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THE POINT OF CONTINUITY PROPERTY: DESCRIPTIVE COMPLEXITY AND ORDINAL INDEX

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Abstract. Let X be a separable Banach space without the Point of Continuity Property. When the set of closed subsets of its closed unit ball is equipped with the standard Effros-Borel structure, the set of those which have the Point of Continuity Property is non-Borel. We also prove that, for any separable Banach space X, the oscillation rank of the identity on X (an ordinal index which quantifies the Point of Continuity Property) is determined by the subspaces of X with a finite-dimensional decomposition. If X does not contain l_1, subspaces with basis suffice. If X^* is separable, one can even restrict to subspaces with shrinking basis.

1. Notations and preliminaries. In this paper, after a first section devoted to notations and preliminaries, we study in the second section the descriptive complexity of the Point of Continuity Property (PCP). We refer the reader to [7] and [14] for the foundations of the descriptive set theory. In a separable Banach space X without PCP, we equip the set $\mathcal{F}(X)$ of closed subsets

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of its closed unit ball B_X, with the standard Effros-Borel structure. Then the subset of $\mathcal{F}(X)$ consisting of elements which have PCP, is non-Borel. For related results, see [1].

In the third section, we recall the definition of the oscillation rank of the identity from (B_X, w) to $(B_X, \| \, \|)$, which is the ordinal index obtained by slicing B_X with weak open subsets (about slicing, see [4]). A separable Banach space X has PCP if, and only if, this rank is a countable ordinal. We show that this rank is the supremum of the ranks of its subspaces with a finite-dimensional decomposition, or of its subspaces with a basis, if X contains no isomorph of l_1, or of its subspaces with a shrinking basis, if X^* is separable. That specifies some results proved by J. Bourgain [2], N. Ghoussoub and B. Maurey [6] and, the second author [10]. (See also [8] and [9] for related results)

We begin by recalling some notations. In the sequel, we will denote by $\omega = \{0, 1, 2, \ldots\}$ the first infinite ordinal, by ω^* the set $\omega \setminus \{0\}$, by ω_1 the first uncountable ordinal and by $\omega^{<\omega}$ the set of all finite sequences in ω, including the empty sequence. If $s \in \omega^{<\omega}$, $|s|$ will be its length, and we write

$$s = (s(0), s(1), s(2), \ldots, s(|s| - 1)).$$

Concatenation is denoted $\frown$, and if $p \in \omega$, we will write $(p) \frown s = p \frown s$ and $s \frown (p) = s \frown p$. When $s \neq \emptyset$, $s-$ is the sequence such that $s = s- \frown p$ for some $p \in \omega$. We have a partial order in $\omega^{<\omega}$ defined by

$$s \leq t \quad \text{if} \quad t = s \quad \text{or} \quad t = s \frown u \quad \text{for some} \quad u \in \omega^{<\omega}.$$

A tree on ω is a subset $\theta \subset \omega^{<\omega}$ such that $t \in \theta$ whenever $s \in \theta$ and $t \leq s$. A branch b of θ is an infinite sequence of elements in ω such that $s \in \theta$ whenever $s \in \omega^{<\omega}$ and $s(i) = b(i)$, $\forall 0 \leq i \leq |s| - 1$.

Let $\mathcal{T}$ be the set of trees on ω. We recall that a tree $\theta \in \mathcal{T}$ is said well founded if it has no branch. We denote by WF the set of well founded trees.

We denote by $\mathrm{ht}(\theta)$ the height of $\theta \in \mathcal{T}$, and we refer to [7] for the definition and for a complete information about the following notions and properties.

Let P a Polish space. A subset A of P is analytic if it is the Borel image of a Borel subset of a Polish space, A is coanalytic (a $\underset{\sim}{\prod}{}_1^1$ set) if $P \setminus A$ is analytic, A is $\underset{\sim}{\prod}{}_1^1$$-$hard if for every Polish space Y and every $\underset{\sim}{\prod}{}_1^1$ set Q in Y, there is a

Borel map $f : Y \to P$ with $Q = f^{-1}(A)$, and A is $\underset{\sim}{\prod}{}^1_1$−complete if A is $\underset{\sim}{\prod}{}^1_1$ and $\underset{\sim}{\prod}{}^1_1$−hard.

It is known that a $\underset{\sim}{\prod}{}^1_1$−hard subset is not analytic, thus non-Borel, and WF is $\underset{\sim}{\prod}{}^1_1$−complete in the Polish space $\mathcal{T}$ equipped with the topology inherited from $2^{\omega^{<\omega}}$.

Let X be a Banach space. If C is a nonempty subset of X, C is said to have the Point of Continuity Property (PCP), if on every closed and bounded subset of C the identity map has some point of weak to norm continuity. So, X has PCP if, and only if, its closed unit ball, B_X, has PCP. We denote by $PCP(X)$ the family of all closed subsets of B_X with PCP. This property is, for example, a very important property to study the relation between Radon-Nikodym and Krein-Milman properties. For a complete information about PCP we refer to [2], [5], and [12].

