Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **24** (1998), 95-98

Serdica
Mathematical Journal

Institute of Mathematics
and Informatics, Bulgarian
Academy of Sciences

A NOTE ON TOTALLY BOUNDED QUASI-UNIFORMITIES

P. Fletcher, W. Hunsaker

Communicated by J. Jayne

Dedicated to the memory of Professor D. Doitchinov

Abstract. We present the original proof, based on the Doitchinov completion, that a totally bounded quiet quasi-uniformity is a uniformity. The proof was obtained about ten years ago, but never published. In the meantime several stronger results have been obtained by more direct arguments [8, 9, 10]. In particular it follows from Künzi's [8] proofs that each totally bounded locally quiet quasi-uniform space is uniform, and recently Déak [10] observed that even each totally bounded Cauchy quasi-uniformity is a uniformity.

1. Introduction. The purpose of this note is to continue the study of D-complete and quiet quasi-uniformities, which were introduced by D. Doitchinov in [2], [3] and [4]. We consider these quasi-uniformities in the class of totally bounded spaces, and our principal results are that every totally bounded D-complete space is compact, and that every totally bounded quiet quasi-uniformity is a uniformity. The first of these results might easily be anticipated because of an analogous result for totally bounded complete quasi-uniform spaces, but the second result

1991 *Mathematics Subject Classification:* 54E15
Key words: D-complete, quite

shows the surprising strength of quiet quasi-uniformities. Throughout this note, all topological spaces are presumed to be T_1 spaces.

2. Preliminaries. We make use of the following definitions and notation, which are due to D. Doitchinov [2]. Let $(X,\mathcal{U})$ be a quasi-uniform space. A filter $\mathcal{G}$ on X is a *D-Cauchy filter* provided that there is a filter $\mathcal{F}$ on X, called a *co-filter* of $\mathcal{G}$, such that for each U in $\mathcal{U}$ there exists an F in $\mathcal{F}$ and a G in $\mathcal{G}$ with $F \times G \subseteq U$. When $\mathcal{F}$ is a co-filter of $\mathcal{G}$, we write $(\mathcal{F},\mathcal{G}) \to 0$; we say that $(X,\mathcal{U})$ is *D-complete* provided every D-Cauchy filter converges. The space $(X,\mathcal{U})$ is *quiet* provided that for each U in $\mathcal{U}$ there is an entourage V in $\mathcal{U}$ such that if $\mathcal{F}$ and $\mathcal{G}$ are filters on X and x and y are points of X such that $V(x) \in \mathcal{G}$ and $V^{-1}(y) \in \mathcal{F}$ and $(\mathcal{F},\mathcal{G}) \to 0$, then $(x,y) \in U$. If V satisfies the above conditions, we say that V is *quiet for* U. A space $(X,\mathcal{U})$ is *uniformly regular* provided that for each $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that for each $x \in X$, $\overline{V(x)} \subseteq U(x)$ [1].

3. Totally bounded spaces.

Proposition 1. *Let $(X,\mathcal{U})$ be a totally bounded D-complete quasi-uniform space. Then $(X,\mathcal{T}(\mathcal{U}))$ is compact.*

Proof. Let $\mathcal{F}$ be an ultrafilter on X and let $U \in \mathcal{U}$. Since $\mathcal{U}$ is totally bounded, there is a finite cover $\{A_i : i = 1,2,\ldots,n\}$ of X such that $A_i \times A_i \subseteq U$ for $i = 1,2,\ldots,n$. There exists k with $1 \le k \le n$ such that $A_k \in \mathcal{F}$. Consequently, $(\mathcal{F},\mathcal{F}) \to 0$ and so $\mathcal{F}$ converges. □

An alternative proof of Proposition 1 may be obtained by observing that every totally bounded quasi-uniformity is Cauchy bounded in the sense of R. Kopperman [7]. The result then follows since every Cauchy-bounded D-complete quasi-uniform space is compact [7, Theorem 6].

Our next proposition obtains an extension of Proposition 1 for the class of regular spaces; although the gap between Propositions 1 and 2 is small, it is significant.

Proposition 2. *Let $(X,\mathcal{U})$ be a D-complete regular quasi-uniform space and suppose that Y is a dense subset of X such that $\mathcal{U}|_{Y\times Y}$ is totally bounded. Then $(X,\mathcal{U})$ is compact.*

Proof. It suffices to show that every open ultrafilter on X converges. Let $\mathcal{F}$ be such a filter. Let $\mathcal{F}|_Y$ be the restriction of $\mathcal{F}$ to Y, let $\mathcal{G}$ be an ultrafilter on Y containing $\mathcal{F}|_Y$ and let $\mathcal{H} = \{H \subseteq X : G \subseteq H \text{ for some } G \in \mathcal{G}\}$. Then $\mathcal{H}$ is a filter on X, and we show that $(\mathcal{H},\mathcal{F}) \to 0$.

