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A PRODUCT TWISTOR SPACE

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Abstract. In previous work a hyperbolic twistor space over a paraquaternionic Kähler manifold was defined, the fibre being the hyperboloid model of the hyperbolic plane with constant curvature -1. Two almost complex structures were defined on this twistor space and their properties studied. In the present paper we consider a twistor space over a paraquaternionic Kähler manifold with fibre given by the hyperboloid of 1-sheet, the anti-de-Sitter plane with constant curvature -1. This twistor space admits two natural almost product structures, more precisely almost para-Hermitian structures, which form the objects of our study.

1. Introduction and hyperbolic twistor spaces. In [2, 3] we introduced a hyperbolic twistor space which we will need to review. The starting point was the following simple observation. In [12] P. Libermann introduced the notion of an *almost quaternionic structure of the second kind* (*presque quaternioniennes de deuxième espèce*). This consists of an almost complex structure J_1

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and an almost product structure, J_2 such that $J_1J_2+J_2J_1=0$. Setting $J_3=J_1J_2$ one has a second almost product structure which also anti-commutes with J_1 and J_2. Now on a manifold M with such a structure, set

$$j = y_1J_1 + y_2J_2 + y_3J_3.$$

Then j is an almost complex structure on M if and only if

$$-y_1^2 + y_2^2 + y_3^2 = -1.$$

This suggested a *hyperbolic twistor space* $\pi : Z \longrightarrow M$ as a hypersurface in the subbundle E of $End(TM)$ spanned by $\{J_1, J_2, J_3\}$ with each fibre being this hyperboloid, noting that either branch may be regarded as a model of the hyperbolic plane. Recall that the classical twistor space over a quaternionic Kähler manifold of dimension ≥ 8 is a bundle over the manifold with the fibre being a sphere in the subbundle of the bundle of endomorphisms determined by the three underlying local almost complex structures defining the quaternionic Kähler structure.

There are a number of examples of almost quaternionic structures of the second kind including the paraquaternionic projective space as described by Blažić [4]. Under certain holonomy assumptions almost quaternionic structures of the second kind become paraquaternionic Kähler (see e.g. Garcia-Rio, Matsushita and Vazquez-Lorenzo [7]). Even more strongly one has the notion of a neutral hyperkähler structure and Kamada [11] has observed that the only compact four-manifolds admitting such a structure are complex tori and primary Kodaira surfaces. We remark that the neutral hyperkähler four-manifolds are Ricci flat and self-dual ([11]).

Also the tangent bundle of a differentiable manifold carries an almost quaternionic structure of the second kind as studied by S. Ianus and C. Udriste [8, 9]; this includes examples where the dimension of the manifold carrying the structure is not necessarily $4n$.

However the most natural setting for this kind of structure is on a manifold M of dimension $4n$ with a neutral metric g, i.e. a semi-Riemannian metric of signature $(2n, 2n)$. One reason for this is that such a metric may be given with respect to which J_1 acts as an isometry on tangent spaces and J_2, J_3 act as anti-isometries; the effect of this is that we may define three fundamental 2-forms Ω_a, $a = 1, 2, 3$, by $\Omega_a(X, Y) = g(X, J_aY)$. Riemannian metrics can be chosen such that $g(J_aX, J_aY) = g(X, Y)$, but then Ω_2 and Ω_3 are symmetric tensor fields instead of 2-forms.

The neutral metric g induces a metric on the fibres of E by $\frac{1}{4n}\mathrm{tr}A^tB$ where A and B are endomorphisms of T_pM and A^t is the adjoint of A with respect to g. This metric on the fibre is of signature $(+--)$, the norm of J_1 being $+1$ and the norms of J_2 and J_3 being -1.

