Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **24** (1998), 339-344

Serdica
Mathematical Journal

Institute of Mathematics
and Informatics, Bulgarian
Academy of Sciences

GENERALIZED PROBLEM OF STARLIKENESS FOR PRODUCTS OF p-VALENT STARLIKE FUNCTIONS[*]

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Communicated by D. Drasin

Abstract. We consider functions of the type $F(z) = z^p \prod_{j=1}^{n} \left[f_j(z)/z^p\right]^{a_j}$ where f_j are p-valent functions starlike of order α_j and a_j are complex numbers. The problem we solve is to find conditions for the centre and the radius of the disc $\{z : |z-\omega| < r\}$, contained in the unit disc $\{z : |z| < 1\}$ and containing the origin, so that its transformation by the function F be a domain starlike with respect to the origin.

For an integer $p \geq 1$ the functions of the form

$$f(z) = z^p + c_{p+1}z^{p+1} + \cdots$$

that are analytic in the unit disc $\mathcal{D} = \{z : |z| < 1\}$ and for which

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha,\ (0 \leq \alpha < p),\ z \in \mathcal{D},$$

[*]Partially supported by grant No. 433/94 NSF of the Ministry of Education and Science of the Republic of Bulgaria

1991 *Mathematics Subject Classification:* 30C45

Key words: univalent, $p-$valent, starlike of order α

are called p-valent functions starlike of order α. The usual notation for the set of these functions is $S_p^*(\alpha)$.

Let now $n \geq 1$ be an integer and $f_j \in S_p^*(\alpha_j),\ j = 1, 2, \ldots, n$. Denote by $\mathcal{F} = \mathcal{F}(p; \alpha_1, \ldots, \alpha_n; a_1, \ldots, a_n)$ the set of functions given by the formula

$$F(z) = z^p \prod_{j=1}^{n} \left[\frac{f_j(z)}{z^p}\right]^{a_j}, \tag{1}$$

where a_j are complex numbers and we chose the branch for which $1^{a_j} = 1$.

In [1] Alexandrov stated and solved the following problem. Let $\mathcal{M}$ be the set of functions of the form

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots$$

that are analytic and univalent in $\mathcal{D}$. Let $\mathcal{B} \subset \mathcal{D}$ be a domain starlike with respect to an inner point ω with smooth boundary given by the function $z(\varphi) = \omega + r(\varphi)\mathrm{e}^{i\varphi}$. To find conditions for the function $r(\varphi)$ such that for each $f \in \mathcal{M}$ the image domain $f(\mathcal{B})$ is starlike with respect to $f(\omega)$.

Here we state a similar problem.

Consider discs $\mathcal{K} = \mathcal{K}(\omega, r) = \{z : |z - \omega| < r\}$. Let $\mathcal{K} \subset \mathcal{D}$ and $0 \in \mathcal{K}$. It is clear *a priori* that

$$0 \leq |\omega| < \frac{1}{2} \quad \text{and} \quad |\omega| < r \leq 1 - |\omega|. \tag{2}$$

The aim of our studies is to find (if necessary) additional conditions for ω and r under which the disc $\mathcal{K}$ will be transformed by all functions in $\mathcal{F}$ onto a domain starlike with respect to the origin.

The shape of the image domain $F(\mathcal{K})$ doesn't depend on rotations of $\mathcal{D}$. Hence without loss of generality we may suppose that $\omega > 0$.

Since the set $\mathcal{F}$ is too large it is convenient to introduce the following exhaustion. Let $M > 0$.

$$\mathcal{F}(M) = \left\{F \in \mathcal{F} :\ \sum_{j=1}^{n} (p - \alpha_j)|a_j| \leq M\right\}.$$

Theorem. *Let the natural number $p \geq 1$ and $M > 0$ be fixed. If*

$$(3) \qquad 0 \leq \omega < \begin{cases} \frac{1}{4}, & \text{if } 0 < M \leq \frac{p}{2} \\ \frac{p}{2(2M+p)}, & \text{if } \frac{p}{2} \leq M \end{cases} \qquad \text{and} \qquad \omega < r \leq \frac{p}{2M+p} - \omega$$

the disc $\mathcal{K}$ is transformed by each function of the class $\mathcal{F}(M)$ onto a domain starlike with respect to the origin.

