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# ON A FIVE-DIAGONAL JACOBI MATRICES AND ORTHOGONAL POLYNOMIALS ON RAYS IN THE COMPLEX PLANE

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ABSTRACT. Systems of orthogonal polynomials on the real line play an important role in the theory of special functions [1]. They find applications in numerous problems of mathematical physics and classical analysis. It is known, that classical polynomials have a number of properties, which uniquely define them. Particularly, one can deduce, that they satisfy the following recurrence: $Jp = \lambda p$, where $p = \begin{pmatrix} p_0 \\ p_1 \\ . \end{pmatrix}$ vector of polynomials $p_k(\lambda)$,

$k = \overline{0, \infty}$, $J$ – symmetric, semi-infinite matrix: $J = \begin{pmatrix} \beta_0 & \alpha_0 & 0 & 0 & . \\ \alpha_0 & \beta_1 & \alpha_1 & 0 & . \\ 0 & \alpha_1 & \beta_2 & \alpha_2 & . \\ . & . & . & . & . \end{pmatrix}$,

$\beta_k$ – real, $\alpha_k > 0$, $k = \overline{0, \infty}$, which have a three diagonal structure [2].

In Chapter 1 of our work we consider the class of polynomials with described above recurrence, where $J$ – is not necessarily symmetric one. The more wide conditions are proposed for $J$. The subject of orthogonality for such systems of polynomials is investigated. The corresponding theorems of orthogonality are obtained.

Chapter 2 of the work is devoted to the solution of some symmetric moments problem. The statement of the problem is a generalization of a well-known moments problem on the real line [3]. The criterion for the real case is known [3]. The generalized problem is found to be closely connected with a semi-infinite fivediagonal matrices (we also call them Jacobi). The criterion of solvability is proved.

**1.** Let $\{p_k(\lambda)\}_{k=0}^{\infty}$ – be set of polynomials, defined in complex plane ($p_k(\lambda)$ is of k-th degree ).

Let us suppose the following:

1) $Jp = \lambda p$, where

$$
J = \begin{pmatrix}
\beta_0 & \alpha_0 & 0 & 0 & . \\
\gamma_0 & \beta_1 & \alpha_1 & 0 & . \\
0 & \gamma_1 & \beta_2 & \alpha_2 & . \\
0 & 0 & \gamma_2 & \beta_3 & . \\
. & . & . & . & .
\end{pmatrix} \quad \text{– semi-infinite threediagonal matrix;}
$$

$$
\beta_k \in C, \gamma_k \in R
$$

$$
\alpha_k > 0, \quad k = \overline{0, \infty}
$$

(1.1)
$$
p = \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ . \end{pmatrix} - \quad \text{vector of} p_k(\lambda).
$$

2) $J^2$ - symmetric matrix, i.e.

(1.2)
$$
(J^2)^* = J^2.
$$

**Note.** All of classical polynomials obviously satisfy the conditions (1.1)(1.2), because for them $J = J^*$.

**Theorem 1.** *If system $\{p_k(\lambda)\}_{k=0}^{\infty}$ satisfy the conditions (1.1)(1.2), then roots of polynomials $p_k(\lambda)$ lie on the real and the imaginary axes in the complex plane.*

The proof of theorem immediately follows from the fact, that roots of $p_N(\lambda)$ are eigenvalues of cutted matrix

$$
J_N = \begin{pmatrix}
\beta_0 & \alpha_0 & 0 & 0 & . \\
\gamma_0 & \beta_1 & \alpha_1 & 0 & . \\
0 & \gamma_1 & \beta_2 & \alpha_2 & . \\
. & . & . & . & . \\
. & . & . & . & \beta_{N-1}
\end{pmatrix}.
$$

Let $\lambda^*$ be a zero of $p_N(\lambda)$, i.e. $p_N(\lambda^*) = 0$ and let $p^N = (p_0, \ldots, p_{N-1})$. Then we have $J_N p^N = \lambda^* p^N$ and therefore $(J_N)^2 p^N = (\lambda^*)^2 p^N$. For $(J_N)^2$ – is symmetric, then $(\lambda^*)^2$ is real.

**Example 1.**

$$J = \begin{pmatrix} \beta i & \frac{1}{\sqrt{2}} & 0 & 0 & . \\ \frac{1}{\sqrt{2}} & -\beta i & \frac{1}{2} & 0 & . \\ 0 & \frac{1}{2} & \beta i & \frac{1}{2} & . \\ 0 & 0 & \frac{1}{2} & . & . \\ . & . & . & . & . \end{pmatrix}, \quad \text{where } \beta \text{ is real parameter.}$$

$$J^2 = \begin{pmatrix} -\beta^2 + \frac{1}{2} & 0 & \frac{1}{2\sqrt{2}} & 0 & . \\ 0 & -\beta^2 + \frac{3}{4} & 0 & \frac{1}{4} & . \\ \frac{1}{2\sqrt{2}} & 0 & -\beta^2 + \frac{1}{2} & 0 & . \\ 0 & \frac{1}{4} & 0 & . & . \\ . & . & . & . & . \end{pmatrix} \quad \text{and } J^2 = (J^2)^*.$$

Put $p_0(\lambda) = \frac{1}{\sqrt{2}}$, $p_1(\lambda) = \lambda - \beta i$, from reccurency (1.1) we can find subsequently all of $p_k(\lambda)$:

$$p_{2k}(\lambda) = T_{2k}(\sqrt{\lambda^2 + \beta^2}) = \cos\left(2k \arccos \sqrt{\lambda^2 + \beta^2}\right)$$

$$p_{2k+1}(\lambda) = \frac{\lambda - \beta i}{\sqrt{\lambda^2 + \beta^2}} T_{2k+1}(\sqrt{\lambda^2 + \beta^2}) = \frac{\lambda - \beta i}{\sqrt{\lambda^2 + \beta^2}} \cos\left((2k+1) \arccos \sqrt{\lambda^2 + \beta^2}\right),$$

$$k = \overline{0, \infty},$$

where $T_n(\lambda) = \cos(n \arccos \lambda)$ is the Chebyshev polynomial of 1-st kind.

Evidently, the roots of $p_k(\lambda)$ lie on the real and the imaginary axes.

**Example 2.**

$$J = \begin{pmatrix} \beta i & \frac{1}{2} & 0 & 0 & . \\ \frac{1}{2} & -\beta i & \frac{1}{2} & 0 & . \\ 0 & \frac{1}{2} & \beta i & \frac{1}{2} & . \\ 0 & 0 & \frac{1}{2} & . & . \\ . & . & . & . & . \end{pmatrix}, \quad \text{where } \beta\text{- is real.}$$

$$J^2 = \begin{pmatrix} -\beta^2 + \frac{1}{4} & 0 & \frac{1}{4} & 0 & . \\ 0 & -\beta^2 + \frac{1}{2} & 0 & \frac{1}{4} & . \\ \frac{1}{4} & 0 & -\beta^2 + \frac{1}{2} & 0 & . \\ 0 & \frac{1}{4} & 0 & . & . \\ . & . & . & . & . \end{pmatrix}$$

Put $p_0(\lambda) = 1$, $p_1(\lambda) = 2(\lambda - \beta i)$, we have:

$$p_{2k}(\lambda) = U_{2k}(\sqrt{\lambda^2 + \beta^2})$$

$$p_{2k+1}(\lambda) = \frac{\lambda - \beta i}{\sqrt{\lambda^2 + \beta^2}} U_{2k+1}(\sqrt{\lambda^2 + \beta^2}),$$

where $U_n(\lambda) = \frac{\sin((n+1)\arccos\lambda)}{\sqrt{1-\lambda^2}}$ is the Chebyshev polynomial of 2-nd kind.

**Example 3.**

$$J = \begin{pmatrix} \beta i & \frac{1}{\sqrt{3}} & 0 & 0 & . \\ \frac{1}{\sqrt{3}} & -\beta i & \frac{2}{\sqrt{15}} & 0 & . \\ 0 & \frac{2}{\sqrt{15}} & \beta i & \frac{3}{\sqrt{35}} & . \\ 0 & 0 & . & . & . \\ . & . & . & . & . \end{pmatrix}$$

($\beta$ real) yields a polynomials connected with Legendre's polynomials $\hat{p}_k(\lambda)$:

$$p_{2k}(\lambda) = \hat{p}_{2k}(\sqrt{\lambda^2 + \beta^2})$$

$$p_{2k+1}(\lambda) = \frac{\lambda - \beta i}{\sqrt{\lambda^2 + \beta^2}} \hat{p}_{2k+1}(\sqrt{\lambda^2 + \beta^2}).$$

       Let us study now the question of orthogonality for described systems. We denote by $T_{n,\beta}(\lambda)$ and $U_{n,\beta}(\lambda)$ polynomials of examples 1 and 2 correspondently.