Alternatively one may choose a Lorentz metric $\langle\,,\rangle$ directly on the fibres of E such that $\langle J_1, J_1\rangle = -1$, $\langle J_2, J_2\rangle = +1$, $\langle J_3, J_3\rangle = +1$. This metric is of signature $(-++)$ and has the advantage of inducing immediately a Riemannian metric of constant curvature -1 on the hyperbolic planes defined by $-y_1^2 + y_2^2 + y_3^2 = -1$, $y_1 > 0$, in each fibre. In [3] we adopted this metric for its geometric attractiveness and as a matter of notation set $\epsilon_1 = -1$ and $\epsilon_2 = \epsilon_3 = +1$. Further, denoting also by π the projection of E onto M, if x_i are local coordinates on M, set $q_i = x_i \circ \pi$. For a section of E we denote its vertical lift to E as a vector field by the superscript ^{v} and frequently utilize the natural identifications of J_a^v with J_a itself and with $\frac{\partial}{\partial y_a}$ in terms of the fibre coordinates y_1, y_2, y_3.

As with the theory of twistor spaces over quaternionic Kähler manifolds, the theory of hyperbolic twistor spaces over paraquaternionic Kähler manifolds of dimension ≥ 8 develops nicely by virtue of the fact that the covariant derivatives of sections of the subbundle of the endomorphism bundle are again sections of the subbundle. In particular we have the machinery of horizontal lifts: Let D denote the Levi-Civita connection of the neutral metric on M. Then the horizontal lift X^h of a vector field X to the bundle E is is given by

$$X^h = \sum_i X^i \frac{\partial}{\partial q^i} - \sum_{a,b=1}^{3} \epsilon_b y_a(\langle D_X J_a, J_b\rangle \circ \pi)\frac{\partial}{\partial y_b}. \tag{1.1}$$

We then defined two almost complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ on the hyperbolic twistor space Z as follows. Acting on horizontal vectors these are the same and given by $\mathcal{J}_1 X^h = \mathcal{J}_2 X^h = (jX)^h$ where as before $j = \sum y_a J_a$. For a vertical vector $V = V^1\frac{\partial}{\partial y_1} + V^2\frac{\partial}{\partial y_2} + V^3\frac{\partial}{\partial y_3}$ tangent to Z, i.e. $\langle \sigma, V\rangle = 0$, let

$$\mathcal{J}_1 V = (y_3V^2 - y_2V^3)\frac{\partial}{\partial y_1} + (y_3V^1 - y_1V^3)\frac{\partial}{\partial y_2} + (y_1V^2 - y_2V^1)\frac{\partial}{\partial y_3} \tag{1.2}$$

and let $\mathcal{J}_2 V$ be the negative of this expression. In particular $\mathcal{J}_k V = (-1)^{k-1}\sigma \times V$, $k = 1, 2$, $\sigma \in Z$ where $\times$ is the vector product determined by the paraquaternionic algebra.

We also defined a semi-Riemannian metric on Z by $h = \pi^* g + \langle\, ,\rangle_v$, $\langle\, ,\rangle_v$ being the restriction of $\langle\, ,\rangle$ to the fibres of Z. It is easy to check that this metric is Hermitian with respect to both $\mathcal{J}_1$ and $\mathcal{J}_2$.

The theory now develops as in the quaternionic Kähler case and we have the following result from [3] quite analogous to the classical twistor space theory; τ denotes the scalar curvature of the base manifold.

Theorem 1. *On the hyperbolic twistor space of a paraquaternionic Kähler manifold of dimension ≥ 8 we have the following: The almost complex structure $\mathcal{J}_1$ is integrable and the Hermitian structure $(\mathcal{J}_1, h)$ is indefinite semi-Kähler. The structure $(\mathcal{J}_1, h)$ is indefinite Kähler if and only if $\tau = -4n(n+2)$. The almost complex structure $\mathcal{J}_2$ is never integrable but $(\mathcal{J}_2, h)$ is indefinite semi-Kähler. The structure $(\mathcal{J}_2, h)$ is indefinite almost Kähler if and only if $\tau = 4n(n+2)$ and indefinite nearly Kähler if and only if $\tau = -2n(n+2)$.*

These numbers are the negatives of what one would have in the usual twistor space over a quaternionic Kähler manifold of dimension ≥ 8. This sign change is due to our choice of metric on the fibres of E. If we take $<,>$ as the $(+--)$ metric we would have the other values, but the fibres of Z would then have a negative definite metric.