Proof. It is well known that for a function $F \in \mathcal{F}(M)$ the image domain $F(\mathcal{K})$ will be starlike with respect to the origin if

$$(4) \qquad \min_{|z-\omega|=r} \operatorname{Re}\left\{\frac{zF'(z)}{F(z)}\right\} \geq 0.$$

From (1) we have

$$\begin{aligned} \min_{|z-\omega|=r} \operatorname{Re}\left\{\frac{zF'(z)}{F(z)}\right\} &= p + \min_{|z-\omega|=r} \sum_{j=1}^{n} \operatorname{Re}\left\{a_j\left(\frac{zf_j'(z)}{f_j(z)} - p\right)\right\} \\ &\geq p + \sum_{j=1}^{n} \min_{|z-\omega|=r} \operatorname{Re}\left\{a_j\left(\frac{zf_j'(z)}{f_j(z)} - p\right)\right\}. \end{aligned}$$

Since $f_j \in S_p^*(\alpha_j)$,

$$a_j\left(\frac{zf_j'(z)}{f_j(z)} - p\right) \prec \frac{2(p-\alpha j)a_j z}{1-z}, \ |z| < 1.$$

By the subordination principle this yields

$$\left|a_j\left(\frac{zf_j'(z)}{f_j(z)} - p\right) - \frac{2(p-\alpha_j)a_j(\omega - \omega^2 + r^2)}{(1-\omega)^2 - r^2}\right| \leq \frac{2(p-\alpha_j)|a_j|r}{(1-\omega)^2 - r^2},$$

$$0 < |z - \omega| < 1 - \omega.$$

Hence

$$\min_{|z-\omega|=r} \operatorname{Re}\left\{a_j\left(\frac{zf_j'(z)}{f_j(z)} - p\right)\right\} \geq \frac{2(\omega - \omega^2 + r^2)(p-\alpha_j)\operatorname{Re}a_j}{(1-\omega)^2 - r^2} - \frac{2(p-\alpha_j)|a_j|r}{(1-\omega)^2 - r^2}.$$

Further we shall deal with a fibering of $\mathcal{F}(M)$. For $m \in (0, M]$

$$\mathcal{F}_m = \left\{F \in \mathcal{F}(M) : \sum_{j=1}^{n}(p-\alpha_j)|a_j| = m\right\}.$$

So $\mathcal{F}(M) = \bigcup_{m\in(0,M]} \mathcal{F}_m$. Now for a function $F \in \mathcal{F}_m$ we can write

$$\min_{|z-\omega|=r} \text{Re}\left\{\frac{zF'(z)}{F(z)}\right\} \geq \frac{(2\mu-p)r^2 - 2mr + (1-\omega)[2\omega\mu + (1-\omega)p]}{(1-\omega)^2 - r^2} \equiv U(r;\mu),$$

where $\mu = \sum_{j=1}^{n}(p-\alpha_j)\text{Re}a_j$. It is clear that $-m \leq \mu \leq m$.

In view of (4) and (2) we shall look for a solution of the equation $U(r;\mu) = 0$ lying in the interval $(\omega, 1-\omega]$. For the discriminant $\Delta(\mu) = m^2 - (1-\omega)(2\mu-p)[2\omega\mu + (1-\omega)p]$ of the numerator we have

$$\min_{-m\leq\mu\leq m} \Delta(\mu) = \Delta(m) = [(1-2\omega)m - (1-\omega)p]^2 \geq 0.$$

