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EXTREMAL SOLUTIONS FOR A CLASS OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. We study, in Carathéodory assumptions, existence, continuation and continuous dependence of extremal solutions for an abstract and rather general class of hereditary differential equations. By some examples we prove that, unlike the nonfunctional case, solved Cauchy problems for hereditary differential equations may not have local extremal solutions.

1. Introduction. It is well known that the theory of differential inequalities finds application in the study of the uniqueness and the continuous dependence for solutions of ordinary and partial differential equations.

In the nonfunctional case several authors considered differential inequalities: we limit ourselves to quote to V. Lakshmikantham–S. Leela [24], J. Szarski [27], W. Walter [29], [30]. In these monographs estimates of solutions of differential inequalities are obtained by extremal solutions of the corresponding differential equations.

The wide theory of functional differential equations has given rise to a natural development of a functional differential inequalities theory, mainly in C^1- setting ([1], [2], [23], [28]). In order to develop an analogous theory for functional differential inequalities in Carathéodory assumptions, we start studying

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the properties of Carathéodory extremal solutions; in particular, in the present paper we study existence, continuation, and continuous dependence of extremal solutions for a class of hereditary differential equations introduced and studied in [10].

As remarked in [10], let us point out that this hereditary structure is rather general and includes several previous formulations as particular case ([14], [15], [16], [17], [18], [19], [20], [21], [22]).

Let E be a closed, connected subset of $\mathbb{R}$ and let $C(E)$ be the space of all continuous functions mapping E into $\mathbb{R}$, with the compact-open topology. Let $\mathcal{C}$ be the set of all closed nonempty subsets of E. Let $\alpha : E \to \mathcal{C}$ denote a lag function that maps every $t \in E$ into a closed set $\alpha(t) \subset (-\infty, t] \cap E$. Given a Carathéodory function $g : E \times C(E) \to \mathbb{R}$ with the Volterra property (with respect to α)

(v_α)

for every $x, y \in C(E)$ and $t \in E$, if $x\mid_{\alpha(t)} = y\mid_{\alpha(t)}$ then $g(t,x) = g(t,y)$,

we study the functional differential equation

$$(*) \qquad \dot{x} = g(t,x)$$

where $t \in E$, $x \in C(E)$.
The hereditary structure does not explicitly appear in the formulation of $(*)$, but it is a consequence of the Volterra property assumed on the function g. The generality of the hereditary structure depends, in addition to the particularly abstract formulation of $(*)$, on the definition of the lag function. We assume that α is not necessarily continuous, nor a compact neither a connected valued map. Such a lag function was introduced and studied in [3], where its generality was illustrated by examples and references ([16], [17], [19], [20], [22]).

About the existence of extremal solutions for the initial value problem related to $(*)$, it should be emphasized that, unlike the nonfunctional case, solved Cauchy problems related to hereditary differential equations may not have extremal solutions, as we shall prove with the Example 5.4.

We prove the existence of the Carathéodory maximal (minimal) solution under the hypothesis, property $(+)$, that the maximum (minimum) between two solutions is again a solution. This property is naturally verified in the nonfunctional case; moreover it also holds, for example, when the past $\alpha(t)$ is contained in $(-\infty, t_0]$ for every $t \in [t_0, t_0 + p_0]$ or when g is monotone with respect to the functional argument. The Example 5.5 proves that the property $(+)$ can be verified even if the function g is not monotone.

The continuous dependence result for the extremal solutions is proved assuming an additional condition, property $(++)$, which is immediately verified both in the nonfunctional case and in monotonicity hypothesis on function g.

Examples 6.10, 6.11 show that the monotonicity condition with respect to the functional argument is strictly stronger than condition (++), even if both of them are not necessary for the continuous dependence of extremal solutions.

2. Notations and definitions. Given a closed, connected subset E of $\mathbb{R}$, let $C(E)$ be the space of all continuous functions mapping E into $\mathbb{R}$, with the compact open topology. Let d indicate a metric that provides such a topology in $C(E)$ (cf. [26], page 31).

Let $\mathcal{C}$ be the set of all closed, nonempty subsets of E. Let $\alpha : E \to \mathcal{C}$ be a set valued function nonnecessarily continuous, which maps every $t \in E$ into a closed set $\alpha(t) \subset (-\infty, t] \cap E$ of the class $\mathcal{C}$. Let $g : E \times C(E) \to \mathbb{R}$ be a function with the Volterra property (with respect to α)

(v_α) for every $x, y \in C(E)$ and $t \in E$, if $x \mid_{\alpha(t)} = y \mid_{\alpha(t)}$ then $g(t,x) = g(t,y)$.

We consider the following differential equation

$$(*) \qquad \dot{x}(t) = g(t,x)$$

where $t \in E$, $x \in C(E)$. The property (v_α) insure that $(*)$ is a functional differential equation.

Given $x \in C(E)$ and $\Omega \in \mathcal{C}$, let $\Gamma(x,\Omega) = \{(t,x(t)) \in \mathbb{R}^2 : t \in \Omega\}$ be the graph of the restriction of x to Ω and let $G = \{\Gamma(x,\Omega) : x \in C(E),\ \Omega \in \mathcal{C}\}$ be the set of all graphs.