We end this section with a brief discussion of the Levi-Civita connection by $\overline{\nabla}$ of the metric $h = \pi^* g + \langle\, ,\rangle$, on E. At a point $\sigma \in Z \subset E$,

$$(\overline{\nabla}_{X^h} Y^h)_\sigma = (D_X Y)^h_\sigma - \frac{1}{2}(R_{XY}\sigma)^v.$$

For sections s and t of E we have

$$(\overline{\nabla}_{X^h} s^v)_\sigma = \frac{1}{2}(\hat{R}_{\sigma s} X)^h + (D_X s)^v_\sigma, \quad (\overline{\nabla}_{s^v} X^h)_\sigma = \frac{1}{2}(\hat{R}_{\sigma s} X)^h, \quad \overline{\nabla}_{s^v} t^v = 0.$$

where $\hat{R}_{\sigma s} X$ is defined by $g(\hat{R}_{\sigma s} X, Y) = h((R_{XY}\sigma)^v, s^v)$.

2. Product twistor spaces. The idea of a hyperbolic twistor space with fibre coming from the two-sheeted hyperboloid raises the question: What about the other hyperboloid, $-y_1^2 + y_2^2 + y_3^2 = +1$? This hyperboloid is a doubly ruled surface and hence has a natural almost product structure. The geometry of this hyperboloid is the following. The position vector is $\sigma = y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} + y_3 \frac{\partial}{\partial y_3}$ and the induced metric is $ds^2 = dy_1^2 - dy_2^2 - dy_3^2$ making the hyperboloid the anti-de Sitter plane with constant curvature -1. We've chosen the induced metric as the hyperboloid inherits a neutral metric from either the induced metric or the Lorentz metric on the fibres of E; the latter option would make the hyperboloid the de-Sitter plane of constant curvature $+1$. The vector fields

$$e_1 = (1+y_1^2)\frac{\partial}{\partial y_1} - (y_3 - y_1 y_2)\frac{\partial}{\partial y_2} + (y_2 + y_1 y_3)\frac{\partial}{\partial y_3},$$

$$e_2 = (1+y_1^2)\frac{\partial}{\partial y_1} + (y_3 + y_1 y_2)\frac{\partial}{\partial y_2} - (y_2 - y_1 y_3)\frac{\partial}{\partial y_3}$$

are both light-like and tangent to the rulings. We note also that $\langle e_1, e_2\rangle = 2(1+y_1^2)$, $\langle \sigma, e_1\rangle = 0$ and $\langle \sigma, e_1\rangle = 0$.

Now on a manifold M with an almost quaternionic structure of the second kind, set

$$p = y_1 J_1 + y_2 J_2 + y_3 J_3.$$

Then p is an almost product structure on M if and only if

$$-y_1^2 + y_2^2 + y_3^2 = +1.$$

This suggests a *product twistor space* $\pi : Z \longrightarrow M$ as a hypersurface in the subbundle E of $End(TM)$ spanned by $\{J_1, J_2, J_3\}$ with each fibre being this doubly ruled hyperboloid viewed as the anti-de-Sitter plane with constant curvature -1.

Lemma. *The Weingarten map of the product twistor space Z as a hypersurface in the bundle space E annihilates horizontal vectors and acts as the identity on vertical vectors.*

P r o o f. The position vector σ is also a normal ν to the hyperboloid in the fibres of E and the Weingarten map of a hypersurface is given by $\overline{\nabla}_X \nu = -\epsilon A X$ where $\epsilon = -1$ if $\langle \nu, \nu\rangle = -1$ as in the present case. Using the summation convention for repeated indices $a, b, c = 1, 2, 3$, we have first for a horizontal lift (equation (1.1)) to a point σ

$$\begin{aligned}
\overline{\nabla}_{X^h} \nu &= \overline{\nabla}_{X^h} y_c \frac{\partial}{\partial y_c} \\
&= -\epsilon_c y_a (\langle D_X J_a, J_c\rangle \circ \pi)\frac{\partial}{\partial y_c} + y_c\Big(\frac{1}{2}(\hat{R}_{\sigma J_c} X)^h + (D_X J_c)^v_\sigma\Big) \\
&= \frac{1}{2}(\hat{R}_{\sigma\sigma} X)^h = 0.
\end{aligned}$$

