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GROUPS WITH RESTRICTED CONJUGACY CLASSES

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Abstract. Let FC^0 be the class of all finite groups, and for each non-negative integer n define by induction the group class FC^{n+1} consisting of all groups G such that for every element x the factor group $G/C_G(\langle x\rangle^G)$ has the property FC^n. Thus FC^1-groups are precisely groups with finite conjugacy classes, and the class FC^n obviously contains all finite groups and all nilpotent groups with class at most n. In this paper the known theory of FC-groups is taken as a model, and it is shown that many properties of FC-groups have an analogue in the class of FC^n-groups.

1. Introduction. A group G is called an *FC-group* if every element of G has finitely many conjugates, or equivalently if for each $x \in G$ the centralizer $C_G(x)$ of x has finite index in G. The theory of FC-groups had a strong development in the second half of last century, and also in recent years many authors have investigated properties of groups with finiteness restrictions on their conjugacy classes (for details we refer in particular to the survey [12]). Groups with finite conjugacy classes can be considered as the most natural tool in order to study

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properties which are common both to finite groups and abelian groups. On the other hand, the attempt to investigate wider classes of groups generalizing both finiteness and nilpotency (like for instance the class of FC-nilpotent groups) has not been satisfactory until now. The aim of this article is to overcome this difficulty, introducing and studying a new group theoretical property which is suitable for our pourposes.

Let $\mathfrak{X}$ be a class of groups. Recall that a group G is called an $\mathfrak{X}C$*-group* (or a group with $\mathfrak{X}$ *conjugacy classes*) if for each element x of G the factor group $G/C_G(\langle x\rangle^G)$ belongs to the class $\mathfrak{X}$. In particular, taking for $\mathfrak{X}$ the class of all finite groups, we obtain in this way the concept of an FC-group defined above; moreover, if for each non-negative integer n we denote by $\mathfrak{N}_n$ the class of nilpotent groups with class at most n, then $\mathfrak{N}_nC = \mathfrak{N}_{n+1}$. The structure of $\mathfrak{X}C$-groups has been studied for several natural choices of the class $\mathfrak{X}$, for instance for the class of all Černikov groups [4, 8] and for that of polycyclic-by-finite groups [2, 3].

Denote by FC^0 the class of all finite groups, and suppose by induction that for some non-negative integer n a group class FC^n has been defined; we denote then by FC^{n+1} the class of all groups G with conjugacy classes in FC^n, i.e. such that for every element x the factor group $G/C_G(\langle x\rangle^G)$ has the property FC^n. Clearly, the conditions FC and FC^1 are equivalent, and for each non-negative integer n the class FC^n contains all finite groups and all nilpotent groups of class at most n. We take the known theory of FC-groups as our model, and show that a number of the properties of FC-groups have an analogue in the class of FC^n-groups. Some of our results will be proved for the wider class FC^∞ consisting of all groups G such that for any element x of G the group $G/C_G(\langle x\rangle^G)$ has the property FC^n for some non-negative integer n.

Most of our notation is standard and can be found in [9]. We shall use the monograph [11] as a general reference for results on FC-groups.

2. Some preliminaries. Our first lemma deals with some closure properties of the class of FC^n-groups, and can be easily proved by induction on n. It follows of course that the same properties also hold for the class of FC^∞-groups.

Lemma 2.1. *For each non-negative integer n, subgroups, homomorphic images and direct products of FC^n-groups are likewise FC^n-groups.*

For each positive integer n let G_n be a finitely generated nilpotent group of class n such that the factor group $G_n/Z_{n-1}(G_n)$ is infinite. Thus G_n is an FC^n-group which does not belong to the class FC^{n-1} (see Section 3, Proposition

3.6), so that

$$FC^0 \subset FC^1 \subset \cdots \subset FC^n \subset FC^{n+1} \subset \cdots$$