       **Theorem 2.** *Let us consider a space of quadratically summable vector-functions with a property of symmetry:*

$$L_s^2([-\sqrt{1-\beta^2}, \sqrt{1-\beta^2}] \cup [-|\beta|i, |\beta|i]; d\sigma) = \{f(\lambda) = (f_1(\lambda), f_2(\lambda)):$$

$$\int\limits_{-\sqrt{1-\beta^2}}^{\sqrt{1-\beta^2}} (f_1, f_2) d\sigma \overline{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}} + \int\limits_{-|\beta|i}^{|\beta|i} (f_1, f_2) d\sigma \overline{\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}} < \infty, \ f_2(-\lambda) = f_1(\lambda)\}$$

*where*

$$d\sigma(\lambda) = \begin{pmatrix} \lambda^2 + \beta^2 & -\beta(\beta - \lambda i) \\ -\beta(\beta + \lambda i) & \lambda^2 + \beta^2 \end{pmatrix} \frac{d\lambda}{|\lambda| \sqrt{\lambda^2 + \beta^2} \sqrt{1 - (\lambda^2 + \beta^2)}}, \lambda \in R$$

$$d\sigma(\lambda) = \begin{pmatrix} \beta(\beta - \lambda i) & -(\lambda^2 + \beta^2) \\ -(\lambda^2 + \beta^2) & \beta(\beta + \lambda i) \end{pmatrix} \frac{d\lambda}{i \mid \lambda \mid \sqrt{\lambda^2 + \beta^2} \sqrt{1 - (\lambda^2 + \beta^2)}},$$

$$\lambda \in (-i\infty, i\infty),$$

$$\beta \in [-1, 1].$$

*We define a scalar product in $L_s^2$ by equality:*

$$\langle f, g \rangle_{L_s^2} = \int\limits_{-\sqrt{1-\beta^2}}^{\sqrt{1-\beta^2}} (f_1, f_2) d\sigma \overline{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}} + \int\limits_{-|\beta|i}^{|\beta|i} (f_1, f_2) d\sigma \overline{\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}},$$

*where $f, g \in L_s^2$. then the following property of orthogonality for $T_{n,\beta}(\lambda)$ holds:*

$$\langle (T_{n,\beta}(\lambda), T_{n,\beta}(-\lambda)), (T_{m,\beta}(\lambda), T_{m,\beta}(-\lambda)) \rangle_{L_s^2} =$$

$$= \int\limits_{-\sqrt{1-\beta^2}}^{\sqrt{1-\beta^2}} (T_{n,\beta}(\lambda), T_{n,\beta}(-\lambda)) d\sigma \overline{\begin{pmatrix} T_{m,\beta}(\lambda) \\ T_{m,\beta}(-\lambda) \end{pmatrix}} +$$

$$+ \int\limits_{-|\beta|i}^{|\beta|i} (T_{n,\beta}(\lambda), T_{n,\beta}(-\lambda)) d\sigma \overline{\begin{pmatrix} T_{m,\beta}(\lambda) \\ T_{m,\beta}(-\lambda) \end{pmatrix}} =$$

(1.3)
$$= \begin{cases} \frac{\pi}{2}\delta_{km}, & k > 1, m > 1 \\ \pi\delta_{km}, & k = 1 \ or \ m = 1 \end{cases}$$

*where $k, m \in N$; $\delta_{km}$ – Cronecer's symbol.*

To prove the theorem one may easily rewrite (1.3) into the sum of separate addents. Then after the change of variable we use the orthogonality of Chebyshev's polynomials $T_n(\lambda)$.

Analogous to above is the next theorem:

**Theorem 3.** *Let us put into consideration a space:*

$$\hat{L}_s^2([-\sqrt{1-\beta^2}, \sqrt{1-\beta^2}] \cup [-|\beta|i, |\beta|i]; d\hat\sigma) = \{f(\lambda) = (f_1(\lambda), f_2(\lambda)) :$$

$$\int\limits_{-\sqrt{1-\beta^2}}^{\sqrt{1-\beta^2}} (f_1, f_2) d\hat\sigma \overline{\left(\begin{array}{c} f_1 \\ f_2 \end{array}\right)} + \int\limits_{-|\beta|i}^{|\beta|i} (f_1, f_2) d\hat\sigma \overline{\left(\begin{array}{c} f_1 \\ f_2 \end{array}\right)} < \infty,\ f_2(-\lambda) = f_1(\lambda)\}$$

$$d\hat\sigma(\lambda) = \left(\begin{array}{cc} \lambda^2 + \beta^2 & -\beta(\beta - \lambda i) \\ -\beta(\beta + \lambda i) & \lambda^2 + \beta^2 \end{array}\right) \frac{\sqrt{1 - (\lambda^2 + \beta^2)}\,d\lambda}{|\lambda|\sqrt{\lambda^2 + \beta^2}},\ \lambda \in R$$

$$d\hat\sigma(\lambda) = \left(\begin{array}{cc} \beta(\beta - \lambda i) & -(\lambda^2 + \beta^2) \\ -(\lambda^2 + \beta^2) & \beta(\beta + \lambda i) \end{array}\right) \frac{\sqrt{1 - (\lambda^2 + \beta^2)}\,d\lambda}{i\,|\lambda|\sqrt{\lambda^2 + \beta^2}},\ \lambda \in (-i\infty, i\infty)$$

$$\beta \in [-1, 1]$$

*with a scalar product*

$$\langle f, g \rangle_{\hat L_s^2} = \int\limits_{-\sqrt{1-\beta^2}}^{\sqrt{1-\beta^2}} (f_1, f_2) d\hat\sigma \overline{\left(\begin{array}{c} g_1 \\ g_2 \end{array}\right)} + \int\limits_{-|\beta|i}^{|\beta|i} (f_1, f_2) d\hat\sigma \overline{\left(\begin{array}{c} g_1 \\ g_2 \end{array}\right)},$$

*where $f, g \in \hat L_s^2$. Then*

$$\langle (U_{n,\beta}(\lambda), U_{n,\beta}(-\lambda)), (U_{m,\beta}(\lambda), U_{m,\beta}(-\lambda)) \rangle_{\hat L_s^2} = \frac{\pi}{2} \delta_{mn}.$$

**Note.** Theorems 2,3 shows, that systems of polynomials $T_{n,\beta}(\lambda), U_{n,\beta}(\lambda)$ form orthogonal system in a special, symmetric $L^2(d\sigma)$ spaces. Appeareance of matrix measure is due to the fact, that J is certainly not symmetric.

These theorems admit a generalization:

**Theorem 4.** *Consider system of polynomials $\{p_k(\lambda)\}_{k=1}^\infty, \lambda \in C$, such that*

$$Jp = \lambda p,$$

*where*

$$J = \left(\begin{array}{ccccc} 0 & \alpha_0 & 0 & 0 & . \\ \alpha_0 & 0 & \alpha_1 & 0 & . \\ 0 & \alpha_1 & 0 & \alpha_2 & . \\ 0 & 0 & \alpha_2 & . & . \\ . & . & . & . & . \end{array}\right)$$

*is symmetric Jacobi matrix; $\alpha_k > 0, k = \overline{0, \infty}$ and*

$$p = \begin{pmatrix} p_0(\lambda) \\ p_1(\lambda) \\ . \end{pmatrix}.$$

*Let $\rho(x)$ be a non-negative measure on the real axis with respect to which the system $p_k(\lambda), k = 0, 1, \ldots$ is orthogonal, i.e.*

$$\int_{-a}^{a} p_n(x) p_m(x) \rho(x) dx = A_n \delta_{nm},$$

*where*

$$0 < a \leq +\infty, \ A_n > 0, \ n, m = \overline{0, \infty}. \ ^1$$

*Consider matrix $J_\beta$ as follows:*

$$J_\beta = J + \begin{pmatrix} \beta i & 0 & 0 & 0 & . \\ 0 & -\beta i & 0 & 0 & . \\ 0 & 0 & \beta i & 0 & . \\ . & . & . & -\beta i & . \\ . & . & . & . & . \end{pmatrix} = J + \beta i \ \mathrm{diag}(1, -1, 1, -1, \ldots),$$

*where $\beta \in [-a, a], \beta < \infty$. If system of polynomials $\{p_{n,\beta}(\lambda)\}_{n=0}^{\infty}, \lambda \in C$, satisfy the relation:*

$$J_\beta p_\beta = \lambda p_\beta,$$

*where*

$$p_\beta = \begin{pmatrix} p_{0,\beta}(\lambda) \\ p_{1,\beta}(\lambda) \\ . \\ . \\ . \end{pmatrix}$$

*then*

$$p_{2k,\beta} = p_{2k}(\sqrt{\lambda^2 + \beta^2}),$$

$$p_{2k+1,\beta} = \frac{\lambda - \beta i}{\sqrt{\lambda^2 + \beta^2}} p_{2k+1}(\sqrt{\lambda^2 + \beta^2})$$

---

[1]such measure always exists (see G.Freud, [2, Theorem 1.5, p. 60]).

*and property of orthonality holds:*

$$
\int\limits_{-\sqrt{a-\beta^2}}^{\sqrt{a-\beta^2}} (p_{n,\beta}(\lambda), p_{n,\beta}(-\lambda)) \begin{pmatrix} \lambda^2 + \beta^2 & -\beta(\beta - \lambda i) \\ -\beta(\beta + \lambda i) & \lambda^2 + \beta^2 \end{pmatrix} \times
$$

$$
\times \overline{\begin{pmatrix} p_{m,\beta}(\lambda) \\ p_{m,\beta}(-\lambda) \end{pmatrix}} \frac{\rho(\sqrt{\lambda^2 + \beta^2})d\lambda}{|\lambda|\,\sqrt{\lambda^2 + \beta^2}} +
$$

$$
+ \int\limits_{-|\beta|i}^{|\beta|i} (p_{n,\beta}(\lambda), p_{n,\beta}(-\lambda)) \begin{pmatrix} \beta(\beta - \lambda i) & -(\lambda^2 + \beta^2) \\ -(\lambda^2 + \beta^2) & \beta(\beta + \lambda i) \end{pmatrix} \times
$$

$$
\times \overline{\begin{pmatrix} p_{m,\beta}(\lambda) \\ p_{m,\beta}(-\lambda) \end{pmatrix}} \frac{\rho(\sqrt{\lambda^2 + \beta^2})d\lambda}{i\,|\lambda|\,\sqrt{\lambda^2 + \beta^2}} =
$$

$$
= 2A_n \delta_{nm}, \ n, m = \overline{0, \infty}.
$$

**Note.** As we'll see, orthogonal polynomials on the real axis, polynomials of a kind $p_{n,\beta}$ from Theorem 4 and polynomials corresponding to an anti-symmetric matrix $J$ of a kind

$$
J = \begin{pmatrix} \beta i & \alpha_0 & 0 & 0 & . \\ -\alpha_0 & -\beta i & \alpha_1 & 0 & . \\ 0 & -\alpha_1 & \beta i & \alpha_2 & . \\ 0 & 0 & -\alpha_2 & -\beta i & . \\ . & . & . & . & . \end{pmatrix},
$$

$\beta \in R, \alpha_k > 0, k = \overline{0, \infty}$ exhaust in fact the whole class of systems of polynomials, satisfying (1.1)(1.2)

**2.** In this chapter we consider the problem of recovering the matrix measure by its known power moments. In case of real axis, to find function $\sigma(\lambda), \lambda \in R : \sigma(\lambda) \geq 0, \int_{-\infty}^{\infty} \lambda^k \sigma(\lambda)d\lambda = s_k, k = \overline{0, \infty}$ ($s_k$ – fixed real) it is necessary and sufficient for $\{s_k\}_{k=0}^{\infty}$ to be positive, H. Hamburger, [3, Theorem 2.1.1, page 43].