Similarly for a vertical tangent vector V

$$\overline{\nabla}_V\nu = \overline{\nabla}_V y_c\frac{\partial}{\partial y_c} = (Vy_c)\frac{\partial}{\partial y_c} = V^c\frac{\partial}{\partial y_c} = V.$$

Thus $AV = V$ for a vertical tangent V and $AX = 0$ for a horizontal tangent X.

We now define two almost product structures $\mathcal{P}_1$ and $\mathcal{P}_2$ on the product twistor space Z as follows. Acting on horizontal vectors these are the same and given by $\mathcal{P}_1(X^h_\sigma) = \mathcal{P}_2(X^h_\sigma) = (pX)^h_\sigma$. For a vertical vector V tangent to Z, let

$$\mathcal{P}_1V = \mathcal{P}_1(V^1e_1 + V^2e_2) = V^2e_2 - V^1e_1$$

and let $\mathcal{P}_2V$ be the negative of this expression. Then $\mathcal{P}_i^2 = I$ giving two almost product structures on the product twistor space Z. If V is written $V = V^1\frac{\partial}{\partial y_1} + V^2\frac{\partial}{\partial y_2} + V^3\frac{\partial}{\partial y_3}$, $\langle\sigma, V\rangle = 0$, the expression for $\mathcal{P}_1V$ is the same as that for $\mathcal{J}_1V$ in equation (1.2) and $\mathcal{P}_1^2V = V$ instead of $-V$ simply because $-y_1^2+y_2^2+y_3^2 = +1$ instead of $-y_1^2 + y_2^2 + y_3^2 = -1$.

Restricting the metric $\langle\,,\rangle$ to the fibres (hyperboloids of 1-sheet) of Z and denoting the restriction by $\langle\,,\rangle_v$, we define a semi-Riemannian metric on Z by $h = \pi^*g + \langle\,,\rangle_v$ with signature $(2n+1, 2n+1)$. With respect to this metric both $\mathcal{P}_1$ and $\mathcal{P}_2$ act as anti-isometries. This latter property implies that each $\mathcal{P}_i$ is an *almost paracomplex structure*, again a structure introduced by P. Libermann [13] in 1952, i.e. each $\mathcal{P}_i$ is an almost product structure whose corresponding eigenspaces are isomorphic.

When an almost paracomplex structure $\mathcal{P}$ is considered along with the anti-isometry property, it is called an *almost para-Hermitian structure* and a *para-Hermitian structure* if $\mathcal{P}$ is integrable. If $\mathcal{P}$ is parallel with respect to the Levi-Civita connection ∇ of the metric, the structure is said to be *para-Kähler*. If the corresponding fundamental 2-form is closed the structure is said to be *almost para-Kähler*. If $(\nabla_X\mathcal{P})X = 0$, the structure is said to be *nearly para-Kähler*. If the fundamental 2-form is coclosed, the almost para-Hermitian structure is said to be *semi-para-Kähler*. Refinements of these and other classes may be introduced; classifications of almost para-Hermitian manifolds were given by C. Bejan [1] and by P. M. Gadea and J. M. Masque [6]. For a general reference to paracomplex geometry see [5].