The closed ball with center $x \in X$ and radius $r > 0$ is denoted by $B(x, r)$. When X is separable, $\mathcal{F}(X)$ is the set of closed subsets of B_X equipped with the Effros-Borel structure (see [3]), thus $\mathcal{F}(X)$ is a standard Borel space (i.e. whose Borel structure is generated by a Polish topology). The Effros-Borel structure is generated by the family

$$\{\{F \in \mathcal{F}(X) : F \cap O \neq \emptyset\} : O \text{ is an open subset of } B_X\}.$$

If $A \subset X$, $\operatorname{span}(A)$ denotes the vector space spanned by A, $\overline{\operatorname{span}}(A)$ its closure, and $\operatorname{diam}(A) = \sup\{\|x - y\| : x, y \in A\}$. Given $x \in X$, $\mathcal{E} \subset X^*$ and $\varepsilon > 0$, we write

$$\Gamma(x, \mathcal{E}, \varepsilon) = \{y \in X : |x^*(x - y)| < \varepsilon \ \forall x^* \in \mathcal{E}\}.$$

We refer to [11] for the definition of a finite-dimensional Schauder decomposition (FDD). A basic sequence $\{x_i\}_{i\in\omega}$ in X is shrinking if

$$(\overline{\operatorname{span}}\{x_i : i \in \omega\})^* = \overline{\operatorname{span}}\{x_i^* : i \in \omega\},$$

where $\{x_i^*\}_{i\in\omega}$ is the sequence of biorthogonal functionals of $\{x_i\}_{i\in\omega}$.

2. Descriptive complexity of PCP. In this section we show a new application of the descriptive theory in Banach spaces. For this, we use the theory of analytic sets, whose foundations are exposed in [7] and [14], to study the descriptive complexity of the point of continuity property in Banach spaces.

Our main goal in this section is the following:

Theorem 1. *Let X be a separable Banach space failing PCP. Then PCP(X) is* $\underset{\sim}{\prod}{}^1_1$*−hard, thus non-Borel.*

To prove this theorem we need the following lemmas and preliminaries:

Lemma 2 [2, Lemme 5.5]. *Let X be a Banach space, $A \subset X$, $x \in \overline{A}^w$. Then for every $d \in \omega$ and for every $\varepsilon > 0$ there is $C \in \mathcal{P}_F(A)$ such that:*

$$\forall \mathcal{E} \in \mathcal{P}_d,\ \Gamma(x, \mathcal{E}, \varepsilon) \cap C \neq \emptyset,$$

where $\mathcal{P}_d$ denotes the set of finite subsets of B_{X^*} with d elements, at most, and $\mathcal{P}_F(A)$ is the set of finite subsets of A.

Now, we define Λ, a family of subsets in $\omega^{<\omega} \times \omega^{<\omega}$ by $M \in \Lambda$ if:

1. $(\emptyset, \emptyset) \in M$.
2. If $(s,t) \in M$, then $|s| = |t|$ and $(s', t') \in M$ for any $s', t' \in \omega^{<\omega}$ such that $|s'| = |t'|$ and $s' < s, t' < t$.
3. $\forall (s,t) \in M$, $\forall m \in \omega$ $\exists p \in \omega^*$ such that $(s \frown m, t \frown i) \in M$ if, and only if, $i < p$.

It is clear that, if $M \in \Lambda$, then for every $s \in \omega^{<\omega}$ there is $t \in \omega^{<\omega}$ such that $(s,t) \in M$.

For every $n \in \omega$, we define Λ_n, a new family of subsets of $\omega^{<\omega} \times \omega^{<\omega}$ by: $M \in \Lambda_n$ if

1. $(\emptyset, \emptyset) \in M$.
2. If $(s,t) \in M$, then $|s| = |t|$ and $(s', t') \in M$ for any $s', t' \in \omega^{<\omega}$ such that $|s'| = |t'|$ and $s' < s, t' < t$.

3. $\forall (s,t) \in M'$, $\forall m \in \omega\ \exists p \in \omega :$

$$(s \frown m, t \frown i) \in M \text{ if, and only if, } i < p, \text{ where}$$

$$M' = \{(s,t) \in M : \exists (s',t') \in M \text{ with } s < s',\ t < t'\}.$$

4. $\forall s \in \omega^{<\omega}$ such that $|s| \leq n$, $\exists t \in \omega^{<\omega} : (s,t) \in M$.

With these preliminaries we can prove the following.

Lemma 3. *Let X be a Banach space failing PCP. Then there are $M \in \Lambda$, $(x_{(s,t)})_{(s,t)\in M} \subset B_X$, $(V_{(s,t)})_{(s,t)\in M}$ weak open subsets, and $\varepsilon > 0$ such that for every $(s,t), (s',t') \in M$ we have:*

i) $\forall m \in \omega$, $\forall \mathcal{E} \in \mathcal{P}_{m+1}$, $\Gamma(x_{(s,t)}, \mathcal{E}, \frac{1}{m+1}) \cap F_{(s,t)}(m) \neq \emptyset$, *where*

$$F_{(s,t)}(m) = \{x_{(s\frown m, t\frown i)} : (s \frown m, t \frown i) \in M\}.$$

ii) $\forall m \in \omega$, $F_{(s,t)}(m) \cap B(x_{(s,t)}, \varepsilon) = \emptyset$.

iii) $x_{(s,t)} \in V_{(s,t)}$, $V_{(s,t)} \cap B(x_{(s-,t-)}, \varepsilon) = \emptyset$ *and,* $s' < s, t' < t$ *implies* $V_{(s,t)} \subset V_{(s',t')}$. *Furthermore, if s and s' or t and t' are incomparable, then* $V_{(s,t)} \cap V_{(s',t')} = \emptyset$.

P r o o f. Let $A \subset B_X$ and $\varepsilon > 0$ be such that every weak open subset of A has diameter, at least, 2ε.

By induction in $n \in \omega$, we construct $M_n \in \Lambda_n$, $(x_{(s,t)})_{(s,t)\in M_n} \subset A$ and $(V_{(s,t)})_{(s,t)\in M_n}$ weak open subsets, such that i) and ii) are satisfied for any $(s,t) \in M'_n$, and iii) is satisfied for any $(s,t), (s',t') \in M_n$.

