Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **24** (1998), 1-4

Serdica
Mathematical Journal

Institute of Mathematics
and Informatics, Bulgarian
Academy of Sciences

ON A CLASS OF D–CAUCHY FILTERS

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Communicated by J. Jayne

Dedicated to the memory of Professor D. Doitchinov

Abstract. In a quasi-uniform space, firmly D-Cauchy filters are introduced and they role in constructing firm extensions is investigated.

Let $\mathcal{U}$ be a quasi-uniformity on a set X. A pair $(\mathfrak{t}, \mathfrak{s})$ of filters in X is said to be *Cauchy* [2, 1.1] iff, for any given entourage $U \in \mathcal{U}$, there are $T \in \mathfrak{t}$ and $S \in \mathfrak{s}$ such that $T \times S \subset U$. According to [4], a filter $\mathfrak{s}$ in X is said to be *D-Cauchy* (Cauchy in [4]) iff it admits a *cofilter*, i.e. a (proper) filter $\mathfrak{t}$ such that $(\mathfrak{t}, \mathfrak{s})$ is a Cauchy filter pair. Our purpose is to investigate a class of D-Cauchy filters and, in particular, its role in extension problems.

1. Preliminaries. For quasi-uniform spaces in general, see the monograph [5]. We denote by $\mathcal{U}^{-1} = \{U^{-1} : U \in \mathcal{U}\}$ the conjugate of $\mathcal{U}$ [5, p. 1] and by $\mathcal{U}^{tp}$ the topology induced by $\mathcal{U}$ [5, p. 3]; we write $\mathcal{U}^{-tp}$ for $(\mathcal{U}^{-1})^{tp}$. If $X \subset Y$, the quasi-uniformity $\mathcal{V}$ on Y is said to be an *extension* of $\mathcal{U}$ iff the restriction

1991 *Mathematics Subject Classification:* 54E15, 54D35
Key words: quasi-uniform space, D-Cauchy filter, firm extension
*Research supported by Hungarian National Foundation for Scientific Research, Grant No T 016094.

$\mathcal{V}|X = \{V \cap (X \times X) : V \in \mathcal{V}\}$ of $\mathcal{V}$ to X equals $\mathcal{U}$ and X is $\mathcal{V}^{tp}$-dense. The extension $\mathcal{V}$ of $\mathcal{U}$ is said to be *firm* [2, p. 50] iff X is dense for $\sup\{\mathcal{V}^{-tp}, \mathcal{V}^{tp}\}$.

For $p \in Y - X$ let $\mathfrak{s}^+(p)$ be a (proper) filter in X. We say that the quasi–uniformity $\mathcal{V}$ on Y is *compatible with* $(\mathcal{U}, \mathfrak{s}^+)$ iff $\mathcal{V}$ is an extension of $\mathcal{U}$ and $\mathfrak{s}^+(p)$ is, for each $p \in Y - X$, the trace on X of the $\mathcal{V}^{tp}$–neighbourhood filter of p. If $\mathfrak{s}^-(p)$ is, again for $p \in Y - X$, a (proper) filter in X, $\mathcal{V}$ is said to be *compatible with* $(\mathcal{U}, \mathfrak{s}^-, \mathfrak{s}^+)$ iff it is compatible with $(\mathcal{U}, \mathfrak{s}^+)$ and $\mathcal{V}^{-1}$ is compatible with $(\mathcal{U}^{-1}, \mathfrak{s}^-)$. It is well-known that, for the existence of $\mathcal{V}$ compatible with $(\mathcal{U}, \mathfrak{s}^+)$, it is necessary that every filter $\mathfrak{s}^+(p)$ should be $(\mathcal{U}-)$ *round* ([1], 1) [1, 1.1]; a $\mathcal{V}$ compatible with $(\mathcal{U}, \mathfrak{s}^-, \mathfrak{s}^+)$ exists iff the filter pairs $(\mathfrak{s}^-(p), \mathfrak{s}^+(p))$ are round and Cauchy, i.e. each $\mathfrak{s}^+(p)$ is $\mathcal{U}$–round, each $\mathfrak{s}^-(p)$ is $\mathcal{U}^{-1}$–round, and each pair $(s^-(p), s^+(p))$ is $\mathcal{U}$–Cauchy [3, 6.1]. Therefore we can easily deduce the following theorem, implicitly contained in [3, 11.2]:

Theorem 1.1 (J. Deák). *A firm extension compatible with* $(\mathcal{U}, \mathfrak{s}^-, \mathfrak{s}^+)$ *exists iff the filter pairs* $(\mathfrak{s}^-(p), \mathfrak{s}^+(p))$ *are linked* [3, 7.1]*, round and Cauchy. If so then there is exactly one firm extension compatible with* $(\mathcal{U}, \mathfrak{s}^-, \mathfrak{s}^+)$*.*

2. Firmly *D*-Cauchy filters. In Theorem 1.1, we have to do with linked Cauchy filter pairs. Therefore it is natural to say that a D–Cauchy filter $\mathfrak{s}$ is *firmly D-Cauchy* (in a quasi-uniform space $(X, \mathcal{U})$) iff it admits a cofilter $\mathfrak{t}$ such that $(\mathfrak{t}, \mathfrak{s})$ is linked (i.e. $T \cap S \neq \emptyset$ whenever $T \in \mathfrak{t}$, $S \in \mathfrak{s}$).