Moreover, the direct product

$$G = \underset{n\in\mathbb{N}}{\mathrm{Dr}}\, G_n$$

is an FC^∞-group which does not have the property FC^n for any n, and hence

$$\bigcup_{n\in\mathbb{N}} FC^n \subset FC^\infty.$$

Let G be a group. The *FC-centre* $F(G)$ of G is the subgroup consisting of all elements of G having only finitely many conjugates. Thus a group G is an FC-group if and only if it coincides with its FC-centre. The *upper FC-central series* of G is defined as the ascending normal series of G whose terms $F_\alpha(G)$ are defined by the positions

$$F_0(G) = \{1\}, \quad F_{\alpha+1}(G)/F_\alpha(G) = F(G/F_\alpha(G))$$

for every ordinal α, and

$$F_\lambda(G) = \bigcup_{\beta<\lambda} F_\beta(G)$$

if λ is a limit ordinal. The group G is called *FC-nilpotent* if $G = F_n(G)$ for some non-negative integer n. It will be proved in Section 3 that if G is any FC^n-group (where n is a positive integer), then $F_n(G) = G$, and so G is FC-nilpotent. On the other hand, our next result shows in particular that if A is any infinite abelian group without elements of order 2 and x is the automorphism of A defined by $a^x = a^{-1}$ for all $a \in A$, the semidirect product $\langle x\rangle \ltimes A$ does not have the property FC^∞. Therefore

$$\bigcup_{n\in\mathbb{N}} FC^n$$

is properly contained in the class of all FC-nilpotent groups.

Lemma 2.2. *Let G be an FC^∞-group containing a finite non-empty subset X such that $C_G(\langle X\rangle^G) = \{1\}$. Then G is finite.*

Proof. Since X is finite and for every $x \in X$ the group $G/C_G(\langle x\rangle^G)$ has the property FC^{n_x} for some non-negative integer n_x, we have that G itself is an FC^n-group for some non-negative integer n. Choose the smallest m such that G has the property FC^m, and assume that $m > 0$. Then $G/C_G(\langle x\rangle^G)$ is an FC^{m-1}-group for each element x of X, and hence also G is an FC^{m-1}-group. This contradiction proves the lemma. $\square$

3. Main results. The first lemma of this section describes the behaviour of commutators in FC^n-groups. It will be very useful for our considerations.

Lemma 3.1. *Let G be an FC^n-group (where n is a positive integer), and let $x_1, \dots, x_t$ be elements of G with $t \leq n$. Then $G/C_G(\langle [x_1, \dots, x_t] \rangle^G)$ is an FC^{n-t}-group.*

Proof. The statement is obvious if $t = 1$. Suppose $t > 1$, and write $y = [x_1, \dots, x_{t-1}]$, so that by induction on t the factor group $G/C_G(\langle y \rangle^G)$ can be assumed to have the property FC^{n-t+1}. Put $C_1 = C_G(\langle y \rangle^G)$ and $C_2 = C_G(\langle x_t \rangle^G)$, and consider the centralizers

$$H/C_1 = C_{G/C_1}(\langle x_t \rangle^G C_1/C_1)$$

and

$$K/C_2 = C_{G/C_2}(\langle y \rangle^G C_2/C_2).$$

Since G/C_1 is an FC^{n-t+1}-group and G/C_2 is an FC^{n-1}-group, it follows that both G/H and G/K are FC^{n-t}-groups, and so also the factor group $G/H \cap K$ has the property FC^{n-t}. Moreover,

$$[H \cap K, \langle y \rangle^G, \langle x_t \rangle^G] = [\langle x_t \rangle^G, H \cap K, \langle y \rangle^G] = \{1\}$$

so that

$$[\langle y \rangle^G, \langle x_t \rangle^G, H \cap K] = \{1\}$$

by the Three Subgroup Lemma, and hence the subgroup $H \cap K$ is contained in the centralizer $C_G([x_1, \dots, x_t])$. Therefore $G/C_G(\langle [x_1, \dots, x_t] \rangle^G)$ is an FC^{n-t}-group, and the lemma is proved. □

Theorem 3.2. *Let G be an FC^n-group (where n is a positive integer). Then the subgroup $\gamma_n(G)$ is contained in the FC-centre of G. In particular, G is FC-nilpotent and its upper FC-central series has length at most n.*

Proof. The statement follows directly from Lemma 3.1, for $t = n$. □

It is well known that the commutator subgroup of any FC-group is periodic, so that in particular the set of all elements of finite order of an FC-group is a subgroup, and torsion-free FC-groups are abelian. Similar properties also hold for FC^n-groups when $n > 1$.

