Vasil Atanasov Popov

January 14, 1942 – May 31, 1990

CURRICULUM VITA

Graduate Education: Sofia University, Bulgaria

Doctor of Sciences Degree in Mathematics, 1977
Ph.D. in Mathematics, 1971; Thesis Advisor: Blagovest Sendov

Undergraduate Education: Sofia University, Bulgaria

B.S. 1965 in Mathematics

Professional Employment:

1984–1990	Corresponding Member of the Bulgarian Academy of Sciences (BAS)
1981–1984	Professor at the Mathematical Institute of the BAS
1974–1981	Senior Scientist at the Mathematical Institute of the BAS
1965–1974	Scientist at the Mathematical Institute of the BAS

Visiting Positions:

Fall – Winter 1970/1971	Steklov Institute, Moscow
Spring 1983	University of South Carolina, Columbia, South Carolina
Fall 1987 – Fall 1988	University of South Carolina, Columbia, South Carolina
Fall 1989 – Spring 1990	Temple University, Philadelphia, Pennsylvania

Conference Organization:

International Conferences on Constructive Theory of Functions held in Blagoevgrad 1979, Varna 1981, Varna 1984, Varna 1987.

Doctoral Students (Ph.D.):

1 Andrey Andreev
2 Vladimir Hristov
3 Pencho Petrushev
4 Georgi Totkov
5 Kamen Ivanov
6 Petar Binev
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9 Ognyan Trifonov
10 Lyudmil Aleksandrov
11 Lyubomir Dechevski

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Our primary goal in this preamble is to highlight the best of Vasil Popov's mathematical achievements and ideas. V. Popov showed his extraordinary talent for mathematics in his early papers in the (typically Bulgarian) area of approximation in the Hausdorff metric. His results in this area are very well presented in the monograph of his advisor Bl. Sendov, "Hausdorff Approximation".

Vasil's mathematical intuition culminated in his results on *rational and nonlinear spline approximation.* He began this research in the late sixties, but continued to work in this area during his entire life and obtained remarkable results. V. Popov and G. Freud were the first to clearly understand that, in contrast to linear approximation schemes, nonlinear approximation requires different spaces and approaches. They showed in [9, 20] that

$$E_n^r(f) \le c_r n^{-r-1} V f^{(r)}, \quad r \ge 0,$$

where $E_n^r(f)$ is the error of uniform approximation to f from splines of degree r with $n+1$ free knots in $[0,1]$ and $Vf^{(r)} := V_0^1 f^{(r)}$ is the variation of $f^{(r)}$. Though not too hard to prove, this result has been a benchmark in nonlinear approximation and, in particular, in free knot spline and rational approximation. V. Popov refined this result and developed further the theory of nonlinear spline approximation in [19, 26, 36, 45, 49, 55].

The problem for obtaining a similar estimate for rational approximation has been attacked by many mathematicians, among them A. Bulanov, G. Freud, A. Gonchar, J. Szabados, P. Szusz, P. Turan. In a series of articles [39, 40, 43, 52, 56, 65] V. Popov developed an ingenious method for rational approximation which enabled him not only to improve the existing results but also to completely solve the problem. In [56], he proved that:

$$R_n(f) \le c_r n^{-r-1} V f^{(r)}, \quad r \ge 1, \tag{1}$$

where $R_n(f)$ is the error of the uniform approximation to f in $[0,1]$ from rational functions of degree at most n. In [56], V. Popov also settled Newman's conjecture for rational approximation of Lipschitz functions:

$$\textit{If} \quad f \in \mathrm{Lip}\ 1 \quad \textit{on} \quad [0,1], \quad \textit{then} \quad R_n(f) = o(n^{-1}).$$

Impressed by his remarkable solution, Donald Newman referred to V. Popov as "a brilliant young Bulgarian mathematician".

It is interesting to briefly describe the development of Popov's method for rational approximation. Following Vasil's notation, let

$$\Phi_n^r := \sup_{V_0^1 f^{(r)} \le 1} R_n(f).$$

Employing the famous result of Newman for rational approximation of $|x|$, V. Popov [39] showed that if

$$\Phi_n^r \le \Psi(n)n^{-r-1}, \;\; \Psi(n) \ge 1, \quad \text{then} \quad \Phi_n^r \le c_1\Psi(\ln^2 n)n^{-r-1}.$$

Iterating this result, starting with Freud's estimate $\Phi_n^r \le c(\ln^2 n)n^{-r-1}$, he was abled to prove that if $V_0^1 f^{(r)} \le 1$, then

$$R_n(f) \le c_{r,k}\frac{\overbrace{\ln\ldots\ln}^{k} n}{n^{r+1}}$$

for an arbitrary $k \ge 1$, but with constant $c_{r,k} \to \infty$ as $k \to \infty$.

In [56], V. Popov further refined his method and managed to remove the logarithmic factor above. Let now

$$\Phi_{n,A}^r := \sup_{\substack{V_0^1 f^{(r)} \le 1 \\ f^{(s)}(0)=0,\, s=0,\ldots,r}} \; \inf_{\substack{q\in R_n \\ \|q\|_{C(-\infty,+\infty)} \le A}} \|f-q\|_{C[0,1]}.$$

Using again Newman's result, V. Popov proved that if for $n \ge n_0$

$$\Phi_{k,k^{2r+4}}^r \le \phi(k)k^{-r-1} \quad \text{with } \phi(k) \ge 1 \text{ for } k = c\ln^3 n, \text{ then}$$

$$\Phi_{n,n^{2r+4}}^r \le \phi(k)n^{-r-1}\left(1+\frac{3}{\ln n}\right)^{r+1},$$

and estimate (1) follows by iteration.

