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p–SEQUENTIAL SPACES AND CLEAVABILITY

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Abstract. We consider some relations between p-sequential-like properties and cleavability of topological spaces. Under a special assumption we give an very easy proof of the following result of A.V. Arhangel'skii (the main result in [1]): if a (countably) compact space X is cleavable over the class of sequential spaces, then X is also sequential.

All spaces in this paper are assumed to be Hausdorff. Recall some definitions that we shall use.

Let $\mathcal{F}$ be a filter on ω. A sequence $(x_n : n \in \omega)$ in a space X, *$\mathcal{F}$-converges* to a point x in X if for every neighbourhood U of x, the set $\{n \in \omega : x_n \in U\}$ belongs to $\mathcal{F}$ [2]. We shall consider p-sequential and p-Fréchet-Urysohn spaces for $p \in \omega^* = \beta\omega \setminus \omega$. A space X is said to be *p-sequential* if for every non-closed subset A of X there exist a point $x \in X \setminus A$ and a sequence (x_n) in A which p-converges to x. X is an *$FU(p)$-space* if for every $A \subset X$ and every $x \in \overline{A}$ there

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is a sequence (x_n) in A which p-converges to x (see [10]; different generalizations of these notions were considered in [7, 8, 3, 4, 11]).

If $\mathcal{P}$ is a class of topological spaces and $\mathcal{M}$ is a class of (continuous) mappings, then a space X is said to be $\mathcal{M}$-*cleavable* (resp. $\mathcal{M}$-*pointwise cleavable*) *over* $\mathcal{P}$ if for every $A \subset X$ (resp. every $x \in X$) there exist $Y \in \mathcal{P}$ and $f \in \mathcal{M}$, $f : X \to Y$, such that $f(X) = Y$ and $f^{-1}f(A) = A$ (resp. $f^{-1}f(x) = \{x\}$) (see [1, 9]).

Definition 1. *Let $p \in \omega^*$.*

(a) ([2]) *A space X is said to be* **p-compact** *provided every sequence in X has a p-limit point. If X is p-compact for every $p \in \omega^*$ one says that X is* ultracompact.

(b) ([5]) *A space X is called* **p-closed** *if every p-compact subspace of X is closed.*

It was remarked in [5] that if a space X admits a continuous bijection onto a p-closed space, then X is p-closed. We give the following (simple, but useful in what follows) generalization of this fact.

Proposition 2. *If a space X is cleavable over the class $\mathcal{K}$ of all p-closed spaces, then X is a p-closed space.*

P r o o f. Let A be a p-compact subspace of X. Choose a p-closed space Y and a continuous mapping $f : X \to Y$ such that $f^{-1}f(A) = A$. The set $f(A)$ is p-compact in Y and thus it is closed. Then the set $f^{-1}f(A)$ is closed in X, i.e. X is a p-closed space. □

Proposition 3. *If a p-compact space X is cleavable over the class of p-closed spaces, then X is p-sequential.*

P r o o f. By the previous proposition X is p-closed. But p-compact p-closed spaces are precisely p-sequential spaces [5]. □

Every p-sequential space is p-closed. Therefore, we have this

Corollary 4. *If a p-compact space X is cleavable over the class of p-sequential spaces, then X is p-sequential.*

Theorem 5. *If a compact space X is cleavable over the class $\mathcal{K}$ of ccc p-sequential spaces, then X is weakly $FU(\omega^*)$-space (i.e. X is a $FU(q)$-space for some $q \in \omega^*$).*

Proof. By Corollary 4 X is p-sequential, so its tightness is countable. On the other hand, every $Y \in \mathcal{K}$ has cardinality $\leq 2^\omega$ because Y is a compact p-sequential space and for such spaces Y we have $|Y| \leq 2^{c(Y)}$ [6, Th. 3]. Hence, X is cleavable over a class of spaces having cardinality $\leq 2^\omega$. According to a known result [1, 9] the cardinality of X is $\leq 2^\omega$. Theorem 3.12 in [3] garantues now that there is a $q \in \omega^*$ for which X is a $FU(q)$-space. □

A similar result is the following one.