Let $V \in \mathcal{U}$ and let $W \in \mathcal{U}$ such that $W^2 \subseteq V$ and $W(x) \in \mathcal{T}(\mathcal{U})$ for each $x \in X$. Then $W \cap (Y \times Y) \in \mathcal{U}|_{Y \times Y}$ and so there is a finite cover $\{A_i : i = 1, 2, \ldots, n\}$ of Y such that for $i = 1, 2, \ldots, n$, $A_i \times A_i \subseteq W \cap (Y \times Y)$. There exists k with $1 \leq k \leq n$ such that $A_k \in \mathcal{G} \subseteq \mathcal{H}$. Since $\mathcal{F}$ is an open ultrafilter, either $W(A_k) \in \mathcal{F}$ or $X - \overline{W(A_k)} \in \mathcal{F}$, and since $A_k \in \mathcal{G}$, $X - \overline{W(A_k)} \notin \mathcal{F}$. Thus $W(A_k) \in \mathcal{F}$ and $A_k \in \mathcal{H}$. Moreover, since $A_k \times A_k \subseteq W \cap (Y \times Y) \subseteq W$, $A_k \times W(A_k) \subseteq W^2 \subseteq V$. Therefore, $(\mathcal{H}, \mathcal{F}) \to 0$ and so $\mathcal{F}$ converges. □

Corollary. *Let $(X,\mathcal{U})$ be a totally bounded quiet quasi-uniform space and let $(\widehat{X},\widehat{\mathcal{U}})$ be its D-completion. Then $\widehat{\mathcal{U}}$ and $\mathcal{U}$ are point symmetric.*

Proof. Since $(\widehat{X},\widehat{\mathcal{U}})$ is compact, the corollary follows from [6, Propositions 2.24 and 2.26]. □

Proposition 3. *Every totally bounded quiet quasi-uniformity is a uniformity.*

Proof. Let $(X,\mathcal{U})$ be a totally bounded quiet quasi-uniform space and let $(\widehat{X},\widehat{\mathcal{U}})$ be its D-completion. Clearly $(X,\mathcal{U}^{-1})$ is totally bounded and by [2, Theorem 5] $(X,\mathcal{U}^{-1})$ is quiet. Thus, by the previous corollary both $\widehat{\mathcal{U}}$ and $\widehat{\mathcal{U}}^{-1}$ are point symmetric. By arguments given by D. Doitchinov in [2], we may assume that $\widehat{\mathcal{U}}$ and $\widehat{\mathcal{U}}^{-1}$ are quasi-uniformities on the same set X and so by [6, Proposition 2.21 (d)] $\mathcal{T}(\widehat{\mathcal{U}}) = \mathcal{T}(\widehat{\mathcal{U}}^{-1}) = \mathcal{T}(\widehat{\mathcal{U}} \vee \widehat{\mathcal{U}}^{-1})$. Since $(X, \mathcal{T}(\mathcal{U}))$ is a compact Hausdorff space, it follows from [6, Theorem 1.20] that $\widehat{\mathcal{U}}$, and hence $\mathcal{U}$, is a uniformity. □

Corollary. *Every totally bounded uniformly regular D-complete Hausdorff quasi-uniformity is a uniformity.*

Proof. Let $(X,\mathcal{U})$ be a totally bounded uniformly regular D-complete quasi-uniform space. In light of Proposition 3, it suffices to show that $\mathcal{U}$ is quiet. By Proposition 1, $\mathcal{T}(\mathcal{U})$ is compact and so $\mathcal{U}$ is point symmetric [6, Propositions 2.24 and 2.26]. But it is known that every point-symmetric, D-complete, uniformly regular quasi-uniform space is quiet [5, Theorem 2.1].

REFERENCES

[1] Á. Császár. Extensions of quasi-uniformities. *Acta. Math. Hungar.* **37** (1981), 121-145.

[2] D. Doitchinov. On completeness of quasi-uniform spaces. *C. R. Acad. Bulgare Sci.* **41**, *7* (1988), 5-8.

[3] D. Doitchinov. On completeness of quasi-metric spaces. *Topology Appl.* **30** (1988), 127-148.

[4] D. Doitchinov. A concept of completeness of quasi-uniform spaces. *Topology Appl.* **38** (1991), 205-217.

[5] P. Fletcher, W. Hunsaker. Uniformly regular quasi-uniformities. *Topology Appl.* **37** (1990), 285-291.

[6] P. Fletcher, W. F. Lindgren. Quasi-Uniform Spaces. Lecture Notes Pure Appl. Math. vol. **77**, Dekker, New York and Basel, 1982.

[7] R. Kopperman. Total boundedness and compactness for filter pairs. *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.* **33** (1990), 25-30.

[8] H. P. A. Künzi. Totally bounded quiet quasi-uniformities. *Topology Proc.* **15** (1990), 113-115.

[9] H. P. A. Künzi, M. Mršević., I. L. Reilly, M. K. Vamanamurthy. Convergence, precompactness and symmetry in quasi-uniform spaces. *Math. Japonica* **38** (1993), 239-253.

[10] J. Deák. Short notes on quasi-uniform spaces. IV: Cauchy type properties. *Acta Math. Hungar.* **70** (4) (1996), 317-327.

P. Fletcher
Mathematics Department
Virginia Tech
Blacksburg, VA 24061-0123
USA

W. Hunsaker
Mathematics Department
Southern Illinois University
Carbondale, IL 62901-4408
USA

Received February 10, 1997