We put $M_0 = \{(\emptyset, \emptyset)\}$, $V_{(\emptyset,\emptyset)} = X$ and $x_{(\emptyset,\emptyset)}$ any element of A.

Let $n > 0$ and suppose the construction is made for $0 \leq i \leq n-1$.

Let $(s,t) \in M_{n-1} :\ |s| = |t| = n-1$. Then

$$x_{(s,t)} \in \overline{V_{(s,t)} \setminus B(x_{(s,t)}, \varepsilon)}^{w},$$

by hypotheses on A. From Lemma 2, there is $C(0) = \{y_0^0, y_1^0, \ldots, y_{n_0}^0\}$, $C(0) \subset V_{(s,t)} \setminus B(x_{(s,t)}, \varepsilon)$ such that

$$\forall \mathcal{E} \in \mathcal{P}_1,\ \Gamma(x_{(s,t)}, \mathcal{E}, 1) \cap C(0) \neq \emptyset.$$

Then there are $V_0^0, V_1^0, \ldots, V_{n_0}^0$ weak neighbourhoods of $y_0^0, y_1^0, \ldots, y_{n_0}^0$ respectively, and W_0 weak neighbourhood of $x_{(s,t)}$, all contained in $V_{(s,t)}$, pairwise disjoint such that $V_j^0 \cap B(x_{(s,t)}, \varepsilon) = \emptyset,\ 0 \le j \le n_0$.

Now, $x_{(s,t)} \in \overline{W_0 \setminus B(x_{(s,t)}, \varepsilon)}^w$ and a new application of Lemma 2 says us that there is $C(1) = \{y_0^1, y_1^1, \ldots, y_{n_1}^1\} \subset W_0 \setminus B(x_{(s,t)}, \varepsilon)$ such that

$$\forall \mathcal{E} \in \mathcal{P}_2,\ \Gamma\left(x_{(s,t)}, \mathcal{E}, \frac{1}{2}\right) \cap C(1) \neq \emptyset.$$

By repeating this construction we obtain for every $m \in \omega$, W_m weak neighbourhood of $x_{(s,t)}$ with $W_m \subset W_{m-1} \subset V_{(s,t)}$, $C(m) = \{y_0^m, y_1^m, \ldots, y_{n_m}^m\} \subset W_{m-1} \setminus B(x_{(s,t)}, \varepsilon)$ and $V_0^m, V_1^m, \ldots, V_{n_m}^m$ weak neighbourhoods of $y_0^m, y_1^m, \ldots, y_{n_m}^m$, respectively, all contained in W_{m-1}, pairwise disjoint with

$$V_j^m \cap B(x_{(s,t)}, \varepsilon) = \emptyset,\ 0 \le j \le n_m.$$

We put $x_{(s\frown m, t\frown j)} = y_j^m,\ V_{(s\frown m, t\frown j)} = V_j^m\ \forall m \in \omega, 0 \le j \le n_m$ and

$$N_{(s,t)} = \{(s \frown m, t \frown j) : m \in \omega,\ 0 \le j \le n_m\}.$$

Doing $M_n = \cup\{N_{(s,t)} : (s,t) \in M_{n-1},\ |s| = |t| = n-1\} \cup M_{n-1}$, we finish the induction on $n \in \omega$.

Finally, it is easy to see that $M = \cup_{n\in\omega} M_n$ verifies the conditions i), ii) and iii) of Lemma, and the proof is complete. $\square$

Proof of Theorem 1. Let M, $(x_{(s,t)})_{(s,t)\in M}$ and $(V_{(s,t)})_{(s,t)\in M}$ be as in Lemma 3.

We fix $\Delta_0, \Delta_1, \ldots, \Delta_n, \ldots$ infinite, pairwise disjoint subsets of ω and let us define $\Psi : \mathcal{T} \to \mathcal{T}$ by

$$\Psi(\theta) = \{t \in \omega^{<\omega} : \exists s \in \theta, |t| = |s| \text{ and } t(i) \in \Delta_{s(i)},\ \forall i < |s|\}.$$

It is easy to see that Ψ is Borel.

Now, denoting

$$M_\theta = \{(s,t) \in M : s \in \Psi(\theta)\},\ \Sigma_\theta = \{x_{(s,t)} : (s,t) \in M_\theta\}.$$

Σ_θ is closed by iii) of Lemma 3.

Let us define

$$f : \mathcal{T} \longrightarrow \mathcal{F}(X);\ f(\theta) = \Sigma_\theta.$$

Let us see that f is Borel. If B is an open subset of B_X, then

$$\{\theta \in \mathcal{T} : \exists (s,t) \in M_\theta \text{ with } x_{(s,t)} \in B\} =$$

$$= \{\theta \in \mathcal{T} : \exists (s,t) \in M \text{ with } s \in \Psi(\theta),\ x_{(s,t)} \in B\} =$$

$$= \bigcup_{\{(s,t)\in M : x_{(s,t)} \in B\}} \{\theta \in \mathcal{T} : s \in \Psi(\theta)\}.$$

So, f is Borel.

Finally, it is sufficient to see that $\Sigma_\theta \in PCP(X)$ if, and only if, $\theta \in WF$, since WF is $\underset{\sim}{\prod}{}^1_1$−complete in $\mathcal{T}$.

If $\theta \in$ WF, $\Psi(\theta) \in$ WF. Let C be a closed subset of Σ_θ. Then, there is $(s,t) \in M$ such that $x_{(s,t)} \in C$ and $x_{(s',t')} \notin C$, for every $(s',t') \in M$ with $s' > s$.