It turns out to be possible to obtain the criterion of solvability in case of measure concentrated on the real and the imaginary axes in the complex plane and having matrix form.

**Definition** (of symmetric moments problem). *Consider following problem of moments:*

*to find matrix measure* $\sigma(\lambda) = \begin{pmatrix} \sigma_1(\lambda) & \sigma_2(\lambda) \\ \sigma_3(\lambda) & \sigma_4(\lambda) \end{pmatrix}, \lambda \in C; \ \sigma_i(\lambda) : C \to C$

*are continuous,* $i = \overline{1,4}$:

1) $\sigma_1(\lambda) = \overline{\sigma_1(\lambda)}, \sigma_4(\lambda) = \overline{\sigma_4(\lambda)}, \sigma_2(\lambda) = \overline{\sigma_3(\lambda)}$;

$\sigma_1(\lambda) \geq 0, \sigma_1(\lambda)\sigma_4(\lambda) - \sigma_2(\lambda)\sigma_3(\lambda) \geq 0$, *where*

(2.1) $$\lambda \in R \cup T \ (T = (-i\infty, i\infty)).$$

*i.e.* $\sigma(\lambda)$ *is symmetric, nonnegative defined matrix for all* $\lambda \in R \cup T$.

2) $\int\limits_{R \cup T} (\lambda^k, (-\lambda)^k)\sigma(\lambda)\tilde{d}\lambda \begin{pmatrix} 1 \\ 1 \end{pmatrix} = s_k, \ k = \overline{0,\infty}$

(2.2) $$\int\limits_{R \cup T} (\lambda^{k-1}, (-\lambda)^{k-1})\sigma(\lambda)\tilde{d}\lambda \overline{\begin{pmatrix} \lambda \\ -\lambda \end{pmatrix}} = m_k, \ k = \overline{1,\infty},$$

*where* $\{s_k\}_{k=0}^\infty, \{m_k\}_{k=1}^\infty$ – *fixed sequences of complex numbers;*

$$\tilde{d}\lambda = \begin{cases} d\lambda, \ \lambda \in R \\ \frac{d\lambda}{i}, \ \lambda \in T \end{cases}$$

*We'll call this problem* **symmetric moments problem** .

The more general statement of the problem is:

**Definition.** (of generalized symmetric moments problem). *The problem is:*

*to find matrix measure* $\tilde{\sigma}(\lambda) = \begin{pmatrix} \tilde{\sigma}_1(\lambda) & \tilde{\sigma}_2(\lambda) \\ \tilde{\sigma}_3(\lambda) & \tilde{\sigma}_4(\lambda) \end{pmatrix}, \ \lambda \in C; \tilde{\sigma}_i(\lambda) : C \to C$

*is a piecewise continuous on the real and the image axis,* $i = \overline{1,4}$:

1) $\tilde{\sigma}(\lambda)$ *is symmetric, monotonically increasing matrix function:*

$$\tilde{\sigma}_1(\lambda) = \overline{\tilde{\sigma}_1(\lambda)}, \tilde{\sigma}_4(\lambda) = \overline{\tilde{\sigma}_4(\lambda)}, \tilde{\sigma}_2(\lambda) = \overline{\tilde{\sigma}_3(\lambda)};$$

$$\tilde{\sigma}(\lambda_2) \geq \tilde{\sigma}(\lambda_1), \ \lambda_2 \geq \lambda_1, \ \lambda_1, \lambda_2 \in R$$

(2.3)  $\quad\quad\quad\quad \tilde{\sigma}(\lambda_2) \geq \tilde{\sigma}(\lambda_1), \ \dfrac{\lambda_2}{i} \geq \dfrac{\lambda_1}{i}, \ \lambda_1, \lambda_2 \in (-i\infty, i\infty)$

$$2) \int\limits_{R\cup T} (\lambda^k, (-\lambda)^k) d\tilde{\sigma}(\lambda) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = s_k, \ k = \overline{0, \infty}$$

(2.4)  $$\int\limits_{R\cup T} (\lambda^{k-1}, (-\lambda)^{k-1}) d\tilde{\sigma}(\lambda) \overline{\begin{pmatrix} \lambda \\ -\lambda \end{pmatrix}} = m_k, \ k = \overline{1, \infty},$$

*where $\{s_k\}_{k=0}^{\infty}, \{m_k\}_{k=1}^{\infty}$ are fixed sequences of complex numbers. We call this problem* **generalized symmetric moments problem**.

**Note.** In the case of absolute continuity of $\tilde{\sigma}(\lambda)$ we arive at the previous statement. The most of results are formulated for symmetric problem but can be easily reformed to the general form.

The next definitions are usefull:

**Definition.** *We call a pair of sequences $\{s_k, m_{k+1}\}_{k=0}^{\infty}, s_k \in C, m_{k+1} \in C, k = \overline{0, \infty}$* **symmetric***, if holds:*

$$\overline{s_{2k+1}} = m_{2k+1};$$

$$\overline{s_{2k}} = s_{2k}, \overline{m_{2k+2}} = m_{2k+2}, \quad k = \overline{0, \infty}.$$

**Definition.** *We call a pair of sequences $\{s_k, m_{k+1}\}_{k=0}^{\infty}, s_k \in C, m_{k+1} \in C, k = \overline{0, \infty}$* **positive** *one , if the following is correct:*

$$\begin{bmatrix} s_0 & s_1 & . & . & s_k \\ m_1 & m_2 & . & . & m_{k+1} \\ s_2 & s_3 & . & . & s_{k+2} \\ . & . & . & . & . \\ m_k & m_{k+1} & . & . & m_{2k} \end{bmatrix} > 0, k = 2l+1; \quad \begin{bmatrix} s_0 & s_1 & . & . & s_k \\ m_1 & m_2 & . & . & m_{k+1} \\ s_2 & s_3 & . & . & s_{k+2} \\ . & . & . & . & . \\ s_k & s_{k+1} & . & . & s_{2k} \end{bmatrix} > 0, k = 2l$$

$$l = \overline{0, \infty}.$$

The following is true

**Assertion.** *If there exists a nontrivial solution $\sigma(\lambda)$ ( $\sigma(\lambda) \neq 0^2$ ) of the symmetric moments problem (2.1)(2.2), then pair $\{s_k, m_{k+1}\}_{k=0}^{\infty}$ is symmetric and positive.*

P r o o f. Let $\sigma(\lambda)$ be solution of problem (2.1)(2.2).

Consider the functional $\sigma(u, v)$:

$$\sigma(u, v) = \int\limits_{R \cup T} (u(\lambda), u(-\lambda))\sigma(\lambda)\tilde{d}\lambda \overline{\left(\begin{array}{c} v(\lambda) \\ v(-\lambda) \end{array}\right)},$$

where

$$u(\lambda), v(\lambda) \in L_s^2(R \cup T, \sigma(\lambda)\tilde{d}\lambda) =$$

$$\left\{ f(\lambda) = (f_1(\lambda), f_2(\lambda)) : \int\limits_{R \cup T} (f_1, f_2))\sigma(\lambda)\tilde{d}\lambda \overline{\left(\begin{array}{c} f_1 \\ f_2 \end{array}\right)} < \infty, \ f_2(-\lambda) = f_1(\lambda) \right\}.$$

Functional $\sigma(u, v)$ is obviously bilinear.

From condition (2.1) we conclude, that:

$$(2.5) \qquad\qquad \sigma(u, u) \geq 0, \ u \in L_s^2$$

$$(2.6) \qquad\qquad \overline{\sigma(u, v)} = \sigma(v, u), \ u, v \in L_s^2.$$

Note, that by virtue of measure $\sigma(\lambda)$ support structure:

$$\sigma(u, \lambda^2 v) = \sigma(\lambda^2 u, v), \ u, v \in L_s^2.$$

Let $R_n(\lambda) = \sum_{k=0}^{n} x_k \lambda^k$ – be an arbitrary polynomial of $n$-th degree, ($x_k \in C, n \in N$).

We have by using (2.5), bilinearity of $\sigma(u, v)$ and (2.2):

$$0 \leq \sigma(R_n(\lambda), R_n(\lambda)) = \sigma(\sum_{k=0}^{n} x_k \lambda^k, \sum_{j=0}^{n} x_j \lambda^j) =$$

$$= \sum_{k,j=0}^{n} x_k \overline{x_j} \sigma(\lambda^k, \lambda^j) = \sum_{j=0}^{n} (\sum_{k=0}^{n} \sigma(\lambda^k, \lambda^j) x_k) \overline{x_j} =$$

---

[2] $\sigma(\lambda)$ differs from zero on set of positive measure in $R \cup T$.

$$
= \begin{cases}
\left\langle \begin{bmatrix} s_0 & s_1 & . & . & s_n \\ m_1 & m_2 & . & . & m_{n+1} \\ . & . & . & . & . \\ m_n & m_{n+1} & . & . & m_{2n} \end{bmatrix}, (x_0, \ldots, x_n) \right\rangle, & n = 2l + 1 \\
\left\langle \begin{bmatrix} s_0 & s_1 & . & . & s_n \\ m_1 & m_2 & . & . & m_{n+1} \\ . & . & . & . & . \\ s_n & s_{n+1} & . & . & s_{2n} \end{bmatrix}, (x_0, \ldots, x_n) \right\rangle, & n = 2l.
\end{cases} \qquad l = \overline{0, \infty}.
$$

From the preceeding equality follows the positivity of $\{s_k, m_{k+1}\}_{k=0}^{\infty}$.

