ and of

$-P_1'''$ are the same on $(-\infty, 0]$. Hence, for all $\lambda \leq 0$ one has

$$0 \geq S_\lambda'''(x) \geq -S_\lambda'''(-x - 2/\sqrt{10}) \tag{10}$$

$S_\lambda'''(x) < 0$ on $(-1/\sqrt{10}, 0]$ and $S_\lambda'''(x) > 0$ on $(-\infty, -1/\sqrt{10})$.

Inequality (10) implies that for all $\lambda \leq 0$ one has

$$S_\lambda''(x) \geq S_\lambda''(-x - 2/\sqrt{10}) \tag{11}$$

Indeed, it suffices to integrate (10) on $[-1/\sqrt{10}, x]$ and to use the fact that for $x = -1/\sqrt{10}$ the values of the left and right side of (10) are the same. Hence, for all $\lambda \leq 0$ one has

$$x_2^2 + 1/\sqrt{10} > -1/\sqrt{10} - x_1^2 \tag{12}$$

Indeed, $S_\lambda''(x)$ decreases with positive values when $x$ grows from $-1/\sqrt{10}$ to 0 or when $x$ decreases from $-1/\sqrt{10}$ to $-2/\sqrt{10}$. Hence, it reaches faster the zero value when $x$ grows than when $x$ decreases.

$7^0$. A second integration leads to

$$S_\lambda'(x) \geq -S_\lambda'(-x - 2/\sqrt{10}) \tag{13}$$

on $[-1/\sqrt{10}, 0]$. Inequality (13) implies after integration that $S_\lambda(x) \geq S_\lambda(-x-2/\sqrt{10})$ on $[-1/\sqrt{10}, 0]$; in particular (for $x = -2/\sqrt{10} - x_1^2$), one has $S_\lambda(-2/\sqrt{10} - x_1^2) \geq S_\lambda(x_1^2)$.

On the other hand, one has (12) and the function $S_\lambda$ increases on $[-1/\sqrt{10}, 0]$. Hence, $S_\lambda(x_2^2) > S_\lambda(-2/\sqrt{10} - x_1^2) \geq S_\lambda(x_1^2)$, i.e. (9) holds.

The proposition is proved. □

**6. The overdetermined strata for $n = 5$ – proof of Theorem 16.** Part 2) of Theorem 16 follows from the equalities (between roots) defining the three strata. Part 1) follows from the following three lemmas:

**Lemma 36.** *All overdetermined strata of* $\Pi$ *are of dimension 0.*

**Lemma 37.** *All overdetermined strata of* $\Pi$ *belong to* $D(2,4)$.

**Lemma 38.** *The only overdetermined strata of dimension* 0 *and belonging to* $D(2,4)$ *are the points* $\Sigma$, $\Phi$ *and* $F$.

Indeed, these points and $E$ are the only ones from $\Pi \cap D(2,4)$ where four sheats of discriminant sets meet at one point, see Fig. 7. The stratum $E$ is not overdetermined, see 2) of Remarks 9. □

P r o o f o f L e m m a 36. $1^0$. To have an overdetermined stratum one needs to have at least two equalities of the form $x_{i_1}^{j_1} = x_{i_2}^{j_2}$. Therefore one has to consider the intersections $D(k_1, k_2) \cap D(k_3, k_4)$, $(k_1, k_2) \neq (k_3, k_4)$.

$2^0$. One has to exclude the possibilities to have overdetermined strata of dimensions 2 or 1. As $D(2,3) \cap \Pi = \{A, B\}$, no overdetermined stratum of dimension 1 or 2 can be contained in $D(2,3)$. The set $D(1,2)$ intersects $\Pi$ along strata of dimension 1 or 0 (the closures of the edges $AB$, $AC$, $BD$); in the absence of intersection with another discriminant set the points of these edges (excluding their vertices) cannot belong to overdetermined strata, i.e. the edges can contain points of overdetermined strata only of dimension 0. Therefore we exclude the sets $D(2,3)$ and $D(1,2)$ from further consideration.