Popov's method has become a basic tool for proving upper bound estimates for rational approximation. Thus, in [60], the exact rate $O(n^{-1})$ of the uniform rational approximation of the class of all uniformly bounded convex and continuous functions is obtained and "small o" effect is established. Jackson type estimates for rational approximation are proved [71]. In [37, 55], V. Popov reveals the strong relations between rational and free knot spline approximation. An interesting method was developed by V. Popov (along with D. Newman and B. Gao) in [102, 103] for rational approximation of convex functions from convex rational functions.

Many of the above mentioned results found their natural place in V. Popov's monograph (with P. Petrushev) "Rational Approximation of Real Functions" [97].

In going further, V. Popov (with R. DeVore and B. Jawerth) [92, 94, 96, 98, 100, 101] proved some fundamental results in nonlinear approximation by box splines and wavelets which substantially influenced the further developments in the theory of spline and wavelet approximation.

Another circle of V. Popov's developments was connected to error estimates of approximation processes which involve function values at certain points but intrinsically employ the integral metric. Examples of such processes are:

- quadrature formulae;
- onesided approximations in integral metrics;
- approximations by discrete operators in integral metrics;
- numerical methods for differential equations.

In such cases the classical integral and uniform moduli ω_k are not suitable enough as measures of smoothness. In order to provide the correct rates of convergence one needs new characteristics, that should satisfy simultaneously a number of conditions, such as they need to be:

- smaller than the uniform moduli because the approximation error is measured in integral metrics;
- bigger than the integral moduli because of the discrete nature of the approximation process;
- close to the integral moduli for $k \geq 2$ and smooth functions.

In [58], V. Popov introduced *the averaged moduli of smoothness* τ_k, which for the L_p norm, $1 \leq p < \infty$, and an integer k are defined by

$$\tau_k(f,\delta)_p = \|\omega_k(f,\cdot,\delta)\|_p,$$

where

$$\omega_k(f,x,\delta) = \sup\left\{|\Delta_h^k f(t)| : t, t+kh \in \left[x - \frac{k\delta}{2}, x + \frac{k\delta}{2}\right]\right\}.$$

Similar type of characteristics for $k = 1$ had earlier appeared in papers of Bl. Sendov, P. P. Korovkin, E. P. Dolzhenko and E. A. Sevastyanov. The complete theory of these moduli has been developed mainly by the Bulgarian group in approximation theory and is the topic of the monograph of V. Popov and Bl. Sendov "Averaged Moduli of Smoothness" [78].

One of the main applications of the averaged moduli of smoothness is in the theory of *one-sided approximation* by trigonometric polynomials or splines with fixed knots [58, 59, 62, 64, 66, 69]. These results are similar to the corresponding direct and inverse theorems due to Jackson, Zygmund, Timan and Stechkin in the unconstrained case and the role of the usual moduli of smoothness ω_k is played by τ_k. For example, it is proved in [62, 58] that

$$\widetilde{E}_n(f)_p \le c_k \tau_k(f, n^{-1})_p \quad \textit{and} \quad \tau_k(f, n^{-1})_p \le c_k n^{-k} \sum_{s=0}^{n} (s+1)^{k-1} \widetilde{E}_s(f)_p,$$

where $\widetilde{E}_n(f)_p$ denotes the best one-sided L_p-approximation of a 2π-periodic function f by trigonometric polynomials of degree n.

Using τ-moduli V. Popov obtained a quantitative theorem of Korovkin type [76], direct theorems for approximation in integral metrics and the saturation classes for a number of discrete operators [70, 85]. A variety of applications of the averaged moduli to different areas of the numerical analysis, such as quadrature formulae and numerical methods for differential equations, are given in [64, 68, 75, 83, 87, 99].

The function spaces generated by the averaged moduli are systematically studied in [79, 81, 82]. In order to investigate their interpolation properties and to obtain embedding theorems for Sobolev and Besov-type spaces and equivalent norms for them, V. Popov introduced the following *one-sided K-functionals*

$$\widetilde{K}_k(f, t)_p = \inf \left\{ \|\varphi - \psi\|_p + t\|\varphi^{(k)}\|_p + t\|\psi^{(k)}\|_p : \varphi, \psi \in W_p^k, \varphi \le f \le \psi \right\}.$$

He showed that the averaged moduli of smoothness are equivalent to the one-sided K-functionals [84]

$$c_k^{-1} \tau_k(f, t)_p \le \widetilde{K}_k(f, t^k)_p \le c_k \tau_k(f, t)_p.$$

Some of the above mentioned results are generalized in the multidimensional case [77, 80, 91, 95].

With his excellent results and strong personality Vasil Popov deeply influenced the Bulgarian school of Approximation Theory and his grateful students and followers (among them the two of us writing these lines).

Kamen Ivanov and Pencho Petrushev