Next, using $(2.5)(2.6)$ and by the equalities

$$
\overline{s_k} = \overline{\sigma(\lambda^k, 1)} = \sigma(1, \lambda^k) = \begin{cases} s_k, & \text{if } k = 2l \\ m_k, & \text{if } k = 2l + 1 \end{cases} ; \quad l = \overline{0, \infty}
$$

$$
\overline{m_{k+1}} = \overline{\sigma(\lambda^k, \lambda)} = \sigma(\lambda, \lambda^k) = \begin{cases} s_{k+1}, & \text{if } k = 2l \\ m_{k+1}, & \text{if } k = 2l + 1 \end{cases} ; \quad l = \overline{0, \infty}
$$

it follows the symmetry of $\{s_k, m_{k+1}\}_{k=0}^{\infty}$. This completes the proof. $\square$

The simplest particular case, when the conditions of symmetry and positivity of $\{s_k, m_{k+1}\}_{k=0}^{\infty}$ will be apparently sufficient for existence of solution $\sigma(\lambda)$ of problem $(2.1)(2.2)$ is the case: $s_k = m_k, k = \overline{1, \infty}$. The conditions of symmetry and positivity in this case are the condition of positivity of $\{s_k\}_{k=0}^{\infty}$ [3] and the solution of $(2.1)(2.2)$ exists : $\sigma(\lambda) = \begin{pmatrix} \sigma_1(\lambda) & 0 \\ 0 & 0 \end{pmatrix}, \sigma_1(\lambda) \geq 0$ with real support.

**Note.** Moments problem $(2.1)(2.2)$ is equivalent to searching for bilinear functional $\sigma(u, v)$ in some linear space $L \supset Lin\{1, \lambda, \lambda^2, \ldots\}$:

1) $\sigma(u, u) \geq 0, \forall u \in L$.

2) $\overline{\sigma(u, v)} = \sigma(v, u), \forall u, v \in L$.

3) $\sigma(u, \lambda^2 v) = \sigma(\lambda^2 u, v), \forall u, v \in L$.

4) $\sigma(\lambda^k, 1) = s_k; \sigma(\lambda^k, \lambda) = m_{k+1}, k = \overline{0, \infty}$

$\{s_k\}_{k=0}^{\infty}, \{m_k\}_{k=1}^{\infty}$ – fixed sequences of complex values.

Suppose, that solution $\sigma(\lambda)$ $(\sigma \neq 0, \lambda \in R \cup T)$ of moments problem $(2.1)(2.2)$ exists. We put into correspondence for measure $\sigma(\lambda)$ some fivediagonal symmetric matrix J and system of polynomials $\{p_n(\lambda)\}_{k=0}^{\infty}$, orthogonal with

respect to measure $\sigma(\lambda)$. Namely, let $p_k(\lambda)$ be as follows:

$$(2.8) \qquad p_k(\lambda) = \begin{bmatrix} s_0 & s_1 & s_2 & . & s_k \\ m_1 & m_2 & m_3 & . & m_{k+1} \\ s_2 & s_3 & s_4 & . & s_{k+2} \\ . & . & . & . & . \\ 1 & \lambda & \lambda^2 & . & \lambda^k \end{bmatrix}, \ k = \overline{0, \infty} \ (p_0 = 1)$$

Then for vector $(p_k(\lambda), p_k(-\lambda))$ is correct:

$$(p_k(\lambda), p_k(-\lambda)) = \begin{bmatrix} s_0 & s_1 & s_2 & . & s_k \\ m_1 & m_2 & m_3 & . & m_{k+1} \\ s_2 & s_3 & s_4 & . & s_{k+2} \\ . & . & . & . & . \\ (1,1) & (\lambda,-\lambda) & (\lambda^2,\lambda^2) & . & (\lambda^k,(-\lambda)^k) \end{bmatrix}, \ k = \overline{0, \infty}.$$

Multiplying previous equality by $\sigma(\lambda)\overline{\begin{pmatrix} \lambda^l \\ (-\lambda)^l \end{pmatrix}} \ l = \overline{0, k-1}$ and integrating we have:

$$\int\limits_{R \cup T} (p_k(\lambda), p_k(-\lambda))\sigma(\lambda)\overline{\begin{pmatrix} \lambda^l \\ (-\lambda)^l \end{pmatrix}}\tilde{d}\lambda = 0, \quad l = \overline{0, k-1}, \ k - \text{fixed}, \ k = \overline{0, \infty}.$$

Sequently, the system of polynomials $\{p_n(\lambda)\}_{k=1}^{\infty}$ form an orthogonal system with respect to measure $\sigma(\lambda)$:

$$\int\limits_{R \cup T} (p_k(\lambda), p_k(-\lambda))\sigma(\lambda)\tilde{d}\lambda\overline{\begin{pmatrix} p_l(\lambda) \\ p_l(-\lambda) \end{pmatrix}} = 0, \ k, l = \overline{0, \infty} \ k \neq l$$

(generally, $l > k$, but in more detail we can use (2.6) for $l \leq k$).

Using the definition of functional $\sigma(u, v)$ from the proof of the preceding assertion, we have

$$\sigma(p_k(\lambda), p_l(\lambda)) = 0 \quad k, l = \overline{0, \infty}, k \neq l.$$

Let us calculate the norm of polynomial $p_k(\lambda)$. $(p_k(\lambda) \in L_s^2(R \cup T, \sigma(\lambda)\tilde{d}\lambda)$, for the preHilbert space $L_s^2$ with scalar product $\sigma(u, v))$

$$\| p_k(\lambda) \|^2 = \sigma(p_k(\lambda), p_k(\lambda)) = \sigma(p_k(\lambda), \triangle_{k-1}\lambda^k) = \overline{\triangle_{k-1}}\sigma(p_k(\lambda), \lambda^k) = \overline{\triangle_{k-1}}\triangle_k,$$

$$k = \overline{0, \infty}.$$

where

$$\triangle_k = \begin{bmatrix} s_0 & s_1 & . & . & s_k \\ m_1 & m_2 & . & . & m_{k+1} \\ s_2 & s_3 & . & . & s_{k+2} \\ . & . & . & . & . \\ m_k & m_{k+1} & . & . & m_{2k} \end{bmatrix}, k = 2l+1; \ \triangle_k = \begin{bmatrix} s_0 & s_1 & . & . & s_k \\ m_1 & m_2 & . & . & m_{k+1} \\ s_2 & s_3 & . & . & s_{k+2} \\ . & . & . & . & . \\ s_k & s_{k+1} & . & . & s_{2k} \end{bmatrix}, k = 2l.$$

$$l = \overline{0, \infty}.$$

From foregoing assertion follows $\triangle_k > 0, k = \overline{0, \infty}$.

Put

(2.9)
$$\hat{p}_k(\lambda) = \frac{1}{\sqrt{\triangle_{k-1}\triangle_k}} p_k(\lambda), \ (\triangle_{-1} = 1)$$

then sequence $\{\hat{p}_k(\lambda)\}_{k=1}^{\infty}$ form an orthonormal system:

$$\sigma(\hat{p}_k(\lambda), \hat{p}_l(\lambda)) = \delta_{kl}, \quad k, l = \overline{0, \infty}.$$

Consider the polynomial $\lambda^2 \hat{p}_k(\lambda), k = \overline{0, \infty}$. It can be expanded into linear combination of polynomials $\hat{p}_0(\lambda), \hat{p}_1(\lambda), \ldots, \hat{p}_{k+2}(\lambda)$:

$$\lambda^2 \hat{p}_k(\lambda) = \sum_{l=0}^{k+2} \xi_l \hat{p}_l(\lambda) \quad (\xi_l \in C, l = \overline{0, k+2}).$$

Multiplying this equality subsequently by $\hat{p}_n(\lambda), n = \overline{0, k+2}$ we have:

$$\sigma(\lambda^2 \hat{p}_k(\lambda), \hat{p}_n(\lambda)) = \sigma(\sum_{l=0}^{k+2} \xi_l \hat{p}_l(\lambda), \hat{p}_n(\lambda)) = \xi_n, \quad n = \overline{0, k+2}.$$

Next, because of $\{\hat{p}_k(\lambda)\}_{k=0}^{\infty}$ orthogonality:

$$\sigma(\lambda^2 \hat{p}_k(\lambda), \hat{p}_n(\lambda)) = \sigma(\hat{p}_k(\lambda), \lambda^2 \hat{p}_n(\lambda)) = 0, \quad n < k - 2$$

then (we put $\hat{p}_{-1}(\lambda) = \hat{p}_{-2}(\lambda) = 0$):

$$\begin{aligned} \lambda^2 \hat{p}_k(\lambda) &= \sum_{l=k-2}^{k+2} \sigma(\lambda^2 \hat{p}_k(\lambda), \hat{p}_l(\lambda)) \hat{p}_l(\lambda) \\ &= \sigma(\lambda^2 \hat{p}_k(\lambda), \hat{p}_{k+2}(\lambda)) \hat{p}_{k+2}(\lambda) + \sigma(\lambda^2 \hat{p}_k(\lambda), \hat{p}_{k+1}(\lambda)) \hat{p}_{k+1}(\lambda) \\ &\quad + \sigma(\lambda^2 \hat{p}_k(\lambda), \hat{p}_k(\lambda)) \hat{p}_k(\lambda) + \sigma(\lambda^2 \hat{p}_k(\lambda), \hat{p}_{k-1}(\lambda)) \hat{p}_{k-1}(\lambda) \\ &\quad + \sigma(\lambda^2 \hat{p}_k(\lambda), \hat{p}_{k-2}(\lambda)) \hat{p}_{k-2}(\lambda) \\ &= \alpha_k \hat{p}_{k+2}(\lambda) + \beta_k \hat{p}_{k+1}(\lambda) + \gamma_k \hat{p}_k(\lambda) + \overline{\beta_{k-1}} \hat{p}_{k-1}(\lambda) + \alpha_{k-2} \hat{p}_{k-2}(\lambda), \end{aligned}$$

where

$$\alpha_k = \sigma(\lambda^2 \hat{p}_k(\lambda), \hat{p}_{k+2}(\lambda))$$

(2.10)
$$\beta_k = \sigma(\lambda^2 \hat{p}_k(\lambda), \hat{p}_{k+1}(\lambda))$$

$$\gamma_k = \sigma(\lambda^2 \hat{p}_k(\lambda), \hat{p}_k(\lambda)), \quad k = \overline{0, \infty} \ (\beta_{-1} = \alpha_{-1} = \alpha_{-2} = 0)$$

and we used the $\sigma(u, v)$ properties (2.4), (2.5).