Corollary 3.3. *Let G be an FC^n-group (where n is a non-negative integer). Then the subgroup $\gamma_{n+1}(G)$ is periodic. In particular, the elements of finite order of G form a subgroup.*

Proof. We may clearly suppose $n > 0$. The subgroup $\gamma_n(G)$ is an FC-group by Theorem 3.2, so that $\gamma_n(G)'$ is periodic and replacing G by $G/\gamma_n(G)'$ it can be assumed without loss of generality that $\gamma_n(G)$ is abelian. Consider elements $x \in \gamma_n(G)$ and $g \in G$. Since $G/C_G(\langle g\rangle^G)$ is an FC^{n-1}-group, by induction on n we obtain that $\gamma_n(G/C_G(\langle g\rangle^G))$ is a periodic group, so that $x^k \in C_G(\langle g\rangle^G)$ for some positive integer k. Thus

$$[x, g]^k = [x^k, g] = 1,$$

and the commutator $[x, g]$ has finite order. Therefore the abelian group $\gamma_{n+1}(G)$ is generated by elements of finite order, and so it is periodic. $\square$

Corollary 3.4. *Let G be a torsion-free FC^n-group. Then G is nilpotent with class at most n.*

Recall that the group class $\mathfrak{X}$ is called a *Schur class* if for any group G such that the factor group $G/Z(G)$ belongs to $\mathfrak{X}$, also the commutator subgroup G' of G is an $\mathfrak{X}$-group. Thus the famous Schur's theorem just states that finite groups form a Schur class. It is known that the group classes determined by the most natural finiteness conditions have the Schur property; for the class of FC-groups this has been proved in [1].

Corollary 3.5. *For each non-negative integer n, the class of all FC^n-groups is a Schur class.*

Proof. If $n = 0$ the statement is the well known Schur's theorem. Suppose $n > 0$, and let G be any group such that $G/Z(G)$ has the property FC^n. Then $G/C_G(\langle g\rangle^G)$ is obviously an FC^n-group for every element g of G, and so G is an FC^{n+1}-group. Let x and y be elements of G; then $G/C_G(\langle [x, y]\rangle^G)$ is an FC^{n-1}-group by Lemma 3.1. It follows that $G'/C_{G'}(\langle a\rangle^{G'})$ has the property FC^{n-1} for every $a \in G'$, and hence G' is an FC^n-group. $\square$

Observe that if G is any group such that $G/Z(G)$ is an FC^n-group for some non-negative integer n, then G obviously has the property FC^{n+1}, and hence the subgroup $\gamma_{n+1}(G)$ is contained in the FC-centre of G.

In relation to the above corollary, we mention also that Maier and Rogerio [7] have recently proved that FC^n-groups form a Dietzmann class for every non-negative integer n. Here a group class $\mathfrak{X}$ is said to be a *Dietzmann class* if it is closed with respect to subgroups and homomorphic images, and for any element x of a group G such that $\langle x\rangle$ and $G/C_G(\langle x\rangle^G)$ are $\mathfrak{X}$-groups, also the normal closure $\langle x\rangle^G$ belongs to $\mathfrak{X}$. Thus the well known Dietzmann's Lemma states that the class $\mathfrak{F}$ of all finite groups is a Dietzmann class.

Clearly, a finitely generated group has finite conjugacy classes if and only if it is central-by-finite; our next result completely describes finitely generated FC^n-groups for any non-negative integer n.