So, by Lemma 3, $V_{(s,t)} \cap C = \{x_{(s,t)}\}$ and $\operatorname{diam}(V_{(s,t)} \cap C) = 0$. Then $\Sigma_\theta \in PCP(X)$.

Conversely, let $\theta \in \mathcal{T} \setminus WF$. Then, there is σ a branch of θ. Let us define

$$C = \{x_{(s,t)} : s(i) \in \Delta_{\sigma(i)}\ \forall i < |s|\}.$$

So, C is a closed subset of Σ_θ.

Let $x_{(s,t)} \in C$ and V be a weak neighbourhood of $x_{(s,t)}$. For some $p, d \in \omega$, $\mathcal{E} \in \mathcal{P}_d$, we have $\Gamma(x_{(s,t)}, \mathcal{E}, \frac{1}{m+1}) \subset V$ for any $m \geq p$.

If $m \in \Delta_{\sigma(|s|)}$, then $F_{(s,t)}(m) \subset C$, thus $\Gamma(x_{(s,t)}, \mathcal{E}, \frac{1}{m+1}) \cap F_{(s,t)}(m)$ is nonempty (Lemma 3, i), $V \cap C \neq \emptyset$ and $\operatorname{diam}(V \cap C) > \varepsilon$ (Lemma 3, ii). Then $\Sigma_\theta \notin PCP(X)$ and the proof is complete. □

Corollary 4. *If X is a Banach space failing PCP and with a separable dual, then PCP(X) is* $\underset{\sim}{\prod}{}^1_1$*−complete.*

P r o o f. It is sufficient to prove that $PCP(X)$ is $\underset{\sim}{\prod}{}^1_1$.

Let $\mathcal{W}$, O be countable bases of open subsets in B_X for the weak and norm topology, respectively. Given $F \in \mathcal{F}(X)$, it is known that $F \notin PCP(X)$ if, and only if, there are $D \in \mathcal{F}(X)$, $D \subset F$, and $\varepsilon > 0$ such that $\operatorname{diam}(W \cap D) > \varepsilon$, for all $W \in \mathcal{W}$ with $W \cap D \neq \emptyset$. So, $F \notin PCP(X)$ if, and only if, there are $\varepsilon > 0$ and $D \in \mathcal{F}(X)$ such that:

1. $D \subset F$
2. For every $W \in \mathcal{W}$ there are $O_1, O_2 \in O$, such that $O_1, O_2 \subset W$, $O_1 \cap D \neq \emptyset$, $O_2 \cap D \neq \emptyset$ and $\|x_2 - x_1\| > \varepsilon$ for any $x_1 \in O_1,\ x_2 \in O_2$.

Then $\mathcal{F}(X) \setminus PCP(X)$ is analytic, since it is a projection of the following Borel set:

$$\bigcup_{\varepsilon \in Q^+} \bigcap_{W \in \mathcal{W}} \bigcup_{(O_1,O_2) \in I} \{(F, D) \in \mathcal{F}(X)^2 : D \subset F,\ O_1 \cap D \neq \emptyset,\ O_2 \cap D \neq \emptyset\}$$

where I is the following subset of $O \times O$:

$$I = \{(O_1, O_2) : O_1, O_2 \subset W \text{ and } \|x_2 - x_1\| > \varepsilon \ \forall x_1 \in O_1, x_2 \in O_2\}$$

So, PCP(X) is coanalytic. $\square$

We don't know if Theorem 1 is true, for the convex point of continuity property (CPCP), the analogous property to PCP, by changing closed and bounded subsets by closed, bounded and convex subsets. By the other hand, it would be interesting to know if there is some separable Banach space X such that PCP(X) is not coanalytic.

3. Ordinal index and PCP. Let X be a separable Banach space. We denote by $\sigma(X)$ the oscillation rank of identity from (B_X, w) into $(B_X, \|\ \|)$ (the ordinal index obtained by slicing B_X with weak open subsets). We recall the definition of this rank. Let C be a weak closed and bounded subset of X and $\varepsilon > 0$. We denote

$$C'_\varepsilon = \{x \in C : \operatorname{diam}(V \cap C) > \varepsilon \ \forall\ V \text{ weak neighbourhood of } x\}.$$

By a transfinite induction, we define a decreasing transfinite sequence $(C_\varepsilon^\alpha)_{\alpha<\omega_1}$ by

$$C_\varepsilon^0 = C \quad\text{and}\quad C_\varepsilon^{\alpha+1} = (C_\varepsilon^\alpha)'_\varepsilon,$$

and, if α is a limit ordinal

$$C_\varepsilon^\alpha = \bigcap_{\beta<\alpha} C_\varepsilon^\beta.$$

Then

$$\sigma_\varepsilon(X) = \begin{cases} \inf\{\alpha : (B_X)_\varepsilon^\alpha = \emptyset\} & \text{if it exists,} \\ \omega_1 & \text{if not} \end{cases}$$

and $\sigma(X) = \sup\{\sigma_\varepsilon(X) : \varepsilon > 0\}$.

It is a classical result that $\sigma(X) < \omega_1$ if X has PCP, and $\sigma(X) = \omega_1$ if not. Furthermore, if X and Y are isomorphic Banach spaces, then $\sigma(X) = \sigma(Y)$. Here, we prove the following

Theorem 5. *Let X be a separable Banach space. Then*

i) $\sigma(X) = \sup\{\sigma(Y) : Y \subset X \text{ subspace with a FDD}\}$

ii) *If X contains no isomorph of l_1,*

$$\sigma(X) = \sup\{\sigma(Y) : Y \subset X \text{ subspace with a basis}\}$$

iii) *If X^* is separable,*

$$\sigma(X) = \sup\{\sigma(Y) : Y \subset X \text{ subspace with a shrinking basis}\}.$$

If X fails PCP, the theorem is known. Indeed, in this case, J. Bourgain proved in [2] that, X contains a subspace with a FDD and without PCP, N. Ghoussoub and B. Maurey proved in [6] that, if X contains no isomorph of l_1, then X contains a subspace with a basis and without PCP, and the second author proved in [10] that, if X is an Asplund space, then X contains a subspace with a shrinking basis and without PCP. To prove the Theorem 5, we will use the arguments of these authors to build for any $\alpha < \sigma(X)$, with the help of a tree on ω whose height is α, a subspace Y with a FDD or a basis such that $\sigma(Y) \geq \alpha$, following the way used by G. Lancien in [8].