From here

$$\lambda^2 \hat{p}_k(\lambda) = \alpha_{k-2} \hat{p}_{k-2}(\lambda) + \overline{\beta_{k-1}} \hat{p}_{k-1}(\lambda) + \gamma_k \hat{p}_k(\lambda) + \beta_k \hat{p}_{k+1}(\lambda) +$$

(2.11)
$$+\alpha_k \hat{p}_{k+2}(\lambda), \quad k = \overline{0, \infty} \ (\beta_{-1} = \alpha_{-1} = \alpha_{-2} = 0).$$

Equality (2.11) is a recurrence for rebuilding of polynomials $\{\hat{p}_k(\lambda)\}$ and can be written in the following way:

$$J\hat{p} = \lambda^2 \hat{p},$$

where

$$J = \begin{pmatrix} \gamma_0 & \beta_0 & \alpha_0 & 0 & 0 & 0 & . \\ \overline{\beta_0} & \gamma_1 & \beta_1 & \alpha_1 & 0 & 0 & . \\ \alpha_0 & \overline{\beta_1} & \gamma_2 & \beta_2 & \alpha_2 & 0 & . \\ 0 & \alpha_1 & \overline{\beta_2} & \gamma_3 & \beta_3 & \alpha_3 & . \\ . & . & . & . & . & . & . \end{pmatrix} - \text{ symmetric fivediagonal matrix.}$$

(2.12) $\quad \hat{p} = \begin{pmatrix} \hat{p}_0(\lambda) \\ \hat{p}_1(\lambda) \\ \hat{p}_2(\lambda) \\ . \end{pmatrix} \quad \hat{p}_0(\lambda) = \dfrac{1}{\sqrt{s_0}}; \hat{p}_1(\lambda) = \dfrac{1}{\sqrt{s_0(s_0 m_2 - s_1 m_1)}}(s_0 \lambda - s_1).$

Without restricting of generality we can put $s_0 = 1$.

So, as we know measure $\sigma(\lambda)$ is a solution of moments problem, we can build fivediagonal matrix J and orthogonal system of polynomials $\{\hat{p}_k(\lambda)\}_{k=0}^{\infty}$. In our constructions, however, we didn't use the $\sigma(\lambda)$ explicitly, but it's moments $\{s_k, m_{k+1}\}_{k=0}^{\infty}$ only.

Now, let we have a pair of sequences $\{s_k, m_{k+1}\}_{k=0}^{\infty}$, which is symmetric and positive. Let us define after (2.8)(2.9) sequences of polynomials $\{p_k(\lambda)\}_{k=0}^{\infty}$

and $\{\hat{p}_k(\lambda)\}_{k=0}^{\infty}$. Next, by means of equalities 2)-4) of the note after the assertion we define the bilinear functional $\sigma(u,v)$ on $Lin\{1,\lambda,\lambda^2,\ldots\}$. With the aid of (2.10)(2.11)(2.12) we define coefficients $\{\alpha_k,\beta_k,\gamma_k\}_{k=0}^{\infty}$ and construct the corresponding matrix (we call it Jacobi matrix) $J$.

A question appears: Is it possible with the help of $J$ and $\{\hat{p}_k(\lambda)\}_{k=0}^{\infty}$ to find measure $\sigma(\lambda)$. We consider a particular case, when $J$ admits extracting of a square root with threediagonal matrix structure. In that case the solution of moments problem will be constructed in a special form. (in fact this case is one-dimensional). After this a basic theorem of solvability will be proved.

**Theorem 5.** *Let the following sequence of polynomials be given :* $p_0 = 1, p_1 = c_1\lambda + b, \ldots, p_k(\lambda) = c_k\lambda^k + \ldots, \ldots$

$$c_k > 0, k = \overline{1,\infty}, b \in C$$

*and holds the relation:*

$$\begin{pmatrix} \gamma_0 & \beta_0 & \alpha_0 & 0 & 0 & . \\ \overline{\beta_0} & \gamma_1 & \beta_1 & \alpha_1 & 0 & . \\ \alpha_0 & \overline{\beta_1} & \gamma_2 & \beta_2 & \alpha_2 & . \\ . & . & . & . & . & . \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ . \end{pmatrix} = \lambda^2 \begin{pmatrix} p_0 \\ p_1 \\ . \\ . \end{pmatrix},$$

*where*

$$\alpha_k > 0, \gamma_k \in R, \beta_k \in C, k = \overline{0,\infty}$$

*For correctness of equality:*

$$\begin{pmatrix} \hat{\beta}_0 & \hat{\alpha}_0 & 0 & 0 & . \\ \hat{\gamma}_0 & \hat{\beta}_1 & \hat{\alpha}_1 & 0 & . \\ 0 & \hat{\gamma}_1 & \hat{\beta}_2 & \hat{\alpha}_2 & . \\ . & . & . & . & . \end{pmatrix} \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ . \end{pmatrix} = \lambda \begin{pmatrix} p_0 \\ p_1 \\ p_2 \\ . \end{pmatrix}$$

*for some set of complex numbers*

$$\hat{\alpha}_k, \hat{\beta}_k, \hat{\gamma}_k, \ k = \overline{0,\infty},$$

*it is necessary and sufficient the following conditions to be satisfied:*
*a)* $\beta_k = 0$ *or* $\dfrac{\overline{\beta_k}}{\beta_k} = \begin{cases} c, & \text{if } k \text{ is even} \\ \frac{1}{c}, & \text{if } k \text{ is odd}. \end{cases}$
*where* $c$ *– some fixed complex value on the unit circle,* $k = \overline{0,\infty}$.

*b)* $\gamma_0 = \frac{b^2+c}{c_1^2}$, *( c from condition a) )*

$$\gamma_k = c(\alpha_{k-1}'^2 + \alpha_k'^2) + \beta_k'^2$$

*where* $\alpha_k' = \begin{cases} \dfrac{1}{c_1}\dfrac{\alpha_1\alpha_3\ldots\alpha_{k-1}}{\alpha_0\alpha_2\ldots\alpha_{k-2}}, & \text{if } k \text{ is even} \\ c_1\dfrac{\alpha_0\alpha_2\ldots\alpha_{k-1}}{\alpha_1\alpha_3\ldots\alpha_{k-2}}, & \text{if } k \text{ is odd.} \end{cases}$ $\quad k = 2, 3, \ldots$

$$\alpha_0' = \frac{1}{c_1}, \alpha_1' = \alpha_0 c_1,$$

$$\beta_k' = \sum_{j=0}^{k-1}(-1)^j \frac{\beta_{k-1-j}}{\alpha_{k-1-j}'} + (-1)^{k+1}\frac{b}{c_1}, \ k = 0, 1, \ldots$$

*If a) and b) are satisfied, then coefficients* $\{\hat{\alpha}_k, \hat{\gamma}_k, \hat{\beta}_k\}$ *can be found as follows:*

$$\hat{\alpha}_k = \alpha_k', \hat{\gamma}_k = c\alpha_k', \hat{\beta}_k = \beta_k', k = \overline{0, \infty}$$

**Note.** The above theorem shows, that for existance of threediagonal square root of fivediagonal matrix, the hard restrictions must fulfill: sequence $\{\alpha_k\}_{k=0}^{\infty}$ may be arbitrary, but argument of $\beta_k$- is strictly fixed, $\{\gamma_k\}_{k=0}^{\infty}$ being uniquely defined after $\{\alpha_k, \beta_k\}_{k=0}^{\infty}, c_1 > 0, b \in C$ *and some* $c \in C$.

**Corollary 1.** *If under conditions of Theorem 5 is correct:* $c = \pm 1$, *then roots of polynomials* $\{p_k(\lambda)\}_{k=0}^{\infty}$ *lie on the real and the imaginary axes in the complex plane.*

**Note.** For an arbitrary fivediagonal matrix the conclusion of previous corollary is not valid.

**Corollary 2.** *Let moments problem (2.1)(2.2) be given. If sequence of coefficients* $\{\alpha_k\}_{k=0}^{\infty}, \{\beta_k\}_{k=0}^{\infty}, \{\gamma_k\}_{k=0}^{\infty}$, *constructed after moments pair of sequences* $\{s_k, m_{k+1}\}_{k=0}^{\infty}$ *(see the reasoning after deducing of polynomials* $\hat{p}_k$ *form and J) satisfies conditions of preceeding theorem with* $c = 1$, *then the solution of moments problem exists.*

To prove this, note that threediagonal matrix is a real symmetric matrix or have a structure of $J_\beta$ matrix from Theorem 4 in this case.

Note, that the case $c = -1$ in above theorem leads to an anti-symmetric matrix

$$J = \begin{pmatrix} \beta i & \alpha_0 & 0 & 0 & . \\ -\alpha_0 & -\beta i & \alpha_1 & 0 & . \\ 0 & -\alpha_1 & \beta i & \alpha_2 & . \\ 0 & 0 & -\alpha_2 & -\beta i & . \\ . & . & . & . & . \end{pmatrix}, \beta \in R, \alpha_k > 0, k = \overline{0, \infty}$$

The following holds

**Theorem 6.** *Let the moments problem (2.3)(2.4) in general form be given. For existance of problem's solution $\sigma(\lambda)$ (with infinite number of points of increasing) it is necessary and sufficient for pair of sequences $\{s_k, m_{k+1}\}_{k=0}^{\infty}$ to be symmetric and positive.*

P r o o f. Necessity was proved for absolutely continuous case (see Assertion) and can be easily carried on general case.