Proposition 3.6. *Let G be a finitely generated group. Then G is an FC^n-group for some non-negative integer n if and only if the factor group $G/Z_n(G)$ is finite.*

Proof. Suppose first that G is a finitely generated FC^n-group with $n > 0$. Since $G/C_G(\langle x\rangle^G)$ is an FC^{n-1}-group for every element x of G, it follows that also $G/Z(G)$ is an FC^{n-1}-group, and so $G/Z_n(G)$ is finite by induction on n. Conversely, if the factor group $G/Z_n(G)$ is finite for some positive integer n, again by induction on n we obtain that $G/Z(G)$ has the property FC^{n-1}. In particular, $G/C_G(\langle x\rangle^G)$ is an FC^{n-1}-group for each element x of G, and hence G is an FC^n-group. □

It was proved by S. N. Černikov [11, Theorem 1.7] that every FC-group is isomorphic to a subgroup of the direct product of a periodic FC-group and a torsion-free abelian group. A similar result cannot be proved in the case of FC^n-groups with $n \geq 2$, since there exists an FC^2-group which cannot be embedded in any direct product of a periodic group and a torsion-free group. To see this, consider a periodic FC-group H and a non-periodic nilpotent group K of class 2 such that $Z(K)$ is periodic, and put $G = H \times K$. If $x = hk$ is any element of G (with $h \in H$ and $k \in K$), then the index $|H : C_H(\langle h\rangle^H)|$ is finite and $C_K(\langle k\rangle^K)$ contains the commutator subgroup K' of K; moreover,

$$C_G(\langle x\rangle^G) = C_H(\langle h\rangle^H) \times C_K(\langle k\rangle^K),$$

so that the factor group $G/C_G(\langle x\rangle^G)$ is finite-by-abelian, and in particular G is an FC^2-group. If N is any torsion-free normal subgroup of G, we have $N \cap Z(K) = \{1\}$, so that $N \cap K = \{1\}$ and hence $N = \{1\}$. Therefore G cannot be embedded into the direct product of a periodic group and a torsion-free group.

Lemma 3.7. *Let G be an FC^n-group (where n is a positive integer), and let X be a finite subset of G. Then $\langle X\rangle^G/Z_{n-1}(\langle X\rangle^G)$ is finitely generated and $\langle X\rangle^G/Z_n(\langle X\rangle^G)$ is finite.*

Proof. If $n = 1$, G is an FC-group and the statement is obvious. Suppose $n > 1$, and put $C = C_G(\langle X\rangle^G)$. As X is finite, the factor group G/C has the property FC^{n-1}, and hence by induction on n we have that the group

$$\frac{\langle X\rangle^G C/C}{Z_{n-2}(\langle X\rangle^G C/C)}$$

is finitely generated and central-by-finite. On the other hand, the group $\langle X\rangle^G C/C$ is isomorphic to $\langle X\rangle^G/Z(\langle X\rangle^G)$, and so also $\langle X\rangle^G/Z_{n-1}(\langle X\rangle^G)$ is a finitely generated central-by-finite group. The lemma is proved. □

It follows from Dietzmann's Lemma that a periodic group has the property FC if and only if it is covered by finite normal subgroups. This can be considered as a special case of the following result.

Theorem 3.8. *Let G be a periodic group. Then G is an FC^n-group for some positive integer n if and only if it has a local system $\mathfrak{L}$ consisting of normal subgroups such that $L/Z_{n-1}(L)$ is finite for every element L of $\mathfrak{L}$.*

Proof. Suppose first that G is an FC^n-group, and let X be any finite subset of G. Then $\langle X\rangle^G/Z_{n-1}(\langle X\rangle^G)$ is finite by Lemma 3.7, and so the local system consisting of the normal closures of all finite subsets of G satisfies the condition of the statement.

Conversely, assume that G has a local system $\mathfrak{L}$ such that every element L of $\mathfrak{L}$ is a normal subgroup of G and $L/Z_{n-1}(L)$ is finite. If $n = 1$, G is covered by its finite normal subgroups, and hence it is an FC-group. Suppose $n > 1$. Let x be any element of G, and consider the set $\mathfrak{L}_x$ of all subgroups $L \in \mathfrak{L}$ such that x belongs to L. If $L \in \mathfrak{L}_x$, the centre $Z(L)$ is contained in $L \cap C_G(\langle x\rangle^G)$, and so $Z_{n-2}(L/L \cap C_G(\langle x\rangle^G))$ has finite index in $L/L \cap C_G(\langle x\rangle^G))$. It follows that

$$\{LC_G(\langle x\rangle^G)/C_G(\langle x\rangle^G) \mid L \in \mathfrak{L}_x\}$$

is a local system of $G/C_G(\langle x\rangle^G)$ consisting of normal subgroups whose $(n-2)$-th term of the upper central series has finite index, and hence by induction on n the group $G/C_G(\langle x\rangle^G)$ has the property FC^{n-1}. Therefore G is an FC^n-group. □

