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DOITCHINOV'S CONSTRUCT OF SUPERTOPOLOGICAL SPACES IS TOPOLOGICAL

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Dedicated to the memory of Professor D. Doitchinov

ABSTRACT. It is shown that the construct of supertopological spaces and continuous maps is topological.

In 1964, Doitchinov [2] introduced supertopological spaces as a generalization of topological spaces. Starting from Hausdorff's approach using neighborhoods of points, Doitchinov gave axioms for neighborhoods of a broader class of subsets.

We recall Doitchinov's definitions. Let X be a set. A *supertopology* on X is a pair $(\mathcal{M}, \mathcal{V})$ where $\mathcal{M}$ is a collection of subsets of X with $\Big\{\{x\} \mid x \in X\Big\} \subset \mathcal{M}$ and $\mathcal{V}\colon \mathcal{M} \longrightarrow \mathcal{P}\Big(\mathcal{P}(X)\Big)$ is a map assigning to each $A \in \mathcal{M}$ a filter $\mathcal{V}(A)$ on X (called the filter of *neighborhoods* of A) subject to the following two axioms:

(G1) if $A \in \mathcal{M}$ and $U \in \mathcal{V}(A)$ then $A \subset U$.

(G2) if $A \in \mathcal{M}$ and $U \in \mathcal{V}(A)$ then there exists $V \in \mathcal{V}(A)$ such that $U \in \mathcal{V}(B)$ whenever $B \in \mathcal{M}$ with $B \subset V$.

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If $\mathbf{X}$ is a supertopological space (a set endowed with a supertopology) then we may denote the supertopology of $\mathbf{X}$ by $(\mathcal{M}_{\mathbf{X}}, \mathcal{V}_{\mathbf{X}})$. If $\mathbf{X}$ and $\mathbf{Y}$ are supertopological spaces then a map $f\colon \mathbf{X} \longrightarrow \mathbf{Y}$ is said to be *continuous* iff $f[\mathcal{M}_{\mathbf{X}}] = \Big\{ f[A] \mid A \in \mathcal{M}_{\mathbf{X}} \Big\} \subset \mathcal{M}_{\mathbf{Y}}$ and for any $A \in \mathcal{M}_{\mathbf{X}}$ and any $V \in \mathcal{V}_{\mathbf{Y}}\Big(f[A]\Big)$ we have $f^{-1}V \in \mathcal{V}_{\mathbf{X}}(A)$. The resulting construct of all supertopological spaces and continuous maps is denoted by **SuperTop**.

For a discussion of **SuperTop** and its relations to the classical contructs **Top** of topological spaces and continuous maps, **Prox** of proximity spaces and δ–maps, and **Unif** of uniform spaces and uniformly continuous maps, we refer to [5]. That paper also contains a study of extensions of topological spaces using supertopologies as a tool (see also [4]).

Our objective here is to demonstrate the following theorem.

Theorem. **SuperTop** *is a topological construct.*

Proof. Let $(\mathbf{X}_i)_{i\in I}$ be a family of supertopological spaces, indexed by an arbitrary class I, X a set, and $(f_i\colon X \longrightarrow \mathbf{X}_i)_{i\in I}$ a family of maps. We will show that the structured source $(X \xrightarrow{f_i} \mathbf{X}_i)_{i\in I}$ has an initial lift $(\mathbf{X} \xrightarrow{f_i} \mathbf{X}_i)_{i\in I}$. To simplify the notation, we write $\mathcal{M}_i = \mathcal{M}_{\mathbf{X}_i}$ and $\mathcal{V}_i = \mathcal{V}_{\mathbf{X}_i}$. We make X into a supertopological space $\mathbf{X}$ by defining

$$\mathcal{M}_{\mathbf{X}} = \{A \subset X \mid f_i[A] \in \mathcal{M}_i \text{ for each } i \in I\}$$

and for $A \in \mathcal{M}_{\mathbf{X}}$ we say that a subset of U of X belongs to $\mathcal{V}_{\mathbf{X}}(A)$ iff there exist a finite subset J of I and a family $(U_i)_{i\in J}$ with $U_i \in \mathcal{V}_i(f_i[A])$ for each $i \in J$ such that

$$\bigcap_{i\in J} f_i^{-1}[U_i] \subset U.$$

First we show that $\mathbf{X} = \Big(X, (\mathcal{M}_{\mathbf{X}}, \mathcal{V}_{\mathbf{X}})\Big)$ is a supertopological space.
Clearly the singletons are in $\mathcal{M}_{\mathbf{X}}$.

Let $U, W \in \mathcal{V}_{\mathbf{X}}(A)$. We must show $U \cap W \in \mathcal{V}_{\mathbf{X}}(A)$. There exist finite subsets $J, L \subset I$ and families $(U_i)_{i\in J}$ and $(W_i)_{i\in L}$ with

$$\begin{array}{lll} U_i \in \mathcal{V}_i(f_i[A]) & \text{for all} & i \in J, \\ W_i \in \mathcal{V}_i(f_i[A]) & \text{for all} & i \in L, \\ \bigcap_{i\in J} f_i^{-1}[U_i] \subset U & \text{and} & \bigcap_{i\in L} f_i^{-1}[W_i] \subset W. \end{array}$$

For $i \in J \cup L$, define

$$S_i = \begin{cases} U_i & \text{if } i \in J\backslash L \\ U_i \cap W_i & \text{if } i \in J \cap L \\ W_i & \text{if } i \in L\backslash J. \end{cases}$$