On the other hand

$$U_r'(r;\mu) = -2 \cdot \frac{mr^2 - 2\mu(1-\omega)r + m(1-\omega)^2}{[(1-\omega)^2 - r^2]^2}$$

and for the discriminant $\Delta_1(\mu)$ of its numerator we have

$$\Delta_1(\mu) = (\mu^2 - m^2)(1-\omega)^2 \leq 0, \text{ when } |\mu| \leq m.$$

It follows that $U_r'(r;\mu) < 0,\ r \neq \pm(1-\omega)$. Hence for $r \neq \pm(1-\omega)$ the function $U(r;\mu)$ is strictly decreasing and possesses two zeros

$$r^{\pm}(\mu) = \frac{m \pm \sqrt{\Delta(\mu)}}{2\mu - p} = \frac{(1-\omega)[2\omega\mu + (1-\omega)p]}{m \mp \sqrt{\Delta(\mu)}}.$$

It is easily seen that $r^-(\mu) \in (-(1-\omega),\ 1-\omega)$. Denoting $\mu_1 = -\frac{1-\omega}{\omega}\frac{p}{2}$ and $\mu_2 = \frac{p}{2}$ and using the Viète formulae we obtain

$$r^-(\mu) + r^+(\mu) = \frac{2m}{2\mu - p}\begin{cases} < 0, & \text{if} \quad \mu < \mu_2 \\ > 0, & \text{if} \quad \mu > \mu_2 \end{cases},$$

$$r^-(\mu).r^+(\mu) = \frac{(1-\omega)[2\omega\mu + (1-\omega)p]}{2\mu - p}\begin{cases} \geq 0, & \text{if} \quad \mu \leq \mu_1 \\ < 0, & \text{if} \quad \mu_1 < \mu < \mu_2 \\ > 0, & \text{if} \quad \mu > \mu_2. \end{cases}$$

We have to avoid the case $r^-(\mu) \leq 0$. Let $m \leq \mu_2$. Since $|\mu_1| > \mu_2$ it follows that $\mu > \mu_1$ and we have $r^-(\mu) > 0$. For $m > \mu_2$ we state the condition $\mu_1 < -m$

which yields $\omega < \dfrac{p}{(2m+p)}$. So for the purpose of our investigation we obtain

$$0 \le \omega < \begin{cases} \dfrac{1}{2}, & \text{if} \quad 0 < m \le \dfrac{p}{2} \\ \dfrac{p}{2m+p}, & \text{if} \quad \dfrac{p}{2} \le m. \end{cases}$$

To study the behavior of $r^-(\mu)$ we consider its derivative

$$\frac{d}{d\mu}\, r^-(\mu) =$$

$$= \frac{2(1-\omega)}{\left[m+\sqrt{\Delta(\mu)}\right]^2 \sqrt{\Delta(\mu)}} \left\{2\omega m[m+\sqrt{\Delta(\mu)}] + (1-\omega)p[2\omega\mu + (1-\omega)p]\right\}.$$

Because of the above restriction on ω we have $\dfrac{d}{d\mu}\, r^-(\mu) > 0$, i.e. $r^-(\mu)$ is an increasing function of μ. Hence for the radius of the disc $\mathcal{K}$, transformed by each function $F \in \mathcal{F}_m$ onto a domain starlike with respect to the origin we have the limitation

$$r \le r^-(-m) = \frac{p}{2m+p} - \omega.$$

In view of the *a priori* condition (2) we obtain

$$0 \le \omega < \begin{cases} \dfrac{1}{4}, & \text{if} \quad 0 < m \le \dfrac{p}{2} \\ \dfrac{p}{2(2m+p)}, & \text{if} \quad \dfrac{p}{2} \le m. \end{cases}$$

The quantity $r^-(-m)$ is a decreasing function of the parameter m, hence we obtain (3).

If we put $p = 1$ and $n = 1$ we obtain a result which contains the result of Świtoniak [3].

If we put $p = 1,\ \omega = 0$ we obtain some of the results of Dimkov [2].

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Received February 4, 1998