A relevant result by Gorčakov [5] proves that any FC-group with trivial centre can be embedded in a direct product of finite groups. A corresponding theorem for FC^n-groups (with $n \geq 2$) does not hold. In fact, such a result would in particular imply that every FC^2-group with trivial centre is an FC-group, and this latter property is in general false, as the following example suggested by Carlo Casolo shows.

Let $A = \underset{i\in I}{\mathrm{Cr}}\langle a_i\rangle$ be the cartesian product of infinitely many groups of order 3, and for each index $j \in I$ let b_j denote the automorphism of A defined by the position

$$(a_i^{n_i})_{i\in I}^{b_j} = (u_i)_{i\in I},$$

where $u_i = a_i^{n_i}$ if $i \neq j$ and $u_j = a_j^{-n_j}$. Then $B = \langle b_i \mid i \in I\rangle$ is a subgroup of exponent 2 of the automorphism group of A, and the semidirect product $G = B \ltimes A$ is a group with trivial centre which is not an FC-group since the element

$(a_i)_{i\in I}$ has infinitely many conjugates. On the other hand, if $x = ab$ is any element of G (with $a \in A$ and $b \in B$), the centralizer $C_A(b) = C_A(x)$ is normal in G and the index $|A : C_A(b)|$ is finite, so that the factor group $G/C_G(\langle x\rangle^G)$ is finite-by-abelian. Therefore G is an FC^2-group.

It is well known that locally soluble (respectively, locally nilpotent) FC-groups are hyperabelian (respectively, hypercentral) with length at most ω and have an abelian (respectively, a central) descending normal series of length at most $\omega + 1$ [11, p. 10–11]. Our next two theorems extend these results to the case of FC^n-groups.

Theorem 3.9. *Let G be an FC^n-group for some positive integer n.*

(a) *If G is locally soluble, then it has an ascending characteristic series with abelian factors of length at most $\omega + (n - 1)$.*

(b) *If G is locally nilpotent, then G is hypercentral and its upper central series has length at most $\omega + (n - 1)$.*

Proof. (a) For each non-negative integer k, let $S_k/Z(\gamma_n(G))$ be the k-th term of the upper socle series of the group $\gamma_n(G)/Z(\gamma_n(G))$. Clearly every S_k is a characteristic subgroup of G, and S_{k+1}/S_k is abelian since G is locally soluble. Moreover, $\gamma_n(G)$ is an FC-group by Theorem 3.2, so that

$$\gamma_n(G) = \bigcup_{k\in\mathbb{N}_0} S_k$$

and the group G has an ascending characteristic series with abelian factors of length at most $\omega + (n - 1)$.

(b) The subgroup $\gamma_n(G)$ of G lies in the FC-centre of G by Theorem 3.2, so that $\gamma_n(G)$ is also contained in $Z_\omega(G)$ (see [9], Theorem 4.38). Therefore $G/Z_\omega(G)$ is nilpotent with class at most $n - 1$, and G is a hypercentral group with $Z_{\omega+(n-1)}(G) = G$. $\square$

Lemma 3.10. *Let G be an FC^n-group for some non-negative integer n. Then the factor group $G/Z_n(G)$ is residually finite.*

Proof. The statement is obvious if $n = 0$. Suppose $n > 0$, and for each element x of G, put

$$Z_x/C_G(\langle x\rangle^G) = Z_{n-1}(G/C_G(\langle x\rangle^G)).$$

As $G/C_G(\langle x\rangle^G)$ is an FC^{n-1}-group, by induction on n we have that G/Z_x is a residually finite group. On the other hand,

$$Z_n(G) = \bigcap_{x\in G} Z_x,$$

and hence $G/Z_n(G)$ is likewise residually finite. $\square$

Theorem 3.11. *Let G be an FC^n-group for some non-negative integer n.*

(a) *If G is locally soluble, then G is hypoabelian and its derived series has length at most $\omega + n$.*