We now review paraquaternionic Kähler geometry following [7]. An almost quaternionic manifold of the second kind M of dimension $4n$ and neutral metric g is said to be *paraquaternionic Kähler* if the bundle E is parallel with respect to the Levi-Civita connection of g. This is equivalent to the existence of local 1-forms α, β and γ such that

$$\begin{aligned}
D_X J_1 &= - \gamma(X)J_2 - \beta(X)J_3,\\
D_X J_2 &= -\gamma(X)J_1 - \alpha(X)J_3,\\
D_X J_3 &= -\beta(X)J_1 + \alpha(X)J_2.
\end{aligned}$$

From the group theoretic point of view this structure corresponds to the linear holonomy group being a subgroup of $Sp(n,\mathbb{R})\cdot Sp(1,\mathbb{R})$, just as a quaternionic Kähler structure corresponds to the linear holonomy group being a subgroup of $Sp(n)\cdot Sp(1)$. For $n=1$ this is not a restriction.

Setting

$$A = 2(d\alpha - \beta\wedge\gamma), \quad B = 2(d\beta + \gamma\wedge\alpha), \quad C = 2(d\gamma + \alpha\wedge\beta)$$

one can easily obtain the following central relations for the action of the curvature tensor:

$$\begin{aligned}
[R_{XY}, J_1] &= - C(X,Y)J_2 - B(X,Y)J_3,\\
[R_{XY}, J_2] &= -C(X,Y)J_1 - A(X,Y)J_3,\\
[R_{XY}, J_3] &= -B(X,Y)J_1 + A(X,Y)J_2.
\end{aligned}$$

Moreover a paraquaternionic Kähler manifold of dimension ≥ 8 is Einstein and A, B, C satisfy

$$A(X,Y) = -\frac{\tau g(X,J_1Y)}{4n(n+2)}, \ B(X,Y) = -\frac{\tau g(X,J_2Y)}{4n(n+2)}, \ C(X,Y) = \frac{\tau g(X,J_3Y)}{4n(n+2)}$$

where τ is the scalar curvature of g.

For the product twistor space Z with its almost product structures $\mathcal{P}_1$ and $\mathcal{P}_2$ and neutral metric h we have the following result.

Theorem 2. *On the product twistor space of a paraquaternionic Kähler manifold of dimension ≥ 8 we have the following: The almost product structure $\mathcal{P}_1$ is integrable and the para-Hermitian structure $(\mathcal{P}_1, h)$ is semi-para-Kähler. The structure $(\mathcal{P}_1, h)$ is para-Kähler if and only if $\tau = -4n(n+2)$. The almost product structure $\mathcal{P}_2$ is never integrable but $(\mathcal{P}_2, h)$ is semi-para-Kähler. The structure $(\mathcal{P}_2, h)$ is almost para-Kähler if and only if $\tau = 4n(n+2)$ and nearly para-Kähler if and only if $\tau = -2n(n+2)$.*

Proof. The major effort of the proof is to compute the covariant derivatives of $\mathcal{P}_i$, $i = 1, 2$. To begin, by the Lemma,

$$(\nabla_{X^h}\mathcal{P}_i)Y^h|_\sigma = \overline{\nabla}_{X^h}\mathcal{P}_iY^h - \mathcal{P}_i\overline{\nabla}_{X^h}Y^h$$

$$= \overline{\nabla}_{X^h}(y_1J_1Y + y_2J_2Y + y_3J_3Y)^h - \mathcal{P}_i(D_XY)^h + \frac{1}{2}\mathcal{P}_i(R_{XY}\sigma)^v$$

and

$$R_{XY}\sigma = y_1(-C(X,Y)J_2 - B(X,Y)J_3) + y_2(-C(X,Y)J_1 - A(X,Y)J_3)$$
$$+y_3(-B(X,Y)J_1 + A(X,Y)J_2).$$