$3^0$. Exclude dimension 2. Recall that some of the discriminant sets are unions of planes, namely, $D(0,3)$, $D(1,3)$, $D(0,4)$, $D(1,4)$, $D(2,4)$. All planes defined by these sets are different, so their intersections by two and with each of the other two discriminant sets $D(0,1)$ and $D(0,2)$ define strata of dimension at most 1. One has $\dim(D(0,1) \cap D(0,2)) = 1$ because the lines building up $D(0,2)$ lie in the planes $a =$const whereas the strata of $\Pi$ are transversal to these planes, see [7]. Hence, there are no overdetermined strata of dimension 2.

$4^0$. Exclude dimension 1. One can check directly that no three of the sets $D(0,3)$, $D(1,3)$, $D(0,4)$, $D(1,4)$, $D(2,4)$ intersect locally along a line, therefore their intersections do not define overdetermined strata of dimension 1.

*Of their intersections by two only the segments $LE$, $EM$, $AV$ and $BR$ belong to $D(0,1)$ and $EF$ belongs to $D(0,2)$ but these are not overdetermined strata, see* 2) *of Remarks* 9. (One has $\{LE \cup EM\} = D(0,4) \cap D(1,4) \cap \Pi$, $AV = D_-(0,3) \cap D_-(1,3) \cap \Pi$,

$BR = D_+(0,3) \cap D_+(1,3) \cap \Pi$, $EF = D(0,4) \cap D(2,4) \cap \Pi$.)

Indeed, one can notice that $D(0,1)$ and $D(0,2)$ do not contain vertical lines which excludes all intersections by two along vertical segments (i.e. $D(1,3) \cap D(1,4)$, $D(1,3) \cap D(2,4)$ and $D(1,4) \cap D(2,4)$). The two segments constituting $D(1,3) \cap D(0,4)$, the two segments constituting $D(1,4) \cap D(0,3)$ and the two segments $D_+(0,3) \cap D_-(1,3)$ and $D_-(0,3) \cap D_+(1,3)$ have only their extremities in common with $D(0,1)$. Their extremities do not belong to $D(0,2)$ (we leave the details here for the reader; use the geometry of $D(0,2)$ and Proposition 32). Hence, only isolated points of these segments can belong to $D(0,1)$ or $D(0,2)$. (We use the fact that if a line intersects $D(0,2) \cap \Pi$ along a segment, then necessarily the endpoints of the segment belong to $D(0,1)$.) Only the extremities of the segments constituting $D(0,3) \cap D(0,4)$ belong to $D(0,1)$; these segments contain the points $D(0,1) \ni (\pm 9/10\sqrt{10}, 0, 0) \notin D(0,2)$, hence, these two segments do not give rise to one-dimensional overdetermined strata either.

The intersection $D(0,3) \cap D(2,4) \cap \Pi$ consists of two segments passing through the point $\Sigma$, see Fig. 7, of which only isolated points can belong to other discriminant sets.

$5^0$. There remains to exclude the possibility to have overdetermined strata of dimension 1 belonging to $D(0,1)$ or $D(0,2)$. If such strata belong to the intersection of two planes constituting the other discriminant sets, then their nonexistence is already proved in $4^0$, so it suffices to consider the one-dimensional strata belonging to $D(0,1) \cap D(0,2)$ and to show that they do not belong locally to any of the planes from the other discriminant sets. This is true –

for the intersections with $D(1,3)$ or $D(1,4)$ see part 3) of Proposition 32 and Fig. 2 and 5;

for the ones with $D(0,4)$ or $D(2,4)$ see Fig. 2, 5 and 7;

for the one with $D(0,3)$ see Fig. 3 and 5.

The lemma is proved. □

P r o o f o f L e m m a 37. To have an overdetermined stratum of dimension 0 one needs at least four independent equalities of the form $x_{i_1}^{j_1} = x_{i_2}^{j_2}$. The proof of the lemma can be deduced from the following statements the (somewhat tedious) verification of which is left for the reader:

1) Nowhere in $\Pi \backslash D(2,4)$ do three planes from the discriminant sets meet at one and the same point excepting the intersection points $D_{\pm}(0,3) \cap D_{\pm}(1,3) \cap D(1,4)$. The latter do not belong to other discriminant sets except $D(0,1)$, hence, they do not define overdetermined strata, see 2) of Remarks 9.