Let us show sufficiency.

Let pair of sequences $\{s_k, m_{k+1}\}_{k=0}^{\infty}$ be symmetric and positive. It is required to construct solution of problem $\sigma(\lambda)$, satisfying conditions (2.3)(2.4).

By described above reasonings, we define sequence of polynomials $\{\hat{p}_k(\lambda)\}_{k=0}^{\infty}$ and matrix $J$. Let $J_N$ be cutted matrix:

$$J_N = \begin{pmatrix} \gamma_0 & \beta_0 & \alpha_0 & 0 & 0 & & . \\ \overline{\beta_0} & \gamma_1 & \beta_1 & \alpha_1 & 0 & & . \\ \alpha_0 & \overline{\beta_1} & \gamma_2 & \beta_2 & \alpha_2 & & . \\ . & . & . & . & . & & \\ & & & 0 & \alpha_{N-3} & \overline{\beta_{N-2}} & \gamma_{N-1} & \beta_{N-1} \\ & & & 0 & 0 & \alpha_{N-2} & \overline{\beta_{N-1}} & \gamma_N \end{pmatrix}$$

– of $(N+1) \times (N+1)$ order.

Next, obviously holds relation:

$$(2.14) \qquad J_N \hat{p}^{N+1} = \lambda^2 \hat{p}^{N+1} - \begin{pmatrix} 0 \\ 0 \\ . \\ 0 \\ R_1(\lambda) \\ R_2(\lambda) \end{pmatrix}, \quad \text{where } \hat{p}^{N+1} = \begin{pmatrix} \hat{p}_0 \\ \hat{p}_1 \\ . \\ . \\ \hat{p}_N \end{pmatrix};$$

$R_1(\lambda), R_2(\lambda)$ – correcting polynomials (first $(N-1)$ rows coincide with recurrence relation (2.12), but last have cutted form):

$$R_1(\lambda) = \lambda^2 \hat{p}_{N-1}(\lambda) - \alpha_{N-3}\hat{p}_{N-3}(\lambda) - \overline{\beta_{N-2}}\hat{p}_{N-2}(\lambda) - \gamma_{N-1}\hat{p}_{N-1}(\lambda) - \beta_{N-1}\hat{p}_N(\lambda);$$

$$R_2(\lambda) = \lambda^2 \hat{p}_N(\lambda) - \alpha_{N-2}\hat{p}_{N-2}(\lambda) - \overline{\beta_{N-1}}\hat{p}_{N-1}(\lambda) - \gamma_N\hat{p}_N(\lambda)$$

of degrees $(N+1)$ and $(N+2)$ correspondingly.

Let us consider the next polynomial:

$$Q(\lambda) = \det \begin{pmatrix} R_1(\lambda) & R_1(-\lambda) \\ R_2(\lambda) & R_2(-\lambda) \end{pmatrix} = R_1(\lambda)R_2(-\lambda) - R_1(-\lambda)R_2(\lambda)$$

Now we'll see, that points of spectrum of $J_N$ are the points, where matrix $Q(\lambda)$ is degenerate.

Let $\lambda$ be zero of $Q(\lambda)$ : $R_1(\lambda) = R_2(\lambda) = 0, |R_1(-\lambda)|^2 + |R_2(-\lambda)|^2 > 0$. Then $\lambda^2$ is point of prime ( at least ) spectrum of $J_N$:

$$J_N\hat{p}_N(\lambda) = \lambda^2\hat{p}_N(\lambda)$$

Analogously, if $\lambda$ – root of $Q(\lambda)$: $|R_1(\lambda)|^2 + |R_2(\lambda)|^2 > 0, R_1(-\lambda) = R_2(-\lambda) = 0$, then $\lambda^2$ is point of prime ( at least ) spectrum of $J_N$:

$$J_N\hat{p}_N(-\lambda) = \lambda^2\hat{p}_N(-\lambda)$$

Let $\lambda$ – root of $Q(\lambda)$: $R_1(\lambda) = R_2(\lambda) = R_1(-\lambda) = R_2(-\lambda) = 0$. In this case $\lambda^2$ is point of double spectrum of $J_N$ (because matrix $J_N$ may have not more then double spectrum):

$$J_N\hat{p}_N(\lambda) = \lambda^2\hat{p}_N(\lambda), \quad J_N\hat{p}_N(-\lambda) = \lambda^2\hat{p}_N(-\lambda)$$

Let now $\lambda$ – arbitrary root of $Q(\lambda)$, be different from considered above and $\lambda \neq 0$. Using the property of determinants, we have: $\exists\alpha(\lambda), \beta(\lambda) : |\alpha(\lambda)|^2 + |\beta(\lambda)|^2 > 0$ – some complex numbers, depending on $\lambda$:

$$\alpha(\lambda)R_1(\lambda) + \beta(\lambda)R_1(-\lambda) = 0$$

$$\alpha(\lambda)R_2(\lambda) + \beta(\lambda)R_2(-\lambda) = 0$$

and $\lambda^2$ is point of prime spectrum ( at least ) of $J_N$:

$$J_N(\alpha(\lambda)\hat{p}_N(\lambda) + \beta(\lambda)\hat{p}_N(-\lambda)) = \lambda^2(\alpha(\lambda)\hat{p}_N(\lambda) + \beta(\lambda)\hat{p}_N(-\lambda))$$

Evidently, that if $\lambda$ is root of $Q(\lambda)$, then $-\lambda$ also is a root. If it is remembered, that $Q(\lambda)$ is of 2N+3 degree we have:

$$Q(\lambda) = A\lambda \prod_{i=0}^{N} \{(\lambda - \lambda_k)(\lambda + \lambda_k)\} = A\lambda \prod_{i=0}^{N} (\lambda^2 - \lambda_k^2), \quad where \ \lambda_k \in C, k = \overline{0, N},$$

$$A - \text{ some real number.}$$

and to any of $\lambda_k$ corresponds a point of spectrum of $J_N$ matrix.

On the other hand, if $\lambda^2 \neq 0$ is eigenvalue of matrix $J_N$ with eigenvector $\vec{a}$:

$$\vec{a} = \begin{pmatrix} 1 \\ a_1 \\ a_2 \\ . \\ a_N \end{pmatrix} \in C^{N+1},$$

then there exist numbers $\alpha, \beta \in C$:

$$\alpha \hat{p}_1(\lambda) + \beta \hat{p}_1(-\lambda) = a_1$$

and therefore, from first (N-1) rows of equality (2.14) follows, that:

$$\alpha \hat{p}(\lambda) + \beta \hat{p}(-\lambda) = \vec{a}$$

and hence: $Q(\lambda) = 0$. (The case, when first element of $\vec{a}$ : $a_0 = 0$ is analogous). If $\lambda^2$ is double eigenvalue it is easy to see, that in this case fulfiles: $R_1(\lambda) = R_1(-\lambda) = R_2(\lambda) = R_2(-\lambda) = 0$ and hence $\lambda^2$ is double root of $Q(\lambda)$.

Special case $\lambda = 0$ is considered analogously.

Note, that fivediagonal matrix $J_N$ can't have more than double spectrum because of recurrence for components of eigenvector.

So, $\lambda_1, \ldots, \lambda_N$ is a set of eigenvalues (with remembered multiplicity) of $J_N$ and then:

$$Q(\lambda) = A\lambda X(\lambda^2), \ A \in R,$$

where $X(\lambda)$ is characteristic polynomial of matrix $J_N$.

Let $\lambda_0, \ldots, \lambda_p, p \leq \left[\frac{N}{2}\right]$ – be double roots of $Q(\lambda)$. Then a set of vectors

$$\vec{c}_i = \begin{pmatrix} 1 \\ \hat{p}_1(\lambda_i) \\ \hat{p}_2(\lambda_i) \\ . \\ \hat{p}_N(\lambda_i) \end{pmatrix}, \quad \vec{\tilde{c}}_i = \begin{pmatrix} 1 \\ \hat{p}_1(-\lambda_i) \\ \hat{p}_2(-\lambda_i) \\ . \\ \hat{p}_N(-\lambda_i) \end{pmatrix}, \ i = \overline{0, p}$$

form, generally, nonorthogonal basis in each of eagen two-dimensional subspaces, corresponding to eigenvalue $\lambda_i^2$ of matrix $J_N$. ($\lambda = 0$ can't be double eigenvalue, or then follows $J_N = 0$).

Next, let $\lambda_{p+1}, \ldots \lambda_l$ – prime roots of $Q(\lambda)$ ($2(p+1) + (l-p) = N+1$). Vectors

$$\vec{c}_i = \alpha(\lambda_i) \begin{pmatrix} 1 \\ \hat{p}_1(\lambda_i) \\ . \\ . \\ \hat{p}_N(\lambda_i) \end{pmatrix} + \beta(\lambda_i) \begin{pmatrix} 1 \\ \hat{p}_1(-\lambda_i) \\ . \\ . \\ \hat{p}_N(-\lambda_i) \end{pmatrix},$$

where

$$\alpha, \beta : \mid \alpha(\lambda_i) \mid^2 + \mid \beta(\lambda_i) \mid^2 > 0, \quad i = \overline{p+1, N}$$

corresponds to one-dimensional eigen subspaces of $J_N$ for eigenvalues $\lambda_i^2$.

Applying a known theorem of the linear algebra, we have:

$$H^{N+1} = \bigoplus_{i=0}^{p} Lin\{\vec{c}_i, \vec{\hat{c}}_i\} \oplus ( \bigoplus_{i=p+1}^{N} Lin\{\vec{c}_i\}),$$

where $H^{N+1}$ is space of complex vectors of dimension $N + 1$, i.e. $(N + 1)$-dimensional complex space is expanding into direct sum of symmetric matrix eigen subspaces.