Endowed the set G with a natural topology (see Section 3), let $\mathcal{Z}$ be the family of all continuous functionals $\mathcal{X} : G \to C(E)$ with the property $\mathcal{X}(\Gamma(x,\Omega)) \mid_\Omega = x \mid_\Omega$ for every $x \in C(E)$, $\Omega \in \mathcal{C}$ and, for every Ω upper bounded, $\mathcal{X}(\Gamma(x,\Omega))(t) = x(\widehat{t})$ for every $t \geq \widehat{t}$, where $\widehat{t} = \sup \Omega$. This family is not empty, as it was proved in [4] (see also Section 3).

Let us denote with $\Pi_E(\cdot)$ and $\Pi_{C(E)}(\cdot)$ the standard projections of the product topology in $E \times C(E)$ into E and $C(E)$, respectively.

Given an open subset $\mathcal{W} \subset E \times C(E)$, let $t_0 \in \Pi_E(\mathcal{W})$ and let p_0 be a fixed positive number such that $[t_0, t_0 + p_0] \subset \Pi_E(\mathcal{W})$. We put

$$I_{(t_0,p_0)} = cl \left(\bigcup_{t_0 \leq t \leq t_0 + p_0} (\alpha(t) \cap (-\infty, t_0]) \right) \cup \{t_0\}.$$

A pair (t_0, ϕ_0) with $t_0 \in \Pi_E(\mathcal{W})$ and $\phi_0 : I_{(t_0,p_0)} \to \mathbb{R}$ continuous function, is called admissible data with respect to $\mathcal{W}$ and α ($(\mathcal{W},\alpha)$-admissible) if there is a functional $\mathcal{X} \in \mathcal{Z}$ such that $(t_0, \mathcal{X}(\Gamma(\phi_0, I_{(t_0,p_0)}))) \in \mathcal{W}$.

Given a $(\mathcal{W},\alpha)$-admissible pair (t_0, ϕ_0), let $\mathcal{Z}_0 \subset \mathcal{Z}$ be the subset of all functionals $\mathcal{X}$ such that $(t_0, \mathcal{X}(\Gamma(\phi_0, I_{(t_0,p_0)}))) \in \mathcal{W}$. Let $g : \mathcal{W} \to \mathbb{R}$ be a function with the Volterra property (v_α).

We consider the following Cauchy problem

$$(\theta) \quad \begin{cases} (\theta_1) & \dot{x}(t) = g(t,x) & \text{a.e. in } [t_0, t_0+p_0] \\ (\theta_2) & x(t) = \phi_0(t) & \text{in } I_{(t_0,p_0)}. \end{cases}$$

Given a $(\mathcal{W},\alpha)$-admissible pair (t_0,ϕ_0), the function $x = x(t_0,\phi_0)$ is said to be a solution of the problem (θ) in the extended sense [in the classic sense] if there is a positive constant p, $p \le p_0$ such that x is defined and continuous on $I_{(t_0,p)} \cup [t_0, t_0+p]$, absolutely continuous on $[t_0,t_0+p]$, coincides with ϕ_0 on $I_{(t_0,p)}$ and there is a $\widehat{\mathcal{X}} \in \mathcal{Z}_0$ such that, almost everywhere [everywhere] in $[t_0,t_0+p]$ $\;(t, \Upsilon_{\widehat{\mathcal{X}}}x) \in \mathcal{W}$ and $\Upsilon_{\widehat{\mathcal{X}}}x$ satisfies the equation (θ_1), where we put

$$\Upsilon_{\widehat{\mathcal{X}}}x = \begin{cases} \widehat{\mathcal{X}}(\Gamma(\phi_0, I_{(t_0,p_0)})), & \text{in } (-\infty, t_0] \cap E \\ x & \text{in } [t_0,t_0+p] \cap E \\ x(t_0+p) & \text{in } [t_0+p, +\infty). \end{cases}$$

Remark 1. Let us point out that the Volterra property assumed on g makes this definition independent of the choice of $\widehat{\mathcal{X}}$ in $\mathcal{Z}_0$.

Indeed, let $x \in C(E)$; if there is a $\widehat{\mathcal{X}} \in \mathcal{Z}_0$ such that

$$[t_0, t_0+p_0] \times \{\Upsilon_{\widehat{\mathcal{X}}}x\} \subset W,$$

then g can be extended to the set

$$\tilde{W} = W \cup \{(t, \Upsilon_{\mathcal{X}}x) : t \in [t_0,t_0+p_0], \mathcal{X} \in \mathcal{Z}_0\}$$

by defining $g(t,\Upsilon_{\mathcal{X}}x) = g(t, \Upsilon_{\widehat{\mathcal{X}}}x)$.
This means that if there is $0 < p \le p_0$ and $\widehat{\mathcal{X}} \in \mathcal{Z}_0$ such that $[t_0,t_0+p] \times \{\Upsilon_{\widehat{\mathcal{X}}}x\} \subset W$ and $\Upsilon_{\widehat{\mathcal{X}}}x$ satisfies the equation (θ_1), then $[t_0,t_0+p] \times \{\Upsilon_{\mathcal{X}}x\} \subset \tilde{W}$ and $\Upsilon_{\mathcal{X}}x$ satisfies the equation (θ_1) for every $\mathcal{X} \in \mathcal{Z}_0$.