2) Nowhere in $\Pi \backslash D(2,4)$ (except the vertices $A$, $B$, $C$, $D$) does a self-intersection point of $D(0,2)$ belong to the intersection of two planes from the discriminant sets or to the intersection of one such plane with $D(0,1)$.

3) Nowhere in $\Pi \backslash D(2,4)$ does a point from $D(0,1) \cap D(0,2)$ belong to the intersection of two planes from the discriminant sets.

4) The set $D(0,2)$ does not intersect transversally an edge of $D(0,1)$. It either contains whole edges ($AC$, $AB$, $DB$) or intersects it at the points $E$, $F$, $\Lambda$, $\Theta$ where $D(0,2)$ has self-intersection.

5) A self-intersection point of $D(0,2)$ does not belong to the closure of an edge of $\Pi$ except for the vertices $A$, $B$, $C$, $D$ and the points $E$, $F$, $\Lambda$, $\Theta$. The points $\Lambda$, $\Theta$ do not belong to any of the planes from the discriminant sets, hence,

they are not overdetermined strata.

6) None of the four vertices $A$, $B$, $C$, $D$ is an overdetermined stratum.

7) A point of an edge, different from $A$, $B$, $C$, $D$ and not in $D(2,4)$ belongs to at most one of the planes from the discriminant sets.

Indeed, if one needs four independent equalities $x_{i_1}^{j_1} = x_{i_2}^{j_2}$ outside $D(2,4)$, one needs either

– the intersection of four planes of the discriminant sets (impossible by 1)), or

– the intersection of three of them and of $D(0,1)$ or $D(0,2)$ (also impossible by 1)), or

– the intersection of two of them with an edge of $\Pi$ or with the set of self-intersection points of $D(0,2)$ or with a point from $D(0,1) \cap D(0,2)$ (impossible respectively by 7), 2) and 3)), or

– a plane from the discriminant sets passing through a vertex of $\Pi$ or through the intersection point of an edge of $\Pi$ with $D(0,2)$ or through a point from $D(0,1)$ which is also a self-intersection point of $D(0,2)$ (see respectively 6), 4) and 2)), or

– an edge of $\Pi$ has a point in common with the set of self-intersection points of $D(0,2)$ (see 5)). □

## 7. Proof of Theorem 2.

**Definition 39.** *Call* component *a maximal open part of* $\Pi$ *with one and the same non-degenerate configuration of the roots of* $P$ *and its derivatives.*

To prove the theorem we count the number of components into which the discriminant sets divide $\Pi$. (We do not raise the question whether these components are connected or not.) As $\Pi$ is invariant under the involution (3), we count only the components which project (in the $ab$-plane) to the left of $EF$ and then multiply their number by 2. We count separately the numbers of components projecting into each of the three parts $A\Delta EA$, $AKICFA$ and $\Delta EF\Delta$.

Theorem 2 is an immediate corollary of the following four lemmas:

**Lemma 40.** *There are* 12 *components of* $\Pi$ *projecting into* $A\Delta EA$*.*

**Lemma 41.** *There are* 25 *components of* $\Pi$ *projecting into* $AKICFA$*.*

**Lemma 42.** *There are* 6 *components of* $\Pi$ *projecting into* $\Delta EF\Delta$ *and with negative* $c$*-coordinate.*