Now

$$\mathcal{P}_iR_{XY}\sigma = (-1)^{i-1}\Big\{C(X,Y)\Big[\frac{y_2 - y_1y_3}{2(1+y_1^2)}e_1 - \frac{y_2 + y_1y_3}{2(1+y_1^2)}e_2\Big]$$
$$+B(X,Y)\Big[\frac{y_3 + y_1y_2}{2(1+y_1^2)}e_1 - \frac{y_3 - y_1y_2}{2(1+y_1^2)}e_2\Big] + A(X,Y)\Big[\frac{1}{2}e_1 + \frac{1}{2}e_2\Big]\Big\}$$
$$= \frac{(-1)^i\tau}{4n(n+2)}\Big\{\Big[-g(X,J_3Y)\frac{y_2 - y_1y_3}{2(1+y_1^2)} + g(X,J_2Y)\frac{y_3 + y_1y_2}{2(1+y_1^2)} + \frac{1}{2}g(X,J_1Y)\Big]e_1$$
$$+\Big[g(X,J_3Y)\frac{y_2 + y_1y_3}{2(1+y_1^2)} - g(X,J_2Y)\frac{y_3 - y_1y_2}{2(1+y_1^2)} + \frac{1}{2}g(X,J_1Y)\Big]e_2\Big\}.$$

Direct expansion of $\overline{\nabla}_{X^h}(y_1J_1Y + y_2J_2Y + y_3J_3Y)^h - \mathcal{P}_i(D_XY)^h$ shows that it is $-\frac{1}{2}(R_{XpY}\sigma)^v$, which is vertical, and expanding this curvature term we have

$$(R_{XpY}\sigma)^v$$
$$= y_1(-C(X, y_1J_1Y + y_2J_2Y + y_3J_3Y)J_2 - B(X, y_1J_1Y + y_2J_2Y + y_3J_3Y)J_3)$$
$$+y_2(-C(X, y_1J_1Y + y_2J_2Y + y_3J_3Y)J_1 - A(X, y_1J_1Y + y_2J_2Y + y_3J_3Y)J_3)$$
$$+y_3(-B(X, y_1J_1Y + y_2J_2Y + y_3J_3Y)J_1 + A(X, y_1J_1Y + y_2J_2Y + y_3J_3Y)J_2)$$
$$= \frac{\tau}{4n(n+2)}\Big\{(-y_1g(X,J_2Y) - y_2g(X,J_1Y))\Big[\frac{y_2 - y_1y_3}{2(1+y_1^2)}e_1 + \frac{y_2 + y_1y_3}{2(1+y_1^2)}e_2\Big]$$
$$+(-y_1g(X,J_3Y) - y_3g(X,J_1Y))\Big[\frac{y_3 + y_1y_2}{2(1+y_1^2)}e_1 + \frac{y_3 - y_1y_2}{2(1+y_1^2)}e_2\Big]$$
$$+(-y_2g(X,J_3Y) + y_3g(X,J_2Y))\Big[-\frac{1}{2}e_1 + \frac{1}{2}e_2\Big]\Big\}.$$

Now since $(\nabla_{X^h}\mathcal{P}_i)Y^h$ is vertical, it is enough to consider its inner product with a vertical tangent vector field V and we have

(2.1) $$h((\nabla_{X^h}\mathcal{P}_2)Y^h, V) = \frac{\tau}{4n(n+2)}[V^1g(X,J_1Y) + V^2g(X,J_2Y) + V^3g(X,J_3Y)],$$

while

$$(\nabla_{X^h}\mathcal{P}_1)Y^h = 0. \tag{2.2}$$

For $(\nabla_{X^h}\mathcal{P}_i)V$, its horizontal part may be found immediately from the above and to show that its vertical part vanishes we show that $(\nabla_{X^h}\mathcal{P}_i)V$ is horizontal. To do this effectively recall that $\mathcal{P}_1$ can be given by equation (1.2) and we may regard this formula as extended to E, i.e. $\mathcal{P}_1 V$ is given by this formula for V tangent to E, even though one no longer has $\mathcal{P}_1^2 = I$. Then