First, we recall some results of [8] and [9]. The following family $(T_\alpha)_{\alpha<\omega_1} \subset \mathcal{T}$ is constructed inductively:

$$T_0 = \{\emptyset\}$$

$$T_{\alpha+1} = \{\emptyset\} \cup \bigcap_{n\in\omega} n \frown T_\alpha \text{ where } n \frown T_\alpha = \{n \frown s : s \in T_\alpha\}$$

$$T_\alpha = \{\emptyset\} \cup \bigcap_{n\in\omega} n \frown T_{\alpha_n} \text{ if } \alpha \text{ is a limit ordinal,}$$

where $(\alpha_n)_{n\in\omega}$ is an enumeration of $\{\beta : \beta < \alpha\}$.

It is not difficult to see that $ht(T_\alpha) = \alpha$, and for any $s \in T_\alpha$, $T_\alpha(s) = T_{h_\alpha(s)}$, where

$$T_\alpha(s) = \{t \in \omega^{<\omega} : s \frown t \in T_\alpha\} \text{ and } h_\alpha(s) = ht(T_\alpha(s)).$$

We denote

$$T'_\alpha = \{s \in T_\alpha : \exists n \in \omega \text{ with } s \frown n \in T_\alpha\}.$$

Lemma 6 ([8, Lemma 3.3], [9, Lemme 2.6])**.** *If $1 \le \alpha < \omega_1$, there is a bijection $\phi_\alpha : \omega \longrightarrow T_\alpha$ which satisfies:*

i) *$\forall s, s' \in T_\alpha$ with $s < s'$ one has that $\phi_\alpha^{-1}(s) < \phi_\alpha^{-1}(s')$.*

ii) *$\forall s \in T'_\alpha$ and $\forall n, p \in \omega$ with $n < p$ one has that $\phi_\alpha^{-1}(s \frown n) < \phi_\alpha^{-1}(s \frown p)$*

The following result is proved in the same way as Proposition 2 in [13] or Proposition 5.4 in [8].

Lemma 7. *Let X be a separable Banach space with PCP. Then, there is $\alpha < \omega_1$ such that $\sigma(X) = \omega^\alpha$.*

We will use the following lemma.

Lemma 8. *Let Y be a separable Banach space, and $\alpha < \omega_1$. Assume that there is $\varepsilon > 0$ and a family $(F_s)_{s\in T_\alpha}$ of nonempty subsets of B_Y, such that for any $s \in T'_\alpha, x \in F_s, n \in \omega$ and for any $\mathcal{E} \subset B_{Y^*}$ with, at most, $n+1$ elements one has*

$$\Gamma\left(x, \mathcal{E}, \frac{1}{n+1}\right) \cap (F_{s\frown n} \setminus B(x, \varepsilon)) \ne \emptyset.$$

Then $F_\emptyset \subset (B_Y)^\alpha_{\frac{\varepsilon}{2}}$, thus $\sigma(Y) \ge \alpha$.

Proof. If $\alpha = 0$, it is clear. We make a transfinite induction. Let $0 < \alpha < \omega_1$, and suppose that there is $\varepsilon > 0$ and $(F_s)_{s \in T_\alpha}$ as in the hypotheses of Lemma. Assume the Lemma true for any ordinal less than α.

If $\alpha = \beta + 1$, then for any $n \in \omega$, $F_{(n)} \subset (B_Y)^{\beta}_{\frac{\varepsilon}{2}}$ since $\beta < \alpha$. Let $x \in F_\emptyset$ and V a weak neighborhood of x. There exists $m \in \omega$ and $\mathcal{E} \subset B_{Y^*}$ with, at most, $m+1$ elements, such that $\Gamma(x, \mathcal{E}, \frac{1}{m+1}) \subset V$. Then

$$\Gamma\left(x, \mathcal{E}, \frac{1}{m+1}\right) \cap (F_{(m)} \setminus B(x, \varepsilon)) \neq \emptyset,$$

$V \cap (B_Y)^{\beta}_{\frac{\varepsilon}{2}} \neq \emptyset$ and $\operatorname{diam}(V \cap (B_Y)^{\beta}_{\frac{\varepsilon}{2}}) > \dfrac{\varepsilon}{2}$.

Consequently, $x \in (B_Y)^{\beta+1}_{\frac{\varepsilon}{2}}$ and $F_\emptyset \subset (B_Y)^{\alpha}_{\frac{\varepsilon}{2}}$.

If α is a limit ordinal, then $F_{(n)} \subset (B_Y)^{\alpha_n}_{\frac{\varepsilon}{2}}$ for any $n \in \omega$, where $(\alpha_n)_{n \in \omega}$ is the enumeration of the ordinals less than α used to build T_α.

Let $x \in F_\emptyset, \beta < \alpha$ and let V be a weak neighborhood of x. There exists $m \in \omega$ and $\mathcal{E} \subset B_{Y^*}$, with, at most, $m+1$ elements, such that $\Gamma(x, \mathcal{E}, \frac{1}{m+1}) \subset V$.