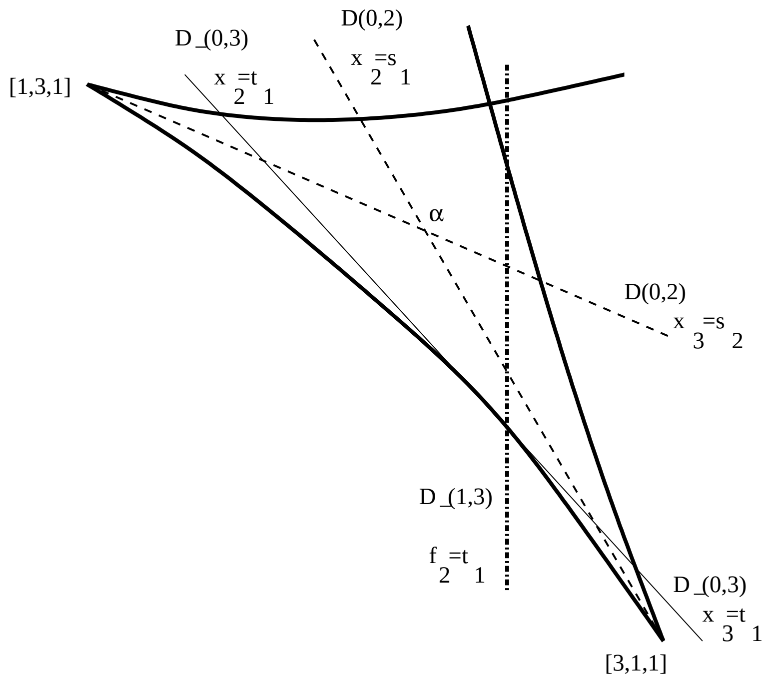


Fig. 8. Components projecting on $A\Delta EA$

**Lemma 43.** *There are* 15 *components of* $\Pi$ *projecting into* $\Delta EF\Delta$ *and with positive* $c$*-coordinate.*

Recall that one of the discriminant sets is $D(0,4) : c = 0$ and that on the figures the positive $c$-axis is directed downward. For convenience in the proofs we use the same notation for the sets $D(i,j)$, $\alpha$, $\beta$, $\gamma$, and for their intersections with the planes $\{a = \text{const}\}$.

P r o o f o f L e m m a 40. We show on Fig. 8 the intersection of $\Pi$ with a vertical plane parallel to the $bc$-one and intersecting the edge $AC$ between $A$ and $I$. On the figure only the set $D(0,4)$ is not shown. The plane $D_+(0,3)$ does not intersect the part of $\Pi$ projecting on $A\Delta EA$ (see Remarks 28) and neither does the surface $\Lambda\Gamma D\Theta B\Lambda \subset D(0,2)$, see Fig. 5 and Lemma 44 below.

There are 6 components to the left of the line representing $D_-(1,3)$. The set $D(0,4)$ intersects them all which makes 12 components projecting on $A\Delta EA$. $\square$

P r o o f o f L e m m a 41. Use again Fig. 8 (like in the proof of Lemma 40). To the right of the line representing $D_-(1,3)$ there are 5 components which are intersected by each of the planes $\{c=0\} = D(0,4)$, $\{b=0\} = D(1,4)$, $TQRSWBZT \subset D_+(0,3)$ and by the surface $\Lambda\Gamma D\Theta B\Lambda \subset D(0,2)$. These surfaces do not intersect each other inside the part of $\Pi$ which is over $AKICFA$. Indeed, every point from $\{c=0\}\cap\Pi$ has a negative $b$-coordinate while every point from $TQRSWBZT\cap\Pi$ has a positive one, see Remarks 28. This means that the three planes do not intersect inside the part of $\Pi$ over $AKICFA$. As for $\Lambda\Gamma D\Theta B\Lambda$, one can apply Lemma 44.

Hence, these four surfaces divide each of the 5 components into 5 parts which makes 25 components of $\Pi$ over $AKICFA$. □

**Lemma 44.** 1) *The part* $\lambda$ *of the surface* $\Lambda\Gamma D\Theta B\Lambda \subset D(0,2)$ *projecting in the ab-plane inside* $A\Lambda CFBA$ *(resp. the part* $\mu$ *of* $\Theta\Xi C\Lambda A\Theta$ *projecting inside* $B\Theta DFAB$*) does not intersect the surface* $TQRYBZT \subset D_+(0,3)$ *(resp.* $AZVWXYA \subset D_-(0,3)$*).*

2) *A point from* $\lambda$ *(resp.* $\mu$*) with given coordinates* $(a,b)$ *has a greater (resp. smaller) c-coordinate than the point from* $TQRYBZT$ *(resp.* $AZVWXYA$*) with the same coordinates* $(a,b)$*.*