(b) *If G is locally nilpotent, then G is hypocentral and its lower central series has length at most $\omega + n$.*

Proof. As the group $G/Z_n(G)$ is residually finite by Lemma 3.10, it is enough to observe that, if X is any residually finite locally soluble (locally nilpotent, respectively) group, then $X^{(\omega)} = \{1\}$ ($\gamma_\omega(X) = \{1\}$, respectively). $\square$

In the last part of this section we will study chief factors and maximal subgroups of FC^∞-groups. Once again, our theorem generalizes known results concerning FC-groups.

Theorem 3.12. *Let G be an FC^∞-group. Then every chief factor of G is finite and every maximal subgroup of G has finite index.*

Proof. As the class of FC^∞-groups is closed under homomorphic images, in order to prove the first part of the statement it is clearly enough to show that every minimal normal subgroup of G is finite. Assume by contradiction that G contains an infinite minimal normal subgroup N, and let g be an element of $G \setminus C_G(N)$. Then $G/C_G(\langle g\rangle^G)$ is an FC^n-group for some non-negative integer n, and by Theorem 3.2 the subgroup $\gamma_n(G/C_G(\langle g\rangle^G)$ lies in the FC-centre of $G/C_G(\langle g\rangle^G)$. On the other hand, N is obviously contained in $\gamma_n(G)$, and hence $NC_G(\langle g\rangle^G)/C_G(\langle g\rangle^G)$ is finite. As $N \cap C_G(\langle g\rangle^G) = \{1\}$, it follows that N is finite, a contradiction.

Assume that G contains a maximal subgroup M such that the index $|G : M|$ is infinite. Then M is not normal in G, and replacing G by the factor group G/M_G we may suppose that M is core-free. Let x be an element of $G \setminus M$, so that $G = \langle x\rangle^G M$ and in particular

$$C_G(\langle x\rangle^G) \cap M = \{1\}.$$

The factor group $G/C_G(\langle x\rangle^G)$ has the property FC^n for some non-negative integer n. Clearly $\gamma_n(G)$ is not contained in M, so that $G = \gamma_n(G)M$ and $\gamma_n(G)$ must be infinite. Thus $\gamma_n(G)$ is not a minimal normal subgroup of G, and hence $\gamma_n(G) \cap M \neq \{1\}$. Let a be a non-trivial element of $\gamma_n(G) \cap M$. By Theorem 3.2 the subgroup $\gamma_n(G/C_G(\langle x\rangle^G))$ is contained in the FC-centre of $G/C_G(\langle x\rangle^G)$, so that $\langle a\rangle^G C_G(\langle x\rangle^G)/C_G(\langle x\rangle^G)$ satisfies the maximal condition on subgroups. It follows that

$$E = \langle a\rangle^G \cap M \simeq (\langle a\rangle^G \cap M)C_G(\langle x\rangle^G)/C_G(\langle x\rangle^G)$$

is a finitely generated non-trivial subgroup of G. Application of Lemma 2.2 yields that the centralizer $C_G(E^G)$ is a non-trivial subgroup of G, so that $G = C_G(E^G)M$ and hence $E^G = E^M \leq M$, contradicting the assumption that M is core-free. This contradiction completes the proof of the theorem. □

4. Subnormal subgroups. A subgroup H of a group G is said to be *serial* in G if there exists a series containing both H and G, i.e. a complete chain Σ of subgroups of G such that if U and V are consecutive terms of Σ, then U is normal in V. If the series between H and G can be chosen to be finite (respectively, well-ordered), we obtain in this way the usual concept of a subnormal (respectively, ascendant) subgroup. A useful result of Hartley (see [6], Lemma 1) shows that if H is a subgroup of a group G and there exists a local system $\mathfrak{L}$ of G such that $H \cap L$ is subnormal in L for each $L \in \mathfrak{L}$, then H is a serial subgroup of G.