In order to characterize firmly D–Cauchy filters, let us recall that, if $\mathfrak{s}$ is any filter in a quasi-uniform space $(X, \mathcal{U})$, there is a finest $\mathcal{U}$–round filter $\mathcal{U}(\mathfrak{s})$ coarser than $\mathfrak{s}$, the *$\mathcal{U}$–envelope* of $\mathfrak{s}$, composed of all sets $U(S)$ where $S \in \mathfrak{s}$, $U \in \mathcal{U}$ [1, 4.6]. In accordance with [1] (but differently from [4] or [5]), let us say that the filter $\mathfrak{s}$ is *Cauchy* in $(X, \mathcal{U})$ iff $(\mathfrak{s}, \mathfrak{s})$ is a Cauchy filter pair, i.e. iff, for any $U \in \mathcal{U}$, there is $S \in \mathfrak{s}$ such that $S \times S \subset U$.

Lemma 2.1. *In a quasi-uniform space* $(X, \mathcal{U})$ *a filter* $\mathfrak{s}$ *is firmly D–Cauchy iff there is a proper Cauchy filter* $\mathfrak{r}$ *such that* $\mathcal{U}(\mathfrak{r}) \subset \mathfrak{s} \subset \mathfrak{r}$*.*

Proof. If $\mathfrak{r}$ is a proper Cauchy filter then $(\mathfrak{r}, \mathfrak{r})$ is a Cauchy filter pair, hence clearly $(\mathfrak{r}, \mathcal{U}(\mathfrak{r}))$ and $(\mathfrak{r}, \mathfrak{s})$ are Cauchy provided $\mathcal{U}(\mathfrak{r}) \subset \mathfrak{s}$. Further, $\mathfrak{s} \subset \mathfrak{r}$ implies that $(\mathfrak{r}, \mathfrak{s})$ is linked.

Suppose on the other hand that $(\mathfrak{t}, \mathfrak{s})$ is linked and Cauchy. Then $\mathfrak{r} = \mathfrak{t}(\cap)\mathfrak{s} = \{T \cap S : T \in \mathfrak{t}, S \in \mathfrak{s}\}$ is a proper Cauchy filter. For any $U \in \mathcal{U}$ and

$R \in \mathfrak{r}$, let $T \in \mathfrak{t}$ and $S \in \mathfrak{s}$ satisfy $T \times S \subset U$ and choose $x \in T \cap S \cap R$. Now $S \subset U(x) \subset U(R)$ so that $U(R) \in \mathfrak{s}$, $\mathcal{U}(\mathfrak{r}) \subset \mathfrak{s}$, and obviously $\mathfrak{s} \subset \mathfrak{r}$. □

Corollary 2.2. *In a quasi-uniform space $(X,\mathcal{U})$, a filter $\mathfrak{s}$ is round and firmly D–Cauchy iff $\mathfrak{s} = \mathcal{U}(\mathfrak{r})$ for some proper Cauchy filter $\mathfrak{r}$.*

Proof. Choose $\mathfrak{r}$ according to Lemma 2.1. By the roundness of $\mathfrak{s}$ we have $\mathfrak{s} \subset \mathcal{U}(\mathfrak{r})$. □

Corollary 2.3. *A round filter pair $(\mathfrak{t},\mathfrak{s})$ is linked and Cauchy iff there is a proper Cauchy filter $\mathfrak{r}$ such that $\mathfrak{t} = \mathcal{U}^{-1}(\mathfrak{r})$, $\mathfrak{s} = \mathcal{U}(\mathfrak{r})$.*

Proof. If $(\mathfrak{t},\mathfrak{s})$ is round, linked and Cauchy then $r = \mathfrak{t}(\cap)\mathfrak{s}$ is a proper Cauchy filter such that $\mathcal{U}(\mathfrak{r}) \subset \mathfrak{s} \subset \mathfrak{r}$. By Corollary 2.2, $\mathfrak{s} = \mathcal{U}(\mathfrak{r})$. Similarly $\mathfrak{t} = \mathcal{U}^{-1}(\mathfrak{r})$.

Conversely, for a proper Cauchy filter $\mathfrak{r}$, the filter pair $(\mathcal{U}^{-1}(\mathfrak{r}),\mathcal{U}(\mathfrak{r}))$ is clearly linked and Cauchy. □

We obtain easily some extension theoretical applications.