As it was remarked in [3], [10], we observe that $I_{(t_0,p_0)}$ contains the set of real numbers on which an initial data must be defined in order to integrate the problem (θ) on $[t_0,t_0+p_0]$.

We shall study the Cauchy problem (θ) for functions g satisfying the Volterra property (v_α) and the Carathéodory condition (c).

A function $g(t,z) : \mathcal{W} \to \mathbb{R}$ is said to satisfy the property (c) if
(c_1) is measurable in t for each fixed z and continuous in z for each fixed t,
(c_2) for any fixed $(t,z) \in \mathcal{W}$, there is a neighborhood $\mathcal{V}$ of (t,z) and a Lebesgue integrable function $m(s)$ such that $\mid g(s,\psi) \mid \le m(s)$ for every $(s,\psi) \in \mathcal{V}$.

A set of functions $\mathcal{G}$ is said to verify the property (c) if each function g of $\mathcal{G}$ satisfies the property (c), and (c_2) holds uniformly with respect to g.

It is immediately verified that, if $g : \mathcal{W} \to \mathbb{R}$ satisfies the property (c) on $\mathcal{W}$, the problem (θ) is equivalent to the following integral equation

$$(\widehat{\theta}) \qquad \begin{aligned} & x(t) = \phi_0(t_0) + \int_{t_0}^{t} g(s,x)ds && \text{in} \quad [t_0, t_0 + p_0] \\ & x(t) = \phi_0(t) && \text{in} \quad I_{(t_0,p_0)}. \end{aligned}$$

The lag function α is not necessarily continuous, nor a compact neither a connected valued map. Such a lag function was introduced and studied in [3], where its generality was illustrated by examples and references (we limit ourselves to refer to [16], [17], [19], [20], [22]).

For problem (θ), existence, continuation, and continuous dependence for extremal solutions are given. The existence theorem is a functional version of a classical result (see theorem 1.2 of [13]); the continuous dependence proof follows the outline of the Theorem 3 of [10].

In [8], [9] the authors proved existence and continuous dependence of extremal solutions, using the method of the under-solutions and over-solutions.

3. A topology on *G*. As in [4], we endow the set $G=\{\Gamma(x,\Omega){:}x{\in}C(E), \Omega{\in}\mathcal{C}\}$ with a topology which represents a localization on compact sets of the Hausdorff metric topology.

Given two elements $\Gamma(x,\Omega), \Gamma(y,\Delta)$ in G and a compact subset K of E, we consider the quasi-distance

$$\rho_K(\Gamma(x,\Omega),\Gamma(y,\Delta)) = \max\{e(\Gamma(x,\Omega\cap K),\Gamma(y,\Delta)), e(\Gamma(y,\Delta\cap K),\Gamma(x,\Omega))\},$$

where $e(A,B) = \sup_{a\in A}\inf_{b\in B}|a-b|$, A, B subsets of $\mathbb{R}^2$, denotes the excess of A over B, with the convention $e(\emptyset, B) = 0$ and $e(A,\emptyset) = +\infty$ if $A \neq \emptyset$.

Definition 3.1. *A sequence $(\Gamma(x_m,\Omega_m))_m \subset G$, is said to be τ-convergent to $\Gamma(x_0,\Omega_0)$ if for every compact subset K of E, the numerical sequence $(\rho_K(\Gamma(x_0,\Omega_0),\Gamma(x_m,\Omega_m)))_m$ converges to* 0.

The space G with the topology induced by the τ-convergence will be denoted by (G,τ).

In [4] the τ-topology in G is studied. Particularly, the connections between this latter, the compact-open, the Attouch-Wets, the Kuratowski, and the graph topology are given.

The τ-topology induces a natural topology on $\mathcal{C}$ (still denoted by τ) which coincides with the topology of the uniform convergence of distance functionals on bounded sets of $\mathbb{R}$.

In [4] the authors proved that the topological space (G, τ) is homeomorphic to the quotient space $[(\mathcal{C}, \tau) \times C(E)]/R$ with respect to the equivalent relation R defined as follows:

$$(\Omega_1, x)R(\Omega_2, y) \quad \text{provided} \quad \Omega_1 = \Omega_2 \quad \text{and} \quad x_{|\Omega_1} = y_{|\Omega_2}.$$

The homeomorphic property of the τ-topology has a great relevance in the study of functional differential equations of type $(*)$. In fact, in [10] this homeomorphism allowed us to get existence, uniqueness, and continuous dependence of the solutions of problem (θ) in (G, τ) by means of a classical fixed point theorem applied to the homeomorphic functional space; here, we apply this same procedure dealing with extremal solutions.