P r o o f. We prove part 2) (only for $TQRYBZT$, for $AZVWXYA$ the proof is analogous), part 1) is an immediate corollary of it. Use 4 of Fig. 4. Recall that the positive $c$-axis is directed downward. The points from $TQRYBZT$ belong to lines $d$ tangent to the sets $\Pi\cap a=$const the points of tangency $\Omega^*$ lying inside the arc $\Omega\Omega_1$ while the points from $\lambda$ belong to the tangent lines passing through $\Omega_1$. (The point $\Omega^*$ belongs to the line $BR$ on Fig. 3.) There remains to observe that the points from $TQRYBZT$ belong to the half-line of $d$ on the same side as $\Omega$ (i.e. to the left of $\Omega^*$, see 4 of Fig. 4) which provides the necessary inequality between the $c$-coordinates. □

P r o o f o f L e m m a 42. The intersection of $\Pi$ with the plane $\{a=0\}$ is represented on Fig. 7. One can see on the figure the points $\Sigma$, $\Phi$ and $F$ defining overdetermined strata. For $a<0$ and close to 0 the six components above the line $EF$ on the figure still exist (remind that the positive $c$-axis is directed downward and that we count only components in the strip between $D(1,4)$ and $D_-(1,3)$, and lying above $D(0,4)$ on the figure).

The two sheats of $D(0,3)$ intersect below $EF$ for $a<0$, see Fig. 9 (one has $c+a/10=0$ at the intersection point, hence, if $a<0$, then $c>0$). Also below $EF$ or outside the strip between $D(1,4)$ and $D(1,3)$ lie the three self-

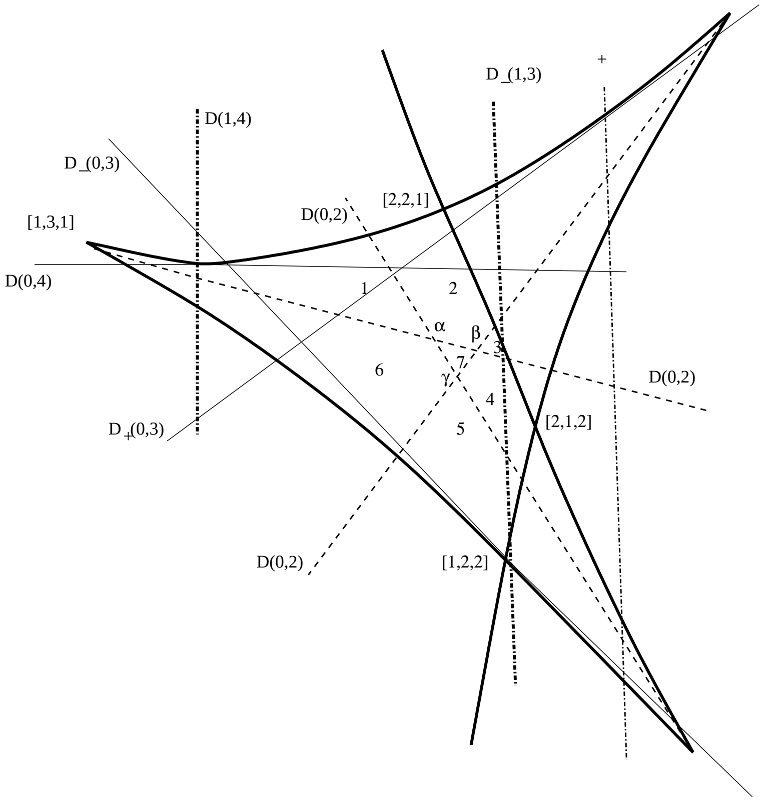


Fig. 9. The discriminant sets for $a = a_0 < 0$ close to 0

intersection points of $D(0,2)$ for $a < 0$; these points belong to the right of the four sectors defined by the intersection of $D_+(0,3)$ and $D_-(0,3)$. All this can be deduced from the slope of the line $\alpha\beta$ and from the concavity of the boundary of $\Pi$, see Fig. 9 representing an intersection of $\Pi$ with $\{a = a_0\}$ for $a_0 < 0$, see also Remarks 30.