Let $l \geq m$ such that $\alpha_l \geq \beta$. Then

$$\Gamma\left(x, \mathcal{E}, \frac{1}{l+1}\right) \cap F_{(l)} \neq \emptyset, \text{ thus } V \cap (B_Y)^{\alpha_l}_{\frac{\varepsilon}{2}} \neq \emptyset,$$

and $V \cap (B_Y)^{\beta}_{\frac{\varepsilon}{2}} \neq \emptyset$. Consequently $x \in (B_Y)^{\beta}_{\frac{\varepsilon}{2}}$ for any $\beta < \alpha$, $x \in (B_Y)^{\alpha}_{\frac{\varepsilon}{2}}$ and $F_\emptyset \subset (B_Y)^{\alpha}_{\frac{\varepsilon}{2}}$. $\square$

The proof of Theorem 5, i) is a slight modification of J. Bourgain's proof about the existence of a subspace with a FDD and without PCP, when X fails PCP ([2]). It follows from the next Lemma.

We fix $M > 0$ and $(\delta_p)_{p \in \omega} \subset \mathbb{R}$ such that $\delta_0 = 0$, $\delta_p > 0$, $\tau_p = \sum\limits_{q=1}^{p} \delta_q < 1$, $\prod\limits_{q=1}^{p} (1 + \delta_q) < M$, for any $p \in \omega^*$ and we denote $r_p = 1 + \tau_p$.

Lemma 9. *Let X be a separable Banach space and $\alpha < \sigma(X)$. Then there exists $\varepsilon > 0$ such that $(B_X)^{\alpha+1}_{\varepsilon} \neq \emptyset$ and there exists a sequence $(F_{\phi_\alpha(p)})_{p \in \omega}$ of finite subsets of X such that, with $A_{\phi_{\alpha(p)}} = \bigcap\limits_{n \leq p} F_{\phi_{\alpha(n)}}$ and $X_p = \operatorname{span}(A_{\phi_{\alpha(p)}})$, we have*

i) $F_{\phi_{\alpha(p)}} \subset (r_p B_X)_\varepsilon^{h_\alpha(\phi_\alpha(p))+1}$.

ii) *For any $n \in \omega, s \in T_\alpha^{'}, x \in F_s$ and $\mathcal{E} \subset B_{X^*}$, with, at most $n+1$ elements, we have*
$$\Gamma\left(x, \mathcal{E}, \frac{1}{n+1}\right) \cap \left(F_{s \frown n} \setminus B\left(x, \frac{\varepsilon}{4}\right)\right) \neq \emptyset.$$

iii) *For any $p \in \omega$, there exists a subspace Y_p of X such that $X_{p+1} = X_p \oplus Y_p$ and the associated projection from X_{p+1} onto X_p has norm less than $1+\delta_p$.*

Proof. We denote $B = B_X$, $h = h_\alpha$ and $\phi = \phi_\alpha$. by Lemma 7, $\alpha + 1 < \sigma(X)$ thus there exists $\varepsilon > 0$ and $x_0^0 \in B_\varepsilon^{\alpha+1}$. We choose $F_{\phi(0)} = \{x_0^0\}$, and we make an induction. Let $p \in \omega^*$ and suppose $F_{\phi(i)}$ built for any $i \leq p$. As X_p is a finite dimensional subspace, there is some finite subset $\mathcal{G} \subset B_{X^*}$ such that for every $u \in X_p$
$$\|u\| \leq (1+\delta_p) \sup\{x^*(u) : x^* \in \mathcal{G}\}.$$
We define $s \in T_\alpha$, $i_0 \in \omega$ and $n \in \omega$ by $\phi(p+1) = s \frown n$ and $\phi(i_0) = s$. Using the Lemma 5.6 of [2] with $2\gamma = \min\{\frac{\varepsilon}{2}, \frac{1}{n+1}, \delta_{p+1}\}$, there exists $\tau > 0$ such that: if $x \in X$ and $|x^*(x)| < \tau$ for any $x^* \in \mathcal{G}$, then there is some $z \in X$ with
$$\|z\| < \gamma \ \text{ and } \ x^*(z) = x^*(x) \ \forall x^* \in \mathcal{G}.$$
Let $x \in F_s$ and $V_x = \Gamma(x, \mathcal{G}, \tau)$.

By hypothesis, $x \in (r_{i_0} B)_\varepsilon^{h(s)+1}$, thus
$$x \in \overline{(r_{i_0} B)_\varepsilon^{h(s)} \setminus B(x, \frac{\varepsilon}{2})}^{w}.$$

Using Lemma 2, there is a finite subset C_x of $(r_{i_0} B)_\varepsilon^{h(s)} \cap V_x$ such that $C_x \cap B(x, \frac{\varepsilon}{2}) = \emptyset$ and $\Gamma(x, \mathcal{E}, \frac{1}{2(n+1)}) \cap C_x \neq \emptyset$ for any $\mathcal{E} \subset B_{X^*}$ with, at most, $n+1$ elements.

If $y \in C_x$, then $y \in V_x$ and $|x^*(x-y)| < \tau$ for any $x^* \in \mathcal{G}$. Thus, there is some $z \in X$ such that $\|z\| < \gamma$ and $x^*(z) = x^*(x-y)$ for any $x^* \in \mathcal{G}$. Denoting $y' = y + z$, we have $x^*(x) = x^*(y')$ for any $x^* \in \mathcal{G}$ and $\|y - y'\| < \gamma$. Let
$$C_x^{'} = \{y' : y \in C_x\}, \quad F_{\phi(p+1)} = \cup_{x \in F_s} C_x^{'}, \quad A_{\phi(p+1)} = A_{\phi(p)} \cup F_{\phi(p+1)}$$
and $X_{p+1} = \operatorname{span}(A_{\phi(p+1)})$. Then iii) is satisfied as in the proof of Proposition 5.4 of [2], with $Y_p = \operatorname{span}\{x - y' : x \in F_s, y' \in C_x^{'}\}$.