It has been proved by Tôgô [10] that any serial subgroup of an FC-group is ascendant with length at most ω; the main theorem of this section generalizes this result to FC^n-groups.

Lemma 4.1. *Let G be a finite-by-nilpotent group. Then every serial subgroup of G is subnormal.*

Proof. Let N be a finite normal subgroup of G such that the factor group G/N is nilpotent. If H is any serial subgroup of G, the index $|HN : H|$ is finite, so that H is subnormal in HN and hence it is also a subnormal subgroup of G. □

Lemma 4.2. *Let G be a locally (finite-by-nilpotent) group. Then the join of any collection of serial subgroups of G is a serial subgroup.*

Proof. Let $(H_i)_{i\in I}$ be a system of serial subgroups of G, and put $H = \langle H_i \mid i \in I\rangle$. Consider any finitely generated subgroup E of G. Then E satisfies the maximal condition on subgroups, and hence $H \cap E$ is contained in $\langle L_{i_1}, \dots, L_{i_t}\rangle$, where $\{i_1, \dots, i_t\}$ is a finite subset of I and L_{i_j} is a suitable finitely generated subgroup of H_{i_j} for each $j \leq t$. Clearly the subgroup $L = \langle E, L_{i_1}, \dots, L_{i_t}\rangle$ is finitely generated, so that it is finite-by-nilpotent and every

$H_i \cap L$ is subnormal in L by Lemma 4.1. It follows that also $\langle H_{i_1} \cap L, \ldots, H_{i_t} \cap L\rangle$ is subnormal in L, and hence the subgroup

$$H \cap E = \langle H_{i_1} \cap L, \ldots, H_{i_t} \cap L\rangle \cap E$$

is subnormal in E. Therefore H is a serial subgroup of G. $\square$

Lemma 4.3. *Let G be a group in which the join of any collection of ascendant subgroups is ascendant, and let F be the FC-centre of G. If H is an ascendant subgroup of G, then there exists an ascending series from H to HF of length at most ω.*

Proof. We may obviously assume that H is properly contained in HF. Put $H_0 = H$, and suppose that for some non-negative integer n an ascendant subgroup H_n of G has been defined such that $H \leq H_n < HF$. Let $\mathfrak{L}_n$ be the local system of F consisting of the normal closures of all finite subsets of F which are not contained in H_n. Let E be any element of $\mathfrak{L}_n$. Since F is the FC-centre of G, E satisfies the maximal condition on subgroups, so that in particular all ascendant subgroups of E are subnormal and the join of any collection of subnormal subgroups of E is likewise subnormal. As H_n is a proper ascendant subgroup of H_nE, there exists another ascendant subgroup K of H_nE such that H_n is a proper normal subgroup of K. It follows that $E \cap K$ is a subnormal subgroup of E normalizing H_n and $E \cap H_n < E \cap K$. Let $X_n(E)$ be the subgroup generated by all subnormal subgroups of E which are contained in $N_G(H_n)$. Thus $X_n(E)$ is a subnormal subgroup of E and

$$E \cap H_n < X_n(E) \leq N_G(H_n).$$

Moreover, if E_1 and E_2 are elements of $\mathfrak{L}_n$ such that $E_1 \leq E_2$, clearly $X_n(E_1)$ is a subnormal subgroup of E_2 normalizing H_n, and hence $X_n(E_1)$ is contained in $X_n(E_2)$. It follows that

$$X_n = \bigcup_{E \in \mathfrak{L}_n} X_n(E)$$

is a subgroup of F normalizing H_n, and H_n is a proper normal subgroup of $H_{n+1} = H_nX_n$. As X_n is generated by subnormal subgroups of G, we also have that H_{n+1} is an ascendant subgroup of G. In this way we can define an ascending series

$$H = H_0 \triangleleft H_1 \triangleleft \ldots \triangleleft H_n \triangleleft H_{n+1} \triangleleft \ldots$$

of ascendant subgroups of G contained in HF. Assume that the subgroup

$$\bigcup_{n \in \mathbb{N}_0} H_n$$

is properly contained in HF. Then there exists an element x of F such that $\langle x\rangle^G$ belongs to each $\mathfrak{L}_n$, and hence