The fundamental tool for proving the homeomorphic result is the existence of continuous functionals $\mathcal{X} \in \mathcal{Z}$ from the topological space (G, τ) into $C(E)$. An example is provided by the 'linear extension' $\mathcal{X}_\ell : G \to C(E)$. Precisely, for every $\Omega \in \mathcal{C}$ let (a, b) be the smallest closed interval, bounded or not, containing Ω and let (a_i, b_i), $i \in \mathbb{N}$ be the open intervals whose union is the complement of Ω in (a, b). For every continuous function $x : \Omega \to \mathbb{R}$, let us consider the linear extension, say $\mathcal{X}_\ell(\Gamma(x, \Omega)) : E \to \mathbb{R}$, obtained by putting $\mathcal{X}_\ell(\Gamma(x, \Omega))(t) = x(a)$ for $t \in (-\infty, a) \cap E$, $\mathcal{X}_\ell(\Gamma(x, \Omega))(t) = x(b)$ for $t \in (b, +\infty) \cap E$ and $\mathcal{X}_\ell(\Gamma(x, \Omega))(t) = x(a_i)$ if $t \in [a_i, \overline{a}_i]$, $\mathcal{X}_\ell(\Gamma(x, \Omega))(t) = x(b_i)$ if $t \in [\overline{b}_i, b_i]$, linear in the interval $[\overline{a}_i, \overline{b}_i]$, where $\overline{a}_i = a_i + \dfrac{b_i - a_i}{3}$, $\overline{b}_i = b_i - \dfrac{b_i - a_i}{3}$.

In particular, we use the following proposition (see [4]).

Proposition 1. *If $(x_n)_n$ is a sequence of functions in $C(E)$ converging to x_0 and $(\Omega_n)_n$ is a sequence in $\mathcal{C}$ τ-converging to Ω_0, then $(\Gamma(x_n, \Omega_n))_n \to \Gamma(x_0, \Omega_0)$ with respect to the τ-topology in G.*

4. Continuation result. Let $\mathcal{W}$ be an open subset of $E \times C(E)$ and let p_0 be a positive constant such that $[t_0, t_0 + p_0] \subset \Pi_E(\mathcal{W})$. Suppose that $g : \mathcal{W} \to \mathbb{R}$ satisfies property (v_α) and (c), and let (t_0, ϕ_0) be a given $(\mathcal{W}, \alpha)$-admissible pair. By virtue of property (c_2), in correspondence with the point $(t_0, \mathcal{X}(\Gamma(\phi_0, I_{(t_0,p_0)})))$ there exists a positive constant not greater than p_0, that we again denote p_0, and a Lebesgue integrable function $m(\cdot)$ such that the ball

$$B(t_0, \phi_0) = B((t_0, \mathcal{X}(\Gamma(\phi_0, I_{(t_0,p_0)}))); p_0) \subset \mathcal{W}$$

and

$$|g(t, \psi)| \le m(t) \quad \text{for every} \quad (t, \psi) \in (t_0, \phi_0). \tag{4.1}$$

Moreover, let $p \leq p_0$ be such that

$$\int_{t_0}^{t+p} m(s)\ ds \leq p_0. \tag{4.2}$$

4.a. Continuation of the solution for a particular lag function. In this section we consider a particular definition of the lag function. For the sake of clarity, we denote this particular lag function by $\tilde{\alpha}$.

Let $\tilde{\alpha} : E \to \mathcal{C}$ be the set valued map defined by $\tilde{\alpha}(t) = (-\infty, t] \cap E$. So, given $t_0 \in E$ and $p_0 \in \mathbb{R}^+$, $I_{(t_0,p_0)} = (-\infty, t_0] \cap E = \tilde{\alpha}(t_0)$. Moreover, a pair (t_0, ϕ_0) with $t_0 \in \Pi_E(\mathcal{W})$ and $\phi_0 : \tilde{\alpha}(t_0) \to \mathbb{R}$ continuous function, is called admissible data with respect to $\mathcal{W}$ and $\tilde{\alpha}$ ($(\mathcal{W}, \tilde{\alpha})$-admissible) if there is $\mathcal{X} \in \mathcal{Z}$ such that $(t_0, \mathcal{X}(\Gamma(\phi_0, \tilde{\alpha}(t_0)))) \in \mathcal{W}$. We note that, by virtue of the definition of $\tilde{\alpha}$ and the property of family $\mathcal{Z}$, all the functionals of $\mathcal{Z}$ map $\Gamma(\phi_0, \tilde{\alpha}(t_0))$ into the same element of $C(E)$. So given a $(\mathcal{W}, \tilde{\alpha})$-admissible pair (t_0, ϕ_0), we have $\mathcal{Z}_0 = \mathcal{Z}$. Therefore, in the present Section 4, $\mathcal{X}$ will denote an arbitrary element of $\mathcal{Z}$. Finally, we denote by $(\tilde{\theta})$ the Cauchy problem related to $\tilde{\alpha}$. Moreover, in this setting for every $x \in C(E)$ we have $\Upsilon_{\mathcal{X}} x = \mathcal{X}(\Gamma(x, \tilde{\alpha}(t_0 + p_0)))$.