Hence, no new components excepting these 6 appear in $\Pi \cap \{c < 0\}$ when $a$ becomes negative. $\square$

P r o o f o f L e m m a 43. $1^0$. For $a_0 < 0$ denote by $\kappa(a_0)$, $\tau(a_0)$ the parts of the sets $\Pi \cap \{c > 0, a = a_0\}$, $\Pi \cap \{c < 0, a = a_0\}$ which are between $D(1,4)$ and $D_-(1,3)$, see Fig. 9 (recall that $D(0,4)$ is defined by $c = 0$ and that the $c$-axis is directed downward).

For $a = a_0 < 0$ and close to 0 the three lines of the set $D(0,2) \cap \{a = a_0\}$ do not intersect within $\tau(a_0)$. Indeed, the segment of the line $\alpha\beta$ lying in the strip between $D(1,4)$ and $D_-(1,3)$ belongs to $\{c > 0\}$ (recall that the slope of the lines tangent to $D(a_0) = D(0,1) \cap \{a = a_0\}$ is a monotonous function of the point on $D(a_0)$, see 2) of Remarks 30; hence, $\alpha\beta$ intersects $D(1,4)$ and $D_-(1,3)$ in $\{c > 0\}$). The intersection point $\gamma$ lies in the half-plane defined by $\alpha\beta$ as shown on Fig. 9, compare with 7 of Fig. 4. The intersection points of the three lines $\alpha\beta$, $\beta\gamma$ and $\alpha\gamma$ coincide only at $\Phi$, therefore there are exactly 7 regions of $\kappa(a_0)$ into which it is divided by these three lines.

The 7 regions can be defined by continuity when $a$ decreases and for some values of $a$ some of them disappear (for $a < -\sqrt{2/5}$ they have all disappeared). Each of the regions can be characterized by being above or below (in the sense of the $c$-coordinate) w.r.t. each of the three lines $\alpha\beta$, $\beta\gamma$ and $\alpha\gamma$ (recall that the lines are never vertical).

$2^0$. To count the number of components mentioned in the lemma one has to see which of these 7 regions is intersected by the sets $D_\pm(0,3)$ and to which of them their intersection point belongs.

**Lemma 45.** 1) *For $a < 0$ the line $D_-(0,3)$ can intersect only regions* 1, 5 *and* 6. *For each of these regions there are values of $a < 0$ for which the line does intersect it.*

2) *The line $D_+(0,3)$ can intersect only regions* 1, 2, 6 *and* 7. *For each of these regions there are values of $a < 0$ for which the line does intersect it.*

3) *For $a < 0$ the intersection point $D_+(0,3) \cap D_-(0,3)$ when it belongs to $\kappa(a)$ belongs to region* 6.

Before proving the lemma count the number of components. Regions 2, 5 and 7 are intersected just by one of the two lines $D_\pm(0,3)$, therefore each of them gives rise to two components. Region 1 gives rise to three and region 6 to four components, and the non-intersected regions 3 and 4 give rise to one component each. This makes $3 \times 2 + 3 + 4 + 2 = 15$ components. $\square$

P r o o f o f L e m m a 45. Prove 1). For small values of $a < 0$ the line $D_-(0,3)$ intersects the face $ABD$ of $\Pi$ at a point whose $b$-coordinate is strictly positive. The segment of this line cut off by $D(1,4)$ and $D_-(1,3)$ lies between the face $ABC$ and the line $\alpha\gamma$. This can be deduced from the concavity of the boundary of $\Pi$, see Proposition 17. Hence, the line $D_-(0,3)$ intersects regions 1, 5 and 6 at least for $a < 0$ small enough and it never intersects regions 2, 3, 4 or 7. This proves 1).

Prove 2). The concavity of the boundary of $\Pi$ implies that the line $D_+(0,3)$ can intersect only regions 1,2,6 and 7, never 3,4, or 5. Choose for $a_0$ the value $a_0^*$ for which the set $\Pi \cap \{a = a_0\}$ contains the point $Z$, see Fig. 3. For such a choice the intersection point of $D_+(0,3)$ and $D_-(0,3)$ belongs to $D_-(1,3) \cap ABC$. On the other hand, the point $\alpha$ lies to the left of $D_-(1,3)$, see part 3) of Proposition 32; for $a = a_0^*$ and for $a = a_0 < 0$ close to 0 the point $\alpha$ is on different sides w.r.t. $D_+(0,3)$. Hence, for some value of $a \in (a_0^*, 0)$ the line $D_+(0,3)$ intersects region 7, hence, 6 and 2 as well. When $a < 0$ is small enough it intersects region 1. This proves 2).