Theorem 2.4. *Let $(X,\mathcal{U})$ be a quasi-uniform space, $X \subset Y$ and $\mathfrak{s}^+(p)$ be a filter in X for $p \in Y - X$. A firm extension compatible with $(\mathcal{U},\mathfrak{s}^+)$ exists iff each $\mathfrak{s}^+(p)$ is round and firmly D–Cauchy.*

Proof. The necessity follows from Theorem 1.1. If each $\mathfrak{s}^+(p)$ is round and firmly D–Cauchy, then, by Corollary 2.2, $\mathfrak{s}^+(p) = \mathcal{U}(\mathfrak{r}_p)$ for a proper Cauchy filter $\mathfrak{r}_p$ (depending on p). Define $\mathfrak{s}^-(p) = \mathcal{U}^{-1}(\mathfrak{r}_p)$ to obtain by Corollary 2.3 a round, linked Cauchy pair $(\mathfrak{s}^-(p),\mathfrak{s}^+(p))$. Now Theorem 1.1 furnishes the existence of the firm extension looked for. □

The filter $\mathfrak{r}$ in Corollary 2.2 is not unique for a given $\mathfrak{s}$: take a (non-T_0) topology $\mathcal{T}$ on X and points $x,y \in X$, $x \neq y$ such that x and y have the same neighbourhood filter $\mathfrak{s}$; for any quasi-uniformity $\mathcal{U}$ inducing $\mathcal{T}$, we have $\mathcal{U}(\dot{x}) = \mathcal{U}(\dot{y}) = \mathfrak{s}$.

However, if $\mathfrak{s}$ is a round and firmly D–Cauchy filter and $\mathfrak{s} = \mathcal{U}(\mathfrak{r})$ for a Cauchy filter $\mathfrak{r}$, then $\mathcal{U}^{-1}(\mathfrak{r})$ does not depend on the choice of $\mathfrak{r}$:

Lemma 2.5. *In a quasi-uniform space $(X,\mathcal{U})$, let $\mathfrak{r}$ and $\mathfrak{r}'$ be proper Cauchy filters such that $\mathcal{U}(\mathfrak{r}) = \mathcal{U}(\mathfrak{r}')$. Then $\mathcal{U}^{-1}(\mathfrak{r}) = \mathcal{U}^{-1}(\mathfrak{r}')$.*

Proof. For $U \in \mathcal{U}$, $R' \in \mathfrak{r}'$, let $U_1 \in \mathcal{U}$ and $R_1 \in \mathfrak{r}$ be such that $U_1^2 \subset U$, $R_1 \times R_1 \subset U_1$. Then $U_1(R_1) \in \mathcal{U}(\mathfrak{r}) = \mathcal{U}(\mathfrak{r}') \subset \mathfrak{r}'$, so there are $x \in U_1(R_1) \cap R'$ and $y \in R_1$ such that $(y,x) \in U_1$, i.e. $(x,y) \in U_1^{-1}$, $y \in U_1^{-1}(x)$, $R_1 \subset U_1^{-1}(y) \subset$

$U^{-1}(x) \subset U^{-1}(R')$. Thus $U^{-1}(R') \in \mathfrak{r}$ and $\mathcal{U}^{-1}(\mathfrak{r}') \subset \mathfrak{r}$; the left hand side being $\mathcal{U}^{-1}$–round, we also have $\mathcal{U}^{-1}(\mathfrak{r}') \subset \mathcal{U}^{-1}(\mathfrak{r})$. Similarly $\mathcal{U}^{-1}(\mathfrak{r}) \subset \mathcal{U}^{-1}(\mathfrak{r}')$. □

Corollary 2.6. *Let* $(X,\mathcal{U})$ *be a quasi-uniform space,* $X \subset Y$ *and, for* $p \in Y - X$, $\mathfrak{s}^+(p)$ *be a round, firmly D–Cauchy filter in* X. *Then there are uniquely determined filters* $\mathfrak{s}^-(p)$ *such that any firm extension* $\mathcal{V}$ *compatible with* $(\mathcal{U}, \mathfrak{s}^+)$ *is compatible with* $(\mathcal{U}, \mathfrak{s}^-, \mathfrak{s}^+)$.

Proof. By Theorem 1.1 if $\mathcal{V}$ is compatible with $(\mathcal{U}, \mathfrak{s}^-, \mathfrak{s}^+)$, then the pairs $(\mathfrak{s}^-(p), \mathfrak{s}^+(p))$ must be linked, round and Cauchy. By Corollary 2.3, there are (proper) Cauchy filters $\mathfrak{r}_p$ such that $\mathfrak{s}^-(p) = \mathcal{U}^{-1}(\mathfrak{r}_p)$, $\mathfrak{s}^+(p) = \mathcal{U}(\mathfrak{r}_p)$. By Lemma 2.5, $\mathfrak{s}^-(p)$ does not depend on the choice of $\mathfrak{r}_p$. □

Corollary 2.7. *Under the hypotheses of Corollary* 2.6 *there is exactly one firm extension compatible with* $(\mathcal{U}, \mathfrak{s}^+)$.

Proof. Corollary 2.6 and Theorem 1.1. □

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Received September 20, 1996