$$\begin{aligned}
(\nabla_{X^h}\mathcal{P}_1)\frac{\partial}{\partial y_1} &= \overline{\nabla}_{X^h}\Big(y_3\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_3}\Big) - \mathcal{P}_1\Big(\frac{1}{2}(\hat{R}_{\sigma J_1}X)^h - \gamma(X)\frac{\partial}{\partial y_2} - \beta(X)\frac{\partial}{\partial y_3}\Big)\\
&= (y_1\beta(X) + y_2\alpha(X))\frac{\partial}{\partial y_2} + y_3\Big(\frac{1}{2}(\hat{R}_{\sigma J_2}X)^h - \gamma(X)\frac{\partial}{\partial y_1} - \alpha(X)\frac{\partial}{\partial y_3}\Big)\\
&+(- y_1\gamma(X) + y_3\alpha(X))\frac{\partial}{\partial y_3} - y_2\Big(\frac{1}{2}(\hat{R}_{\sigma J_3}X)^h - \beta(X)\frac{\partial}{\partial y_1} + \alpha(X)\frac{\partial}{\partial y_2}\Big)\\
&-\frac{1}{2}(p\hat{R}_{\sigma J_1}X)^h + \gamma(X)\Big(y_3\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial y_3}\Big) + \beta(X)\Big(- y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2}\Big)\\
&= \frac{1}{2}y_3(\hat{R}_{\sigma J_2}X)^h - \frac{1}{2}y_2(\hat{R}_{\sigma J_3}X)^h - \frac{1}{2}(p\hat{R}_{\sigma J_1}X)^h
\end{aligned}$$

which is horizontal. The proofs for $\dfrac{\partial}{\partial y_2}$ and $\dfrac{\partial}{\partial y_3}$ and for $\mathcal{P}_2$ are similar.

Similarly treating $(\nabla_V\mathcal{P}_i)X^h$ we find that

$$\begin{aligned}
&h((\nabla_V\mathcal{P}_i)X^h, Y^h)\\
&= \frac{4n(n+2)+\tau}{4n(n+2)}[V^1g(J_1X,Y) + V^2g(J_2X,Y) + V^3g(J_3X,Y)]. \qquad (2.3)
\end{aligned}$$

Finally for vertical tangent vectors V and W

$$(\nabla_V\mathcal{P}_i)W = \overline{\nabla}_V\mathcal{P}_iW - \langle AV, \mathcal{P}_iW\rangle\nu - \mathcal{P}_i\overline{\nabla}_VW$$

noting that the extension of $\mathcal{P}_i$ to E annihilates ν. Treating the terms separately for $\mathcal{P}_1$

$$\begin{aligned}
\overline{\nabla}_V\mathcal{P}_1W &= (V^3W^2 + y_3VW^2 - V^2W^3 - y_2VW^3)\frac{\partial}{\partial y_1}\\
&+(V^3W^1 + y_3VW^1 - V^1W^3 - y_1VW^3)\frac{\partial}{\partial y_2}
\end{aligned}$$

$$+(V^1W^2 + y_1VW^2 - V^2W^1 - y_2VW^1)\frac{\partial}{\partial y_3},$$

$$\mathcal{P}_1\overline{\nabla}_V W = (VW^1)\Big(y_3\frac{\partial}{\partial y_2} - y_2\frac{\partial}{\partial y_3}\Big) + (VW^2)\Big(y_3\frac{\partial}{\partial y_1} + y_1\frac{\partial}{\partial y_3}\Big)$$
$$+(VW^3)\Big(-y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2}\Big)$$

and using $\langle \mathbf{y}, V\rangle = \langle \mathbf{y}, W\rangle = 0$ and $-y_1^2 + y_2^2 + y_3^2 = 1$

$$y_1\langle V, \mathcal{P}_1W\rangle = V^2W^3 - V^3W^2,$$
$$y_2\langle V, \mathcal{P}_1W\rangle = V^1W^3 - V^3W^1,$$
$$y_3\langle V, \mathcal{P}_1W\rangle = V^1W^2 - V^2W^1.$$

Combining these we have $(\nabla_V\mathcal{P}_1)W = 0$ and similarly $(\nabla_V\mathcal{P}_2)W = 0$.