By a transfinite induction, we obtain the following

Claim. Let $\beta < \omega_1$ and $a > 0$. Then

$$(aB)^{\beta}_{\varepsilon} + rB \subset ((a+r)B)^{\beta}_{\varepsilon}.$$

Now, we have $\Gamma(x, \mathcal{E}, \frac{1}{n+1}) \cap C'_x \neq \emptyset$ if $\mathcal{E}$ has, at most, $n+1$ elements. Furthermore $C'_x \cap B(x, \frac{\varepsilon}{4}) = \emptyset$ and, using the claim, $r_{i_0} \leq r_p$ and $h(\phi(p+1)) < h(s)$.

Finally, we obtain

$$C'_x \subset (r_{i_0}B)^{h(s)}_{\varepsilon} + B(0, \delta_{p+1}) \subset [(r_p + \delta_{p+1})B]^{h(s)}_{\varepsilon} \subset (r_{p+1}B)^{h(\phi(p+1))+1}_{\varepsilon}.$$

Consequently i) and ii) are satisfied, and the Lemma follows. $\square$

P r o o f o f T h e o r e m 5. i). Let $\alpha < \sigma(X)$ and let $(X_p)_{p\in\omega}$ be the sequence of finite dimensional spaces built in Lemma 9, and $Y = \overline{\bigcap_{p\in\omega} X_p}$. Then, from Lemma 9, iii) it follows that Y has a FDD with constant less than $\prod_{p\in\omega}(1+\delta_p)$. By Lemma 8 and Lemma 9, it is not difficult to obtain that $F_{\emptyset} \subset (2B_Y)^{\alpha+1}_{\frac{\varepsilon}{8}}$, since $r_p < 2$, thus $(B_Y)^{\alpha+1}_{\frac{\varepsilon}{16}} \neq \emptyset$, with $\varepsilon > 0$ fixed by Lemma 9. Consequently, $\sigma(Y) > \alpha$. As $\sigma(Y) \leq \sigma(X)$ if Y is a subspace of X, i) of Theorem 5 follows. $\square$

We have ii) and iii) of Theorem 5 as a consequence of the following.

Lemma 10. *Let X be a separable Banach space which contains no isomorph of l_1 and let $\alpha < \sigma(X)$. Then there exists $\varepsilon > 0$ such that $(B_X)^{\alpha}_{\varepsilon} \neq \emptyset$ and there exists $(x_s)_{s\in T_\alpha} \subset B_X$ which satisfies:*

i) $\forall s \in T_\alpha,\ x_s \in (B_X)^{h_\alpha(s)}_{\varepsilon}$.

ii) $\forall n \in \omega,\ \forall s \in T'_\alpha,\ \|x_{s\frown n} - x_s\| > \dfrac{\varepsilon}{2}$.

iii) $\forall s \in T'_\alpha,\ w - \lim\limits_{n}(x_{s\frown n} - x_s) = 0$.

iv) $(x_{\phi_\alpha(n)} - x_{\phi_\alpha(n)-})_{n\in\omega^*}$ *is a basic sequence, (shrinking if, in addition, X^* is separable).*

P r o o f o f T h e o r e m 5, ii) a n d iii). For any $\alpha < \sigma(X)$ there is $\varepsilon > 0$ and $(x_s)_{s\in T_\alpha}$ as in Lemma 10. Let $Y = \overline{\text{span}}\{x_s : s \in T_\alpha\}$. By a transfinite induction, we obtain the following

Claim. (See Lemme 2.8 of [9]). Let $\beta < \omega_1$, $\varepsilon > 0$ and Z be a separable Banach space. Suppose that there exists a family $(z_s)_{s\in T_\alpha} \subset B_Z$ which satisfies:

a) $\forall s \in T_\beta^{'},\ \forall n \in \omega,\ \|z_{s\frown n} - z_s\| > \varepsilon.$

b) $\forall s \in T_\beta^{'},\ w - \lim\limits_n (z_{s\frown n} - z_s) = 0.$

Then $z_\emptyset \in (B_Z)_\varepsilon^\beta$.

Thus $(B_Y)^\alpha_{\frac{\varepsilon}{2}} \neq \emptyset$ and $\sigma(Y) \geq \alpha$. By iv) of Lemma 10, Y has a basis (shrinking basis if X^* is separable), and ii) and iii) of Theorem 5 follow. $\square$

P r o o f o f L e m m a 10. We denote $B = B_X$, $h = h_\alpha$ and $\phi = \phi_\alpha$. By Lemma 7, $\alpha+1 < \sigma(X)$, hence there exists $\varepsilon > 0$ such that $B_\varepsilon^{\alpha+1} \neq \emptyset$. We choose $x_{\phi(0)} = x_\emptyset \in B_\varepsilon^{\alpha+1}$, so $x_{\phi(0)} \in B_\varepsilon^\alpha = B_\varepsilon^{h(\phi(0))}$. Since X contains no isomorph of l_1 and $x_{\phi(0)} \in \overline{B_\varepsilon^\alpha \setminus B(x_{\phi(0)}, \frac{\varepsilon}{2})}^w$, there exists

$$(x_i^0)_{i\in\omega} \subset B_\varepsilon^\alpha \setminus B(x_{\phi(0)}, \frac{\varepsilon}{2})$$

such that $w - \lim\limits_i x_i^0 = x_{\phi(0)}$. Let $c > 0$ and $(c_i)_{i\in\omega} \subset (0,1)$ such that $\prod\limits_{i=2}^{+\infty} (1 - c_i)^{-1} < 1 + c$. Following pages 301 and 302 of [6], it is not difficult to build, by induction for any $k \in \omega$, $x_{\phi(k)} \in B$ and $(x_i^k)_{i\in\omega}$ a sequence in B such that

a) $x_{\phi(k)} \in B_\varepsilon^{h(\phi(k))+1}$.