Denote $E_i = Lin\{\vec{c}_i, \vec{\hat{c}}_i\}, i = \overline{0, p}; \quad E_i = Lin\{\vec{c}_i\}, i = \overline{p+1, l}$. Then

$$H^{N+1} = \bigoplus_{i=0}^{l} E_i.$$

Choose in every $E_i$ orthogonal basis:

$$\vec{u}_i = \frac{\vec{c}_i}{\parallel \vec{c}_i \parallel}, \ \vec{\hat{u}}_i = \frac{\vec{\hat{c}}_i - (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\parallel \vec{c}_i \parallel}) \frac{\vec{c}_i}{\parallel \vec{c}_i \parallel}}{\parallel \vec{\hat{c}}_i - (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\parallel \vec{c}_i \parallel}) \frac{\vec{c}_i}{\parallel \vec{c}_i \parallel} \parallel}, \ i = \overline{0, p}$$

$$\vec{u}_i = \frac{\vec{c}_i}{\parallel \vec{c}_i \parallel}, \ i = \overline{p+1, l}$$

For arbitrary vector $\vec{x} \in H^{N+1}$ is correct an expansion:

$$\vec{x} = \sum_{i=0}^{p} (\vec{x}, \vec{u}_i)\vec{u}_i + \sum_{i=0}^{p} (\vec{x}, \vec{\hat{u}}_i)\vec{\hat{u}}_i + \sum_{i=p+1}^{l} (\vec{x}, \vec{u}_i)\vec{u}_i$$

Let $\vec{y}_i \in H^{N+1}$, then for scalar product we have:

$$(\vec{x}, \vec{y}) = \sum_{i=0}^{p} (\vec{x}, \vec{u}_i)\overline{(\vec{y}, \vec{u}_i)} + \sum_{i=0}^{p} (\vec{x}, \vec{\hat{u}}_i)\overline{(\vec{y}, \vec{\hat{u}}_i)} + \sum_{i=p+1}^{l} (\vec{x}, \vec{u}_i)\overline{(\vec{y}, \vec{u}_i)}$$

Substituting expression for $\vec{u}_i, \vec{\hat{u}}_i$ into last equality:

$$(\vec{x}, \vec{y}) = \sum_{i=0}^{p} (\vec{x}, \frac{\vec{c}_i}{\parallel \vec{c}_i \parallel})\overline{(\vec{y}, \frac{\vec{c}_i}{\parallel \vec{c}_i \parallel}}\|\}+$$

$$+ \sum_{i=0}^{p} (\vec{x}, \frac{\vec{\hat{c}}_i - (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\|\vec{c}_i\|})\frac{\vec{c}_i}{\|\vec{c}_i\|}}{\parallel \vec{\hat{c}}_i - (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\|\vec{c}_i\|})\frac{\vec{c}_i}{\|\vec{c}_i\|} \parallel})\overline{(\vec{y}, \frac{\vec{\hat{c}}_i - (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\|\vec{c}_i\|})\frac{\vec{c}_i}{\|\vec{c}_i\|}}{\parallel \vec{\hat{c}}_i - (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\|\vec{c}_i\|})\frac{\vec{c}_i}{\|\vec{c}_i\|} \parallel}})+$$

$$+ \sum_{i=p+1}^{l} (\vec{x}, \frac{\vec{c}_i}{\parallel \vec{c}_i \parallel})\overline{(\vec{y}, \frac{\vec{c}_i}{\parallel \vec{c}_i \parallel})} = \sum_{i=0}^{p} \frac{1}{\parallel \vec{c}_i \parallel^2}(\vec{x}, \vec{c}_i)\overline{(\vec{y}, \vec{c}_i)}+$$

$$+ \sum_{i=0}^{p} \frac{1}{\parallel \vec{\hat{c}}_i - (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\|\vec{c}_i\|})\frac{\vec{c}_i}{\|\vec{c}_i\|} \parallel^2}\{(\vec{x}, \vec{\hat{c}}_i)\overline{(\vec{y}, \vec{\hat{c}}_i)} - (\vec{x}, \vec{\hat{c}}_i)\overline{(\vec{y}, (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\parallel \vec{c}_i \parallel})\frac{\vec{c}_i}{\parallel \vec{c}_i \parallel})}-$$

$$-(\vec{x}, (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\parallel \vec{c}_i \parallel})\frac{\vec{c}_i}{\parallel \vec{c}_i \parallel})\overline{(\vec{y}, \vec{\hat{c}}_i)} + (\vec{x}, (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\parallel \vec{c}_i \parallel})\frac{\vec{c}_i}{\parallel \vec{c}_i \parallel})\overline{(\vec{y}, (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\parallel \vec{c}_i \parallel})\frac{\vec{c}_i}{\parallel \vec{c}_i \parallel})}\}+$$

$$+ \sum_{i=p+1}^{l} \frac{1}{\parallel \vec{c}_i \parallel^2}(\vec{x}, \vec{c}_i)\overline{(\vec{y}, \vec{c}_i)} =$$

$$= \sum_{i=0}^{p} \{\frac{1}{\parallel \vec{c}_i \parallel^2} + \frac{1}{\parallel \vec{\hat{c}}_i - (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\|\vec{c}_i\|})\frac{\vec{c}_i}{\|\vec{c}_i\|} \parallel^2} \frac{1}{\parallel \vec{c}_i \parallel^2} \mid (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\parallel \vec{c}_i \parallel}) \mid^2\}(\vec{x}, \vec{c}_i)\overline{(\vec{y}, \vec{c}_i)}+$$

$$+ \sum_{i=0}^{p} \frac{1}{\parallel \vec{\hat{c}}_i - (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\|\vec{c}_i\|})\frac{\vec{c}_i}{\|\vec{c}_i\|} \parallel^2}$$

$$(\vec{x}, \vec{\hat{c}}_i)(\vec{y}, \vec{\hat{c}}_i) - \sum_{i=0}^{p} \frac{1}{\parallel \vec{\hat{c}}_i - (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\|\vec{c}_i\|})\frac{\vec{c}_i}{\|\vec{c}_i\|} \parallel^2} \left\{(\vec{\hat{c}}_i, \frac{\vec{c}_i}{\parallel \vec{c}_i \parallel})\frac{1}{\parallel \vec{c}_i \parallel}(\vec{x}, \vec{\hat{c}}_i)\overline{(\vec{y}, \vec{c}_i)}+\right.$$

$$\left.+\overline{(\vec{\hat{c}}_i, \frac{\vec{c}_i}{\parallel \vec{c}_i \parallel})}\frac{1}{\parallel \vec{c}_i \parallel}(\vec{x}, \vec{c}_i)(\vec{y}, \vec{\hat{c}}_i)\right\} + \sum_{i=p+1}^{l} \frac{1}{\parallel \vec{c}_i \parallel^2}(\vec{x}, \vec{c}_i)\overline{(\vec{y}, \vec{c}_i)} =$$

$$= \sum_{i=0}^{p} ((\vec{x}, \vec{c}_i), (\vec{x}, \vec{\hat{c}}_i)) \left( \begin{array}{cc} \frac{1}{\|\vec{c}_i\|^2}(1 + \frac{|(\vec{\hat{c}}_i, \vec{c}_i)|^2}{\|\vec{c}_i\|^2 \|\vec{\hat{c}}_i - (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\|\vec{c}_i\|})\frac{\vec{c}_i}{\|\vec{c}_i\|}\|^2}) & -\overline{\frac{(\vec{\hat{c}}_i, \vec{c}_i)}{\|\vec{c}_i\|^2 \|\vec{\hat{c}}_i - (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\|\vec{c}_i\|})\frac{\vec{c}_i}{\|\vec{c}_i\|}\|^2}} \\ -\frac{(\vec{\hat{c}}_i, \vec{c}_i)}{\|\vec{c}_i\|^2 \|\vec{\hat{c}}_i - (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\|\vec{c}_i\|})\frac{\vec{c}_i}{\|\vec{c}_i\|}\|^2} & \frac{1}{\|\vec{\hat{c}}_i - (\vec{\hat{c}}_i, \frac{\vec{c}_i}{\|\vec{c}_i\|})\frac{\vec{c}_i}{\|\vec{c}_i\|}\|^2} \end{array} \right) \times$$

$$\times \overline{\left( (\vec{y}, \vec{c}_i)(\vec{y}, \vec{\hat{c}}_i) \right)} + \sum_{i=p+1}^{l} \frac{1}{\| \vec{c}_i \|^2} (\vec{x}, \vec{c}_i) \overline{(\vec{y}, \vec{c}_i)}$$

Let $P^N$ be the space of polynomials of degree not greater than $N$. Construct mapping $f : H^{N+1} \to P^N$ as follows :

$$\vec{x} = (x_0, x_1, \ldots, x_n) \in H^{N+1} \xrightarrow{f} f(\vec{x})(\lambda) = \sum_{i=0}^{N} x_i \hat{p}_i(\lambda) \in P^N.$$

For linear span of polynomials $\hat{p}_i(\lambda)$ coincides with $P^N$, then it is mapping on $P^N$.