Using these computations we can now easily complete the proof of Theorem 2. That the almost para-Hermitian structure $(\mathcal{P}_1, h)$ is para-Kähler if and only if $\tau = -4n(n+2)$ follows immediately from the above relations, especially equations (2.2) and (2.3). To show the integrability of $\mathcal{P}_1$ first recall that the Nijenhuis tensor of $\mathcal{P}_1$ may be written in terms of the connection ∇ as

(2.4) $$[\mathcal{P}_1,\mathcal{P}_1](X,Y) = \mathcal{P}_1(\nabla_Y\mathcal{P}_1)X - (\nabla_{\mathcal{P}_1Y}\mathcal{P}_1)X - \mathcal{P}_1(\nabla_X\mathcal{P}_1)Y + (\nabla_{\mathcal{P}_1X}\mathcal{P}_1)Y.$$

The cases $[\mathcal{P}_1,\mathcal{P}_1](X^h, Y^h) = 0$ and $[\mathcal{P}_1,\mathcal{P}_1](V,W) = 0$ are immediate. For $[\mathcal{P}_1,\mathcal{P}_1](V, X^h)$, observe that the first two terms of the expansion (2.4) vanish while the remaining two are horizontal. Thus it is enough to compute

$$h([\mathcal{P}_1,\mathcal{P}_1](V,X^h), Y^h) = h((\nabla_V\mathcal{P}_1)X^h, (pY)^h) + h((\nabla_{\mathcal{P}_1V}\mathcal{P}_1)X^H, Y^h);$$

upon expansion using (2.3) the two terms will cancel.

For the almost para-Hermitian structure $(\mathcal{P}_2, h)$, to see that it is almost para-Kähler if and only if $\tau = 4n(n+2)$, the key case to consider is

$$h((\nabla_{X^h}\mathcal{P}_2)Y^h, V) + h((\nabla_V\mathcal{P}_2)X^h, Y^h) + h((\nabla_{Y^h}\mathcal{P}_2)V, X^h)$$
$$= \frac{\tau - 4n(n+2)}{4n(n+2)}[V^1g(X, J_1Y) + V^2g(X, J_2Y) + V^3g(X, J_3Y)].$$

To see that $(\mathcal{P}_2, h)$ is nearly para-Kähler if and only if $\tau = -2n(n+2)$, note that $h((\nabla_{X^h}\mathcal{P}_2)Y^h + (\nabla_{Y^h}\mathcal{P}_2)X^h, V) = 0$ by the skew-symmetry in equation (2.1) and by equations (2.1) and (2.3)

$$h((\nabla_{X^h}\mathcal{P}_2)V + (\nabla_V\mathcal{P}_2)X^h, Y^h)$$

$$= \frac{4n(n+2)+2\tau}{4n(n+2)}[V^1 g(J_1X,Y) + V^2 g(J_2X,Y) + V^3 g(J_3X,Y)].$$

To show the non-integrability of $\mathcal{P}_2$ compute $h([\mathcal{P}_2,\mathcal{P}_2](V,X^h),Y^h)$ at the point $(0,0,1)$ with $V = e_1 = \frac{\partial}{\partial y_1} - \frac{\partial}{\partial y_2}$. The first term in the expansion (2.4) yields

$$-h((\nabla_{X^h}\mathcal{P}_2)V,(pY)^h) = -\frac{\tau}{4n(n+2)}[g(X,J_2Y) - g(X,J_1Y)].$$

Proceeding in this way with the other terms

$$h([\mathcal{P}_2,\mathcal{P}_2](V,X^h),Y^h) = 2[g(X,J_2Y) - g(X,J_1Y)]$$

which is not identically zero, e.g take $X = J_2Y$.

Also by the above computation, $(\nabla_{X^h}\mathcal{P}_i)X^h = 0$ for any $X \in TM$ and $(\nabla_V\mathcal{P}_i)V = 0$ for any vertical vector V, $i = 1,2$. Therefore $(\mathcal{P}_i,h)$ is indefinite semi-para-Kähler.

Finally we remark that, motivated by other considerations, Jensen and Rigoli [10] developed for neutral 4-dimensional manifolds, a similar analogue of the classical twistor space, called the reflector space.

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