Then $S_i \in \mathcal{V}_i(f_i[A])$ for each $i \in J \cup L$ and

$$\bigcap_{i \in J \cup L} f_i^{-1}[S_i] \subset U \cap W.$$

Therefore, $\mathcal{V}_{\mathbf{X}}(A)$ is a filter on X. Obviously, axiom (G1) is satisfied for $\mathbf{X}$. Let $A \in \mathcal{M}_{\mathbf{X}}$ and $U \in \mathcal{V}_{\mathbf{X}}(A)$. There is a finite subset $J \subset I$ and a family $(U_i)_{i \in J}$ with

$$U_i \in \mathcal{V}_i(f_i[A]) \text{ for each } i \in I$$

and

$$\bigcap_{i \in J} f_i^{-1}[U_i] \subset U.$$

For each $i \in J$, by axiom (G2) for $\mathbf{X}_i$, there exists $V_i \in \mathcal{V}_i(f_i[A])$ such that $U_i \in \mathcal{V}_i(B)$ whenever $B \in \mathcal{M}_i$ with $B \subset V_i$. Then $f_i[A] \subset V_i \subset U_i$ for each $i \in J$. Let $V = \bigcap_{i \in J} f_i^{-1}[V_i]$.

Then $V \in \mathcal{V}_{\mathbf{X}}(A)$ and clearly $U \in \mathcal{V}_{\mathbf{X}}(B)$ whenever $B \in \mathcal{M}_{\mathbf{X}}$ with $B \subset V$. Therefore $\mathbf{X}$ is indeed a supertopological space and each $f_i : \mathbf{X} \longrightarrow \mathbf{X}_i$ is a continuous map.

We must show that $(f_i : \mathbf{X} \longrightarrow \mathbf{X}_i)_{i \in I}$ is an initial source in **SuperTop**.

Let $\mathbf{Y}$ be a supertopological space and let $g : \mathbf{Y} \longrightarrow \mathbf{X}$ be a map such that for all $i \in I$, $f_i \circ g : \mathbf{Y} \longrightarrow \mathbf{X}_i$ is continuous.

We must show that $g : \mathbf{Y} \longrightarrow \mathbf{X}$ is continuous. That $g[\mathcal{M}_{\mathbf{Y}}] \subset \mathcal{M}_{\mathbf{X}}$ is immediate. Let $A \in \mathcal{M}_{\mathbf{Y}}$ and $U \in \mathcal{V}_{\mathbf{X}}(g[A])$. Then there exist a finite subset $J \subset I$ and a family $(U_i)_{i \in J}$ with

$$U_i \in \mathcal{V}_i\Big(f_i\Big[g[A]\Big]\Big) \text{ for each } i \in I$$

and

$$\bigcap_{i \in J} f_i^{-1}[U_i] \subset U.$$

Since $f_i \circ g$ is continuous, we have $g^{-1}\Big[f_i^{-1}[U_i]\Big] \in \mathcal{V}_{\mathbf{Y}}(A)$ for each $i \in I$. Therefore

$$g^{-1}\Big(\bigcap_{i \in J} f_i^{-1}[U_i]\Big) = \bigcap_{i \in J} g^{-1}\Big[f_i^{-1}[U_i]\Big] \in \mathcal{V}_{\mathbf{Y}}(A).$$

Hence $g^{-1}[U] \in \mathcal{V}_{\mathbf{Y}}(A)$. It follows that $g : \mathbf{Y} \longrightarrow \mathbf{X}$ is continuous. $\square$

Remarks.

(1) The above theorem implies via purely categorical arguments (see, e.g., [1, §21]) that **SuperTop** is an extremely pleasant construct, e.g., it has concrete products, concrete coproducts, subspaces (defined already by Doitchi-

nov), and quotients, the order relation of the set of all supertopologies on a given set (defined by Doitchinov) is that of a complete lattice, etc.

(2) As Doitchinov has pointed out already in [2] the full subconstruct of **SuperTop**, consisting of all supertopological spaces $\mathbf{X}$ with $\mathcal{M}_{\mathbf{X}} = \Big\{\{x\} \mid x \in X\Big\}$, is concretely isomorphic to **Top**. Since this subconstruct is, obviously, bicoreflective in **SuperTop**, the above theorem implies immediately the well known fact that **Top** is a topological construct as well. Moreover, this subconstruct is simultaneously epireflective (but not bireflective) in **SuperTop**. A corresponding reflection of a supertopological space $\mathbf{X}$ is obtained as the final lift of the natural projection $\mathbf{X} \longrightarrow X/\varrho$ where ϱ is the equivalence relation on X generated by the relation $\{M \times M \mid M \in \mathcal{M}_{\mathbf{X}}\}$.

(3) **SuperTop** fails to be cartesian closed. To see this, recall that there exist quotient maps $q_1 : X_1 \longrightarrow Y_1$ and $q_2 : X_2 \longrightarrow Y_2$ in **Top** such that $q_1 \times q_2 : X_1 \times X_2 \longrightarrow Y_1 \times Y_2$ fails to be a quotient map. By the observation (2) above, quotients and products in **Top** are formed as in **SuperTop**. Thus q_1 and q_2 are quotients in **SuperTop** but $q_1 \times q_2$ fails to be so. This implies that **SuperTop** is not cartesian closed.

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