$$\langle x\rangle^G \cap H_0 < \langle x\rangle^G \cap H_1 < \langle x\rangle^G \cap H_2 < \ldots < \langle x\rangle^G \cap H_n < \ldots$$

is an infinite ascending series of subgroups of $\langle x\rangle^G$, a contradiction. Therefore

$$HF = \bigcup_{n\in\mathbb{N}_0} H_n,$$

and the lemma is proved. □

Theorem 4.4. *Let G be an FC^n-group, where n is a positive integer. Then every serial subgroup H of G is ascendant, and there exists an ascending series from H to G of length at most $\omega + (n-1)$.*

Proof. Since G is an FC^n-group, it follows from Theorem 3.2 that the subgroup $\gamma_n(G)$ is contained in the FC-centre of G; moreover, $L = \gamma_{n+1}(G)$ is periodic by Corollary 3.3, so that it has an ascending G-invariant series

$$\{1\} = L_0 < L_1 < \ldots < L_\tau = L$$

with finite factors. The group G is locally (finite-by-nilpotent) by Proposition 3.6, and hence Lemma 4.2 yields that for each ordinal $\alpha < \tau$ the product HL_α is a serial subgroup of G, so that HL_α is subnormal in $HL_{\alpha+1}$ as the index $|HL_{\alpha+1} : HL_\alpha|$ is finite. Therefore the subgroup H is ascendant in HL, and so even in G. A second application of Lemma 4.2 yields now that the join of any collection of ascendant subgroups of G is likewise ascendant. As $\gamma_n(G)$ is a subgroup of the FC-centre of G, it follows from Lemma 4.3 that there exists an ascending series from H to $H\gamma_n(G)$ of type at most ω; on the other hand, $H\gamma_n(G)$ is subnormal in G with defect at most $n-1$, and hence there exists an ascending series from H to G of type at most $\omega + (n-1)$. □

Corollary 4.5. *Let G be an FC^n-group, where n is a non-negative integer. If all subgroups of G are serial, then G is hypercentral.*

Proof. It follows from Theorem 4.4 that every subgroup of G is ascendant. Thus G is locally nilpotent, and so even hypercentral by Theorem 3.9. □

Our last result shows that if we restrict our attention to groups in which normality is a transitive relation, the group properties introduced in this paper are equivalent to the property FC.

Theorem 4.6. *Let G be an FC^∞-group in which normality is a transitive relation. Then G is an FC-group.*

Proof. Suppose first that G is an FC^2-group. If G is periodic, by Theorem 3.8 there exists a local system $\mathfrak{L}$ consisting of normal subgroups of G such that $L/Z(L)$ is finite for each $L \in \mathfrak{L}$. In particular, any element L of $\mathfrak{L}$ is a periodic FC-group, and so it has a local system consisting of finite normal subgroups. Since all subnormal subgroups of G are normal, we obtain that G itself is covered by finite normal subgroups, so that G is an FC-group. Assume now that the FC^2-group G is not periodic, and let x be any element of G. Consider a finitely generated non-periodic subgroup E of G such that $x \in E$. Then $E^G/Z_2(E^G)$ is finite by Lemma 3.7, so that $\gamma_3(E^G)$ is finite and $E^G/\gamma_3(E^G)$ is a non-periodic abelian group. Moreover, $G/C_G(E^G/\gamma_3(E^G))$ is isomorphic to a group of power automorphisms of $E^G/\gamma_3(E^G)$, and hence it is finite; since $\gamma_3(E^G)$ is finite, it follows that x has finitely many conjugates in G, and so G is an FC-group. Therefore all FC^2-groups in which normality is a transitive relation are FC-groups, and an iterated application of this fact yields that for every non-negative integer n any FC^n-group with the property T is an FC-group.

Suppose now that G is an FC^∞-group. If x is any element of G, the factor group $G/C_G(\langle x\rangle^G)$ has the property FC^n for some non-negative integer n, so that $G/C_G(\langle x\rangle^G)$ is an FC-group. Therefore G is an FC^2-group, and hence it is even an FC-group. $\square$

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