b) $(x_i^k)_{i\in\omega} \subset B_\varepsilon^{h(\phi(k))} \setminus B(x_{\phi(k)}, \frac{\varepsilon}{2})$ and $w - \lim\limits_i x_i^k = x_{\phi(k)}$.

c) $(x_{\phi(i_k)\frown j})_{j=0}^{m_k}$ is a subsequence of $(x_i^{i_k})_{i\in\omega}$ where $m_k \in \omega$ and $i_k \in \omega$ are defined by $\phi(k) = \phi(i_k) \frown m_k$.

d) If $k > 1$, with $v_i = x_{\phi(i)} - x_{\phi(i)-}$, then $\|y + av_k\| \geq 1 - c_k$ for any $a \in \mathbb{R}$ and any $y \in \text{span}\{v_i : i < k\}$ such that $\|y\| = 1$.

Then, by d), the sequence (v_k) is basic, and $(x_{\phi(k)})_{k\in\omega}$ satisfies the required conditions.

If X^* is separable, by the Zippin theorem ([15]), X embeds in a Banach space Z with a shrinking basis $(z_i)_{i\in\omega}$ which can be supposed monotone. We denote Π_i^j the natural projection from Z onto span$\{z_k : i \le k \le j\}$ when $i \le j$. Let $(x_l^*)_{l\in\omega}$ be a dense sequence in X^*. It is not difficult to build, by induction for any $k \in \omega$, $x_{\phi(k)} \in B_\varepsilon^{h(\phi(k))}$ and $(N_k)_{k\in\omega}$ a strictly increasing sequence in ω such that, when $k \ge 1$, denoting $v_k = x_{\phi(k)} - x_{\phi(k)-}$ one has:

a) $|x_l^*(v_k)| < \dfrac{1}{2^k}$ whenever $0 \le l \le k$.

b) $\|v_k\| > \dfrac{\varepsilon}{2}$

c) $\|\Pi_{N_{k-1}}^{N_k-1}(v_k)\| > \dfrac{\varepsilon}{4}$ and $\|v_k - \Pi_{N_{k-1}}^{N_k-1}(v_k)\| \le \dfrac{1}{4}\dfrac{1}{2^k}\dfrac{\varepsilon}{4}$.

By c), the sequence $(v_k)_{k\in\omega^*}$ is equivalent to a basic block of a shrinking basis (see [11, Prop. 1.a.9]), thus $(v_k)_{k\in\omega}$ is a shrinking basic sequence, and $(x_s)_{s\in T_\alpha}$ satisfies the required conditions. $\square$

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REFERENCES

[1] B. Bossard. Codages des espaces de Banach separables. Familles analytiques ou coanalytiques d'espaces de Banach. *C. R. Acad. Sci. Paris, Sér. I* **316** (1993), 1005-1010.

[2] J. Bourgain. La proprieté de Radon-Nikodym. Publications Mathématiques de l'Université Pierre et Marie Curie, vol. **36**, 1979.

[3] J. D. R. Christensen. Topology and Borel Structure. North-Holland, Math. Stud., vol. **10**, 1974.

[4] R. Deville, G. Godefroy, V. Zizler. Smoothness and Renormings in Banach Spaces. Pitman Monographs and surveys in pure and applied Mathematics, vol. **64**, 1993.

[5] N. Ghoussoub, G. Godefroy, B. Maurey, W. Schachermayer. Some topological and geometrical structures in Banach spaces. *Mem. Amer. Math. Soc.* **70**, *378* (1987).

[6] N. Ghoussoub, B. Maurey. $G_\delta-$embeddings in Hilbert spaces II. *J. Func. Anal.* **78** (1988), 271-305.

[7] A. S. Kechris, A. Louveau. Descriptive set Theory and the Structure of Sets of Uniqueness. London Mathematical Society, Lecture Notes series, vol. **128**, 1987.

[8] G. Lancien. On the Szlenk index and the weak$^*-$dentability index. *Quart. J. Math. Oxford ser. (2)* **47** (1996), 59-71.

[9] G. Lancien. Théorie de l'indice et problèmes de renormage en géometrie des espaces de Banach. Thèse de doctorat de l'Université Paris 6, 1992.

[10] G. López. Point of continuity property in Asplund spaces. Publications math. de l'Université Pierre et Marie Curie, vol. **35**, 1996 (to appear).

[11] J. Lindenstrauss, L. Tzafriri. Classical Banach Spaces I. Springer-Verlag, 1977.

[12] W. Schachermayer. The Radon-Nikodym property and the Krein-Milman property are equivalent for strongly regular sets. *Trans. Amer. Math. Soc.* **303**, *2* (1987), 673-687.

[13] A. Sersouri. Lavrientiev index for Banach spaces. *C. R. Acad. Sci. Paris. Sér. I. Math.* **309**, *2* (1989), 95-99.

[14] M. Zinsmeister. Les dérivations analytiques. Prépublications de l'Université Paris-Sud, 86, T 10, 1986.

[15] M. Zippin. Banach spaces with separable duals. *Trans. Amer. Math. Soc.* **1** (1988), 371-379.

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