Theorem 4.1 (continuation)**.** *Suppose that $g : \mathcal{W} \to \mathbb{R}$ satisfies properties $(v_{\tilde{\alpha}})$ and (c). If x_0 is a solution of $(\tilde{\theta})$ on $[t_0, t_0 + \beta_0]$ passing through (t_0, ϕ_0), with $\beta_0 < p$, then the solution x_0 can be extended to the whole interval $[t_0, t_0 + p]$, i.e. there exists a solution $\tilde{x}$ of $(\tilde{\theta})$ passing through (t_0, ϕ_0) on $[t_0, t_0 + p]$ such that $\tilde{x}(t) = x_0(t)$ for every $t \in \tilde{\alpha}(t_0) \cup [t_0, t_0 + \beta_0]$.*

Proof. Let $T \subset [\beta_0, p] \times G$ be the set defined by:

$(t, \Gamma(x, \Omega)) \in T$ iff

(a_1) $\Omega = I_{(t_0,t)} \cup [t_0, t_0 + t]$,
(a_2) x is a continuation of x_0 in Ω,
(a_3) for every $(s, \Gamma(y, [t_0, t_0 + s])) \in T$, with $s < t$, we have that x is a continuation

of y in Ω.

By definition $(\beta_0, \Gamma(x_0, \Omega)) \in T$.

Let $\overline{p} = \sup\{t : (t, \Gamma(x, \Omega)) \in T$, for some $\Gamma(x, \Omega) \in G\}$.

Let $((t_n, \Gamma(x_n, \Omega_n)))_n \in T$ be a sequence such that $(t_n)_n$ converges to $\overline{p}$. We define a continuous function $\tilde{x}$ on $[t_0, t_0 + \overline{p}[$ by

$$\tilde{x}(t) = x_n(t), \quad \text{if } t \in \Omega_n, n \geq 1.$$

Now, for any two points $t_1 < t_2$ in $[t_0, t_0 + \overline{p}[$ it follows that $|\tilde{x}(t_1) - \tilde{x}(t_2)| \leq \int_{t_1}^{t_2} m(s)\, ds$.

This implies, by the Cauchy theorem for convergence, that $\tilde{x}$ can be extended with continuity to $[t_0, t_0+\overline{p}]$. We shall show that $\tilde{x}$ is a solution of problem $(\tilde{\theta})$ passing through (t_0, ϕ_0) on $[t_0, t_0+\overline{p}]$.

It is sufficient to prove $\tilde{x}(t_0+\overline{p}) = \phi_0(t_0) + \int\limits_{t_0}^{t_0+\overline{p}} g(s, \mathcal{X}(\Gamma(\tilde{x}, \tilde{\alpha}(t_0+\overline{p})))) \, ds$.

Indeed, by definition, we have

$$\tilde{x}(t_0+\lambda_n) = x_{n-1}(t_0+\lambda_{n-1}) + \int_{t_0+\lambda_{n-1}}^{t_0+\lambda_n} g(s, \mathcal{X}(\Gamma(x_n, \tilde{\alpha}(t_0+\lambda_n)))) \, ds, \tag{4.3}$$

and by means of the property $(v_{\tilde{\alpha}})$, we can rewrite (4.3) into the form

$$\tilde{x}(t_0+\lambda_n) = \phi_0(t_0) + \int_{t_0}^{t_0+\lambda_n} g(s, \mathcal{X}(\Gamma(\tilde{x}, \tilde{\alpha}(t_0+\lambda_n)))) \, ds. \tag{4.4}$$

By construction, the first term of (4.4) converges to $\tilde{x}(t_0+\overline{p})$ and, by (4.2) and the Lebesgue dominate convergence theorem, the second term of (4.4) converges to $\int\limits_{t_0}^{t_0+\overline{p}} g(s, \mathcal{X}(\Gamma(\tilde{x}, \tilde{\alpha}(t_0+\overline{p})))) \, ds$ as n diverges.

This proves that $(\overline{p}, \Gamma(\tilde{x}, I_{(t_0,t_0+\overline{p})} \cup [t_0, t_0+\overline{p}])) \in T$. Now assume that $\overline{p} < p$. Since $(t_0+\overline{p}, \tilde{x})$ is an admissible pair for problem $(\tilde{\theta})$ on $[t_0+\overline{p}, t_0+p_0]$, in light of the existence Theorem 2 of [10], there is a solution x of $(\tilde{\theta})$ on $[t_0+\overline{p}, t_0+\overline{p}+\epsilon]$ with $\epsilon > 0$, passing through $(t_0+\overline{p}, \tilde{x})$.

Thus, $(t_0+\overline{p}+\epsilon, \Gamma(x, [t_0, t_0+\overline{p}+\epsilon])) \in T$, and we get a contradiction. □

4.b. General continuation result. In this section we will deal with the general formulation of lag function α. We observe that if (t_0, ϕ_0) is $(\mathcal{W}, \alpha)$-admissible, then $(t_0, \mathcal{X}(\Gamma(\phi_0, I_{(t_0,p)}))|_{\tilde{\alpha}(t_0)})$ is $(\mathcal{W}, \tilde{\alpha})$-admissible for every $\mathcal{X} \in \mathcal{Z}_0$.