Prove 3). The intersection point of $D_+(0,3)$ and $D_-(0,3)$ belongs to the plane $c + a/10 = 0$ (see their equations in Subsection 5.3); it follows from 1) and 2) that it can belong only to the closures of regions 1 or 6. For $a < 0$ small enough the points $\alpha$ and $\beta$ are close to the points defined by the systems

$$100c + 7a = 0, \ 100c + 34a \pm \sqrt{3/10}(100b - 21) = 0,$$

see (8). Hence, for these values of $a$ the points $\alpha$ and $\beta$ have smaller $c$-coordinates than $D_+(0,3) \cap D_-(0,3)$ and the point from the line $\alpha\beta$ with the same $b$-coordinate as $D_+(0,3) \cap D_-(0,3)$ has an even smaller $c$-coordinate than it, see the slope of the line $\alpha\beta$ on Fig. 9. This means that for $a < 0$ small enough the intersection point $D_+(0,3) \cap D_-(0,3)$ belongs to region 6.

Prove that this is the case for all $a < 0$. Find all values of $a \neq 0$ for which this intersection point belongs to $D(0,2)$ (in order to leave region 6 and enter into region 1 the intersection point must cross $D(0,2)$). To this end one has to solve the system consisting of equation

$$(14) \quad \begin{vmatrix} 100c + 34a & (3/10)(100b - 21) - 9a^2 & -a(100b - 21)/10 \\ 100b - 21 & 100c + 34a & -9a^2 \\ 90a & 100b - 21 & 100c + 7a \end{vmatrix} = 0$$

see (8), and the equations $c + a/10 = 0$, $b = 9/100$ derived from the ones defining the planes $D_\pm(0,3)$. Replace in the determinant $c$ by $-a/10$ and $b$ by $9/100$. This gives the equation

$$\begin{vmatrix} 24a & -36/10 - 9a^2 & 12a/10 \\ -12 & 24a & -9a^2 \\ 90a & -12 & -3a \end{vmatrix} = 0$$

which is equivalent to $a(675a^4 - 340a^2 + 28) = 0$. Its roots are 0, $\pm\sqrt{2/5}$, $\pm\sqrt{14/135}$. The root $-\sqrt{2/5}$ has to be excluded from consideration because $\Pi \cap \{a = -\sqrt{2/5}\} = \{A\}$.

For $a = -\sqrt{14/135}$ one has $c = (1/10)\sqrt{14/135}$, $b = 9/100$ and one obtains the polynomial

$$\begin{aligned} P^* &= x^5 - x^3 - \sqrt{14/135}x^2 + (9/100)x + (1/10)\sqrt{14/135} \\ &= (x^2 - 1/10)(x^3 - (9/10)x - \sqrt{14/135}) \end{aligned}$$

with roots $x_1 = (\sqrt{7/30} - \sqrt{87/30})/2$, $x_2 = -\sqrt{7/30}$, $x_3 = -1/\sqrt{10}$, $x_4 = 1/\sqrt{10}$, $x_5 = (\sqrt{7/30} + \sqrt{87/30})/2$.

For $P^*$ one has $x_2 = s_1$, $x_3 = t_1$, $x_4 = t_2$ which means that $P^* \notin \kappa(a)$. Indeed, if $P^*$ were in $\kappa(a)$, then one should have $x_3 = s_2$, not $x_2 = s_1$ (because the intersection point must belong to $A\Gamma F\Xi BEA$, see the beginning of Subsection 5.4). In fact, the projection in the $ab$-plane of the point $P^*$ belongs to $AKICFA$ and $D_+(0,3)$ intersects $D_-(0,3)$ to the right of $D_-(1,3)$, see Fig. 9.

Hence, the intersection point $D_+(0,3) \cap D_-(0,3)$ can belong only to region 6. The lemma is proved. $\square$

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