Theorem 6. *If a separable p-compact space X is cleavable over the class $\mathcal{K}$ of p-closed spaces, then X is a weakly $FU(\omega^*)$-space (and p-sequential).*

Proof. X is a p-compact p-closed space (and so p-sequential). By the formula $|X| \leq d(X)^\omega$ for every p-compact p-closed space X, we conclude $|X| \leq 2^\omega$. Since X is a p-seqeuential space, its tightness is countable. Again by Theorem 3.12 in [3] we have that X is a weakly $FU(\omega^*)$-space. □

Theorem 7. *If a space X is closed pointwise cleavable over the class of $FU(p)$-spaces, then X is also a $FU(p)$-space.*

Proof. Let A be a subset of X and $x \in \overline{A}$. Choose a $FU(p)$-space Y and a closed continuous mapping $f : X \to Y$ such that $f^{-1}f(x) = \{x\}$. There is a sequence $(y_n) \subset f(A)$ which p-converges to $f(x) \in \overline{f(A)} \setminus f(A)$. For every $n \in \omega$ take a point $x_n \in f^{-1}(y_n) \cap A$. Then the sequence $(x_n) \subset A$ p-converges to x. Indeed, let U be any neighbourhood of x. Since f is closed and $\{x\} = f^{-1}f(x)$ there is a neighbourhood V of $f(x)$ such that $f^{-1}(V) \subset U$. Because of $\{n \in \omega : y_n \in V\} \in p$ and $\{n \in \omega : x_n \in U\} \supset \{n \in \omega : y_n \in V\}$ we have that the set $\{n \in \omega : x_n \in U\} \in p$, i.e. (x_n) p-converges to x. □

We need now the following lemma.

Lemma 8. *Every countably compact p-sequential space X is p-compact.*

P r o o f. Let (x_n) be a sequence in X. Since X is countably compact, there exists an accumulation point x of this sequence. The set $A = \{x_n : n \in \omega\} \cup \{x\}$ is not closed as $x \in \overline{A} \setminus A$. Since X is p-sequential, there is a sequence $(a_k) \subset A$, $a_k = x_{n_k}$, which p-converges to a point $y \in \overline{A} \setminus A$. This means that for every neighbourhood U of y the set $\{n_k : a_k = x_{n_k} \in U\}$ belongs to p. Clearly, then because of $\{n \in \omega : x_n \in U\} \supset \{n_k : x_{n_k} \in U\}$ we have $\{n \in \omega : x_n \in U\} \in p$. Therefore, (x_n) p-converges to y and X is p-compact. $\square$

Recall that a space X is called *ω-bounded* if the closure of every countable subset of X is compact.

Theorem 9. *If a countably compact regular space X is closed pointwise cleavable over the class $\mathcal{C}$ of Fréchet-Urysohn spaces, then X is ω-bounded.*

P r o o f. Every $Y \in \mathcal{C}$ is a $FU(p)$-space for every $p \in \omega^*$ [9]. By Theorem 7, X is also a $FU(p)$-space for every $p \in \omega^*$. Therefore, by Lemma 8, X is p-compact for every $p \in \omega^*$, i.e. X is ultracompact. According to a result of Bernstein [1, Thms 3.4 and 3.5] (see also [12]) it follows that X is ω-bounded. $\square$

The following theorem is a special case of Theorem 23 in [2] but with very easy proof. The *Novak number* $n(X)$ of a space X is the smallest cardinality of a family of nowhere dense subsets of X covering X.

Theorem 10 $(n(\omega^*) > c)$**.** *If a ultracompact space X is cleavable over the class $\mathcal{K}$ of sequential spaces, then X is also sequential.*

P r o o f. Every $Y \in \mathcal{K}$ is a p-sequential space for each $p \in \omega^*$, so that every $Y \in \mathcal{K}$ is p-closed for each $p \in \omega^*$. By Proposition 2 X is also p-closed for every $p \in \omega^*$. Therefore, X is a p-closed p-compact space for all $p \in \omega^*$, hence X is p-sequential for all $p \in \omega^*$. By a result of Malykhin (which states that under $n(\omega^*) > c$ a space is sequential if and only if it is p-sequential for every $p \in \omega^*$; see Theorem 1.10 in [5]), this means that X is sequential. $\square$

Every ultracompact space is compact, so that we have

Corollary 11 $(n(\omega^*) > c)$**.** *If a compact space is cleavable over the class of sequential spaces, then X is also sequential.*

We end by a question regarding p-compact spaces. Every ω-bounded space is p-compact for every $p \in \omega^*$. A.V. Arhangel′skii has remarked that if an

ω-bounded space is cleavable over the class of spaces of countable tightness, then it itself has countable tightness [1]. So, the following question is natural.

Question 12. *Let a p-compact space X be cleavable over the class of Hausdorff spaces of countable tightness. Is the tightness of X countable?*

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