Theorem 4.2. *Suppose that $g : \mathcal{W} \to \mathbb{R}$ satisfies properties (v_α) and (c). If x is a solution of (θ) passing through (t_0, ϕ_0) on $[t_0, t_0+\beta_0]$ with $\beta_0 < p$, then the solution x can be extended to the whole interval $[t_0, t_0+p]$.*

Proof. For any $\mathcal{X} \in \mathcal{Z}_0$, the function

$$x_0(t) = \begin{cases} \mathcal{X}(\Gamma(\phi_0, I_{(t_0,p)}))(t), & \text{if } t \in \tilde{\alpha}(t_0); \\ x(t), & \text{if } t \in [t_0, t_0+\beta_0] \end{cases}$$

is a solution of $(\tilde{\theta})$ on $[t_0, t_0+\beta_0]$ passing through $(t_0, \mathcal{X}(\Gamma(\phi_0, I_{(t_0,p)}))|_{\tilde{\alpha}(t_0)})$.

By virtue of Theorem 4.1, there exists a solution $\tilde{x}$ of $(\tilde{\theta})$ on $[t_0, t_0+p]$ passing through $(t_0, \mathcal{X}(\Gamma(\phi_0, I_{(t_0,p)})))$ such that $\tilde{x} = x_0$ on $\tilde{\alpha}(t_0+\beta_0)$.

It is trivial to verify that $\tilde{x}$ is also a solution of (θ) on $[t_0, t_0+p]$ passing through (t_0, ϕ_0) with $\tilde{x} = x$ on $I_{(t_0,\beta_0)} \cup [t_0, t_0+\beta_0]$. $\square$

5. Existence of extremal solutions.

Definition 5.1. *If $x_M(x_m)$ is a solution of (θ) passing through (t_0, ϕ_0), existing on some interval $[t_0, t_0+p_0]$, with the property that, every other solution x of (θ) passing through (t_0, ϕ_0) and existing on $[t_0, t_0+p]$, with $p \le p_0$, is such that $x(t) \le x_M(t)$ $(x(t) \ge x_m(t))$ in $[t_0, t_0+p]$, then x_M (x_m) is called a maximal (minimal) solution of (θ) on the interval $[t_0, t_0+p_0]$ passing through (t_0, ϕ_0).*

Clearly, the functions x_M and x_m, if they exist, must be unique.

About the existence, it should be remarked that, unlike the nonfunctional case, solved Cauchy problem related to hereditary differential equations may not have extremal solutions (see Example 5.4). The existence of the extremal solutions for problem (θ) has been proved under the further following condition:

$$(+) \quad \begin{array}{l} \text{if } x, y \text{ are two solutions of problem } (\theta) \text{ in } [t_0, t_0+p_0], \text{ then} \\ (x \vee y)(t) = \max\{x(t), y(t)\}, (x \wedge y)(t) = \min\{x(t), y(t)\}, t \in [t_0, t_0+p_0] \\ \text{is also a solution of problem}(\theta)\text{in}[t_0, t_0+p_0]. \end{array}$$

Remark 5.2. In the nonfunctional case property $(+)$ is naturally verified. Moreover property $(+)$ holds in the following remarkable cases:

$$\alpha(t) \subset (-\infty, t_0], \quad \text{for every} \quad t \in [t_0, t_0+p_0]; \tag{5.1}$$

$$g \text{ is monotone with respect to the functional argument.} \tag{5.2}$$

The following result is a functional version of Theorem 1.2 of [13].

Theorem 5.3. *Let $g : \mathcal{W} \to \mathbb{R}$ be a function satisfying the properties (c), and (v_α). If problem (θ) verifies property $(+)$ then, for every $(\mathcal{W}, \alpha)$-admissible pair (t_0, ϕ_0), $t_0 \in \Pi_E(\mathcal{W})$, $\phi_0 : I_{(t_0,p)} \to \mathbb{R}$, there exist the extremal solutions x_m, x_M of problem (θ) on $[t_0, t_0+p]$ passing through (t_0, ϕ_0).*

Proof. We prove only the existence of x_M; analogously one can prove the existence of x_m.

By the continuation theorem 4.2, all the solutions $\mathcal{F} = \{x\}$ of (θ) passing through (t_0, ϕ_0) exist on $[t_0, t_0+p]$ and satisfy

$$|x(t_1) - x(t_2)| \le \left| \int_{t_1}^{t_2} m(s)\, ds \right| \qquad \text{for every} \quad t_1, t_2 \in [t_0, t_0+p]. \tag{5.3}$$

From (5.3) it follows that the set $\mathcal{F}$ is an equicontinuous set of functions. Further, from (5.3), putting $t_1 = t_0$, the set $\mathcal{F}$ is uniformly bounded on $[t_0, t_0+p]$.

Let Φ be the function defined by $\Phi(t) = \sup_{x\in\mathcal{F}}\{x(t)\}$. Clearly, Φ exists and is uniformly continuous on $[t_0, t_0 + p]$. It will be shown that $\Phi \in \mathcal{F}$, hence $\Phi = x_M$.

Fixed $\epsilon > 0$, let δ_ϵ be the positive number related to the equiuniform continuity of the family $\mathcal{F}$ and the uniform continuity of Φ.

Subdivide the interval $[t_0, t_0 + p]$ into n intervals by the points $t_0 = \tau_0 < \tau_1 < \cdots < \tau_n = t_0 + p$ in such a way that $\max(\tau_{i+1} - \tau_i) < \delta_\epsilon$.