Using previous equality:

$$(\vec{x}, \vec{y})_{H^{N+1}} = \sum_{k=0}^{p} (f(\vec{x})(\lambda_k), f(\vec{x})(-\lambda_k)) \sigma_N(\lambda_k) \overline{\left( \begin{array}{c} f(\vec{y})(\lambda_k) \\ f(\vec{y})(-\lambda_k) \end{array} \right)} +$$

$$+ \sum_{k=p+1}^{l} (f(\vec{x})(\lambda_k), f(\vec{x})(-\lambda_k)) \hat{\sigma}_N(\lambda_k) \overline{\left( \begin{array}{c} f(\vec{y})(\lambda_k) \\ f(\vec{y})(-\lambda_k) \end{array} \right)},$$

where

$$\sigma_N(\lambda_k) = \left( \begin{array}{cc} \frac{1}{\|\vec{c}_k\|^2}(1 + \frac{|(\vec{\hat{c}}_k, \vec{c}_k)|^2}{\|\vec{c}_k\|^2 \|\vec{\hat{c}}_k - (\vec{\hat{c}}_k, \frac{\vec{c}_k}{\|\vec{c}_k\|})\frac{\vec{c}_k}{\|\vec{c}_k\|}\|^2}) & -\overline{\frac{(\vec{\hat{c}}_k, \vec{c}_k)}{\|\vec{c}_k\|^2 \|\vec{\hat{c}}_k - (\vec{\hat{c}}_k, \frac{\vec{c}_k}{\|\vec{c}_k\|})\frac{\vec{c}_k}{\|\vec{c}_k\|}\|^2}} \\ -\frac{(\vec{\hat{c}}_k, \vec{c}_k)}{\|\vec{c}_k\|^2 \|\vec{\hat{c}}_k - (\vec{\hat{c}}_k, \frac{\vec{c}_k}{\|\vec{c}_k\|})\frac{\vec{c}_k}{\|\vec{c}_k\|}\|^2} & \frac{1}{\|\vec{\hat{c}}_k - (\vec{\hat{c}}_k, \frac{\vec{c}_k}{\|\vec{c}_k\|})\frac{\vec{c}_k}{\|\vec{c}_k\|}\|^2} \end{array} \right)$$

$$\hat{\sigma}_N(\lambda_k) = \left( \begin{array}{cc} | \alpha(\lambda_k) |^2 & \alpha(\lambda_k)\overline{\beta(\lambda_k)} \\ \beta(\lambda_k)\overline{\alpha(\lambda_k)} & | \beta(\lambda_k) |^2 \end{array} \right)$$

(we used, that:

$$(\vec{x}, \vec{c}_k)\overline{(\vec{y}, \vec{c}_k)} =$$

$$= (\vec{x}, \alpha(\lambda_k)\hat{p}^{N+1}(\lambda_k) + \beta(\lambda_k)\hat{p}^{N+1}(-\lambda_k))\overline{(\vec{y}, \alpha(\lambda_k)\hat{p}^{N+1}(\lambda_k) + \beta(\lambda_k)\hat{p}^{N+1}(-\lambda_k))} =$$

$$= | \alpha(\lambda_k) |^2 (\vec{x}, \hat{p}^{N+1}(\lambda_k))\overline{(\vec{y}, \hat{p}^{N+1}(\lambda_k))} + \alpha(\lambda_k)\overline{\beta(\lambda_k)}(\vec{x}, \hat{p}^{N+1}(\lambda_k))\overline{(\vec{y}, \hat{p}^{N+1}(-\lambda_k))}$$

$$+\beta(\lambda_k)\overline{\alpha(\lambda_k)}(\vec{x},\hat{p}^{N+1}(-\lambda_k))\overline{(\vec{y},\hat{p}^{N+1}(\lambda_k))}+$$

$$+\mid\beta(\lambda_k)\mid^2(\vec{x},\hat{p}^{N+1}(-\lambda_k))\overline{(\vec{y},\hat{p}^{N+1}(-\lambda_k))}\quad k=\overline{p+1,l})$$

Preceeding equality we can rewrite in following way:

$$(\vec{x},\vec{y})_{H^{N+1}}=\int\limits_{R\cup T}(f(\vec{x})(\lambda),f(\vec{x})(-\lambda))d(\sigma'_N(\lambda)+\hat{\sigma}'_N(\lambda))\overline{\left(\begin{array}{c}f(\vec{y})(\lambda)\\f(\vec{y})(-\lambda)\end{array}\right)}$$

where $\sigma'_N(\lambda)$ is a piecewise constant matrix function $(2\times2)$ with jumps at points $\lambda_k$, $k=\overline{0,p}$:

$$\sigma'_N(\lambda_k+0)-\sigma'_N(\lambda_k-0)=\sigma_N(\lambda_k)\ ;\ \lambda_k-\ \text{real}$$

$$\sigma'_N(\lambda_k+i0)-\sigma'_N(\lambda_k-i0)=\sigma_N(\lambda_k)\ ;\ \lambda_k-\ \text{imaginary}.$$

$\hat{\sigma}'_N(\lambda)-$ piecewise constant matrix function $(2\times2)$ with jumps at points $\lambda_k$, $k=\overline{p+1,l}$:

$$\hat{\sigma}'_N(\lambda_k+0)-\hat{\sigma}'_N(\lambda_k-0)=\hat{\sigma}_N(\lambda_k)\ ;\ \lambda_k-\ \text{real}$$

$$\hat{\sigma}'_N(\lambda_k+0)-\hat{\sigma}'_N(\lambda_k-0)=\hat{\sigma}_N(\lambda_k)\ ;\ \lambda_k-\ \text{imaginary}.$$

$$\sigma'_N(-\infty)=\hat{\sigma}'_N(-\infty)=\sigma'_N(-i\infty)=\hat{\sigma}'_N(-i\infty)=0.$$

Consider next functional

$$\sigma_N(u,v)=\int\limits_{R\cup T}(u(\lambda),u(-\lambda))d\tilde{\sigma}_N(\lambda)\overline{\left(\begin{array}{c}v(\lambda)\\v(-\lambda)\end{array}\right)},$$

$$\tilde{\sigma}_N(\lambda)=\sigma'_N(\lambda)+\hat{\sigma}'_N(\lambda);$$

$$u,v\in L^2_{\tilde{\sigma}_N}=\{u(\lambda):\int\limits_{R\cup T}(u(\lambda),u(-\lambda))d\tilde{\sigma}_N(\lambda)\overline{\left(\begin{array}{c}u(\lambda)\\u(-\lambda)\end{array}\right)}<\infty\}\supset$$

$$Lin\{\hat{p}_0(\lambda),\ldots,\hat{p}_N(\lambda)\}$$

Since matrix-function $\tilde{\sigma}_N(\lambda)$ satisfies conditions (2.3) from statement of moments problem, we easily conclude, that for functional $\sigma_N(u,v)$ properties 1)–3) from note to assertion hold (where $L=L^2_{\tilde{\sigma}_N}$ in our case).
Next, because of:

$$\lambda^k=\sum_{i=0}^k\xi_i\hat{p}_i(\lambda),\quad k:0\le k\le N\ -\ fixed,\ \xi_k>0,\xi_i\in C,i=\overline{0,k-1}.$$

$$\lambda = \eta_0 \hat{p}_0(\lambda) + \eta_1 \hat{p}_1(\lambda), \quad \eta_1 > 0, \eta_0 \in C.$$

then

$$\sigma_N(\lambda^k, 1) = \sigma_N(\sum_{i=0}^{k} \xi_i \hat{p}_i(\lambda), \hat{p}_0(\lambda)) = \sum_{i=0}^{k} \xi_i \sigma_N(\hat{p}_i(\lambda), \hat{p}_0(\lambda)) = \xi_0 = \sigma(\lambda^k, 1) = s_k,$$

where $\sigma(u, v)$ - required functional:

$$\sigma(u, v) = \int\limits_{R \cup T} (u(\lambda), u(-\lambda)) d\tilde{\sigma}(\lambda) \overline{\left( \begin{array}{c} v(\lambda) \\ v(-\lambda) \end{array} \right)},$$

$\tilde{\sigma}(\lambda)$ – solution of moments problem, $u, v \in L^2(d\tilde{\sigma}(\lambda))$;

$$\sigma_N(\lambda^k, \lambda) = \sigma_N(\sum_{i=0}^{k} \xi_i \hat{p}_i(\lambda), \eta_1 \hat{p}_1(\lambda) + \eta_0 \hat{p}_0(\lambda)) = \sum_{i=0}^{k} \xi_i (\overline{\eta_1} \sigma_N(\hat{p}_i(\lambda), \hat{p}_1(\lambda)) +$$

$$+ \overline{\eta_0} \sigma_N(\hat{p}_i(\lambda), \hat{p}_0(\lambda))) = \xi_1 \overline{\eta_1} + \xi_0 \overline{\eta_0} = \sigma(\lambda^k, \lambda) = m_{k+1}.$$

That means, hence:

$$\sigma_N(\lambda^k, 1) = s_k,$$

$$\sigma_N(\lambda^k, \lambda) = m_{k+1}; \quad 0 \le k \le N.$$

Sequently, matrix function $\tilde{\sigma}_N$ satisfies condition (2.4) of moments problem for $0 \le k \le N$.

Observing that

$$\left( \tilde{\sigma}_N(\lambda)( \begin{array}{c} 1 \\ 1 \end{array} ), (1, 1) \right) \le \int\limits_{R \cup T} (1, 1) d\tilde{\sigma}_N(\lambda) \left( \begin{array}{c} 1 \\ 1 \end{array} \right) = s_0,$$

and following standart arguments for components of $\tilde{\sigma}_N(\lambda)$, based on a Theorems of E. Helly, see [6, Theorem 2, p. 420, Theorem 3,p. 422], see also [4], we choose a subsequence $\tilde{\sigma}_{N_k}$ and obtain a measure $\tilde{\sigma}(\lambda)$ that is a solution of our problem (2.3),(2.4). The proof is completed. □

**Note.** It is possible the generalization of the moments problem, that leads to polynomials on pencil of lines with centre at zero. Measure matrix will be of dimension $(n \times n)$, $n \ge 1$ and the orthogonality will be as follows:

$$\int\limits_{P} (p_k(\lambda), p_k(\lambda e), \dots, p_k(\lambda e^{n-1})) d\tilde{\sigma}(\lambda) \left( \begin{array}{c} p_l(\lambda) \\ p_l(\lambda e) \\ \dots \\ p_l(\lambda e^{n-1}) \end{array} \right) = \delta_{kl}, \quad k, l = \overline{0, \infty}$$

$$e - \text{ primitive } n\text{-th order root of unity.}$$

$$P = \{\lambda \in C : \lambda^n \in R\}$$

$\{p_k(\lambda)\}$ – system of polynomials, corresponding to $n$-diagonal symmetric matrix $J$. Also, it is of interest using of $\tilde{\sigma}(\lambda)$ in spectral problems of differential operators theory (analogue of: $\frac{d^2}{dt^2}u = Lu$, where $L$ – threediagonal [5]).

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