For every $\tau_i, (i = 1, \ldots, n)$, choose a solution $x_i \in \mathcal{F}$ so that

$$0 \leq \Phi(\tau_i) - x_i(\tau_i) < \epsilon.$$

This is possible from the definition of Φ. Now, for the given ϵ, define the function x_ϵ as

$$x_\epsilon(t) = \max_{0<i\leq n} x_i(t) \qquad t \in [t_0, t_0 + p].$$

By property (+), x_ϵ is a solution of (θ) passing through (t_0, ϕ_0), and having the property

$$(5.4) \qquad 0 \leq \Phi(\tau_i) - x_\epsilon(\tau_i) < \epsilon \qquad (i = 0, 1, \ldots, n).$$

Since the variation of Φ and x_ϵ in each interval $[\tau_i, \tau_{i+1}]$ is less that ϵ, it results from (5.4)

$$(5.5) \qquad 0 \leq \Phi(t) - x_\epsilon(t) < 3\epsilon \qquad \text{for every } \ t \in [t_0, t_0 + p].$$

Letting $\epsilon = \dfrac{1}{k}(k \in N)$, one obtain a sequence x_k of solutions which converges uniformly, unless of passing to a subsequence, to Φ on $[t_0, t_0 + p]$.

By the integral representation of x_k and the Lebesgue dominate convergence theorem, we have

$$\Phi(t) = \phi_0(t_0) + \int_{t_0}^{t} g(s, \Phi)ds \qquad t \in [t_0, t_0 + p],$$

that is, Φ is a solution of (θ) passing through (t_0, ϕ_0). $\square$

Now, the Example 5.4 proves that the problem (θ) may not admit, in general, local extremal solutions if the property (+) is not satisfied; while the Example 5.5 shows that the property (+) is strictly weaker than the monotonicity condition with respect to the functional argument of g.

Example 5.4. Let $g : [0,1] \times C([0,1]) \to \mathbb{R}$ and $\alpha : [0,1] \to [0,1]$ be defined by

$$g(t,x) = \begin{cases} \sqrt{\left| x(t) - x\left(\frac{1}{5^n}\right) \right|}, & (t,x) \in \left[\frac{1}{5^n}, \frac{2}{5^n}\right[\times C([0,1]),\ n \in \mathbb{N}; \\ -\sqrt{\left| x\left(t - \frac{1}{5^n}\right) - x\left(\frac{1}{5^n}\right) \right|}, & (t,x) \in \left[\frac{2}{5^n}, \frac{3}{5^n}\right[\times C([0,1]),\ n \in \mathbb{N}; \\ -\sqrt{\left| x\left(t - \frac{2}{5^n}\right) - x\left(\frac{1}{5^n}\right) \right|}, & (t,x) \in \left[\frac{3}{5^n}, \frac{4}{5^n}\right[\times C([0,1]),\ n \in \mathbb{N}; \\ \sqrt{\left| x\left(t - \frac{3}{5^n}\right) - x\left(\frac{1}{5^n}\right) \right|}, & (t,x) \in \left[\frac{4}{5^n}, \frac{1}{5^{n-1}}\right[\times C([0,1]),\ n \in \mathbb{N}; \\ 0, & \text{otherwise,} \end{cases}$$

$$\alpha(t) = \begin{cases} 0, & \text{if } \ t = 0; \\ \left\{ t - \frac{i-1}{5^n}, \frac{1}{5^n} \right\}, & \text{if } t \in \left[\frac{i}{5^n}, \frac{i+1}{5^n}\right[, i \in \{1,2,3,4\}; n \in \mathbb{N}; \\ \frac{2}{5}, & \text{if } t = 1. \end{cases}$$

We consider the following Cauchy problem

$$(\theta) \qquad \begin{cases} \dot{x}(t) = g(t,x), & \text{a.e. in } [0,1]; \\ x(0) = 0; \end{cases}$$

Assume that x_M is the maximal solution of problem (θ) in $[0,\epsilon]$, $\epsilon > 0$. Let $n > 1$ be such that $\left[\frac{1}{5^n}, \frac{1}{5^{n-1}}\right] \subset [0,\epsilon]$. Put $x_0 = x_M\left(\frac{1}{5^n}\right)$, the expression of x_M in $\left[\frac{1}{5^n}, \frac{1}{5^{n-1}}\right]$ is forced to be the following

$$x_M(t) = \begin{cases} x_0 + \dfrac{(t - \frac{1}{5^n})^2}{4}, & \text{in } \left[\frac{1}{5^n}, \frac{2}{5^n}\right]; \\ x_0 + \dfrac{1}{4 \cdot 5^{2n}} - \dfrac{(t - \frac{2}{5^n})^2}{4}, & \text{in } \left[\frac{2}{5^n}, \frac{3}{5^n}\right]; \\ x_0 - \dfrac{(t - \frac{3}{5^n})^2}{4}, & \text{in } \left[\frac{3}{5^n}, \frac{4}{5^n}\right]; \\ x_0 - \dfrac{1}{4 \cdot 5^{2n}} + \dfrac{(t - \frac{4}{5^n})^2}{4}, & \text{in } \left[\frac{4}{5^n}, \frac{1}{5^{n-1}}\right]. \end{cases}$$

It follows that $x_M\left(\frac{1}{5^n}\right) = x_0$, for every $n \in \mathbb{N}$, with $\frac{1}{5^{n-1}} \leq \epsilon$.

Therefore, x_0 is forced to be equal zero. So, $x_M(t) < 0$ in $\left]\frac{3}{5^n}, \frac{1}{5^{n-1}}\right[$, for every $n \in \mathbb{N}^+$, with $\frac{1}{5^{n-1}} \leq \epsilon$.

This is a contradiction, because the function $x(t) \equiv 0$ on $[0,1]$ is also a solution of problem (θ).

By taking $-g(t,x)$ in place of $g(t,x)$, the same reasoning shows that there is not minimal solution for problem (θ).

Example 5.5. Let $g : [0,2] \times C([0,2]) \to \mathbb{R}$ and $\alpha : [0,2] \to [0,2]$ be defined by

$$g(t,x) = \begin{cases} \sqrt{|\ x(t)\ |}, & t \in [0,1[\times C([0,2]); \\ -\sqrt{|\ x(t-1) - x(0)\ |}, & t \in [1,2] \times C([0,2]), \end{cases}$$

$$\alpha(t) = \begin{cases} 0, & \text{if } \ 0 \leq t < 1; \\ \{0, t-1\}, & \text{if } \ t \in [1,2]. \end{cases}$$

We consider the following Cauchy problem

$$(\theta^*) \qquad \begin{cases} \dot{x}(t) = g(t,x), & \text{a.e. in } \ [0,2]; \\ x(0) = 0. \end{cases}$$

The function g clearly does not satisfy the monotonicity condition with respect to the functional argument, but the property (+) holds.

Indeed, every solution of (θ^*) is of the following form:

$$x_c(t) = \begin{cases} 0, & \text{if } \ t \in [0,c]; \\ \dfrac{(t-c)^2}{4}, & \text{if } \ t \in [c,1]; \\ \dfrac{(1-c)^2}{4}, & \text{if } \ t \in [1,1+c]; \\ \dfrac{(1-c)^2}{4} - \dfrac{(t-1-c)^2}{4}, & \text{if } \ t \in [1+c,2], \end{cases}$$

with $c \in [0,1]$.

Therefore, since the set of the solutions of (θ^*) is totally ordered with respect to the relation

$$x_1 \leq x_2 \ \text{ iff } \ x_1(t) \leq x_2(t) \qquad \text{for every } t \in [0,2]$$

then the property (+) is trivially satisfied.

6. Continuous dependence of extremal solutions. In this section we will deal only with maximal solutions, but analogous results hold also for minimal solutions.

The continuous dependence result for the extremal solutions of problem (θ) needs some additional conditions with respect to the hypotheses of Theorem 5.3. In particular, we assume the following condition.

Given two triples (t_1, ϕ_1, g_1) and (t_2, ϕ_2, g_2), let x_i, $i = 1, 2$, be the maximal solution of the Cauchy problem

$$(\theta_i) \qquad \begin{aligned} &\dot{x}_i(t) = g_i(t, x_i) && \text{a.e. in} \quad [t_i, t_i + p_0] \\ &x_i(t) = \phi_i(t) && \text{in} \quad I_{(t_i, p_0)}. \end{aligned}$$

We say that the triples (t_1, ϕ_1, g_1) and (t_2, ϕ_2, g_2) satisfy the property (++) if, under the conditions

(d.1) $\quad | \, t_1 - t_2 \, | < p_0$,

(d.2) $\quad g_1(t, x) < g_2(t, x)$ for every $(t, x) \in \mathcal{W}$,

(d.3) $\quad \mathcal{X}(\Gamma(\phi_1, I_{(t_1, p_0)}))(t) < \mathcal{X}(\Gamma(\phi_2, I_{(t_2, p_0)}))(t)$ for every $t \in E$,

(d.4) $\quad \phi_1(t_1) + \int_{t_1}^{t} g_1(s, x_1)\, ds < \phi_2(t)$ for every $t \in [t_1, t_2]$, if $t_1 < t_2$,

(d.5) $\quad \phi_2(t_2) + \int_{t_2}^{t} g_2(s, x_2)\, ds < \phi_1(t)$ for every $t \in [t_2, t_1]$, if $t_2 < t_1$,

then we have

$$\mathcal{X}(\Gamma(x_1, I_{(t_1, p_0)} \cup [t_1, t_1 + p_0]))(t) < \mathcal{X}(\Gamma(x_2, I_{(t_2, p_0)} \cup [t_2, t_2 + p_0]))(t)$$

for every $t \in (-\infty, t_1 \wedge t_2 + p_0]$.[1]

Remark 6.1. In the nonfunctional case property (++) is naturally verified.

Now, we prove that if g_1, g_2 are monotone with respect to the functional argument then property (++) is verified.

Indeed, by contradiction, let t^* be the smallest point in $(-\infty, t_1 \wedge t_2 + p_0]$ that verifies $x_1(t^*) = x_2(t^*)$. In light of conditions (d.3), (d.4) and (d.5) $t_1 \vee t_2 < t^*$, and let $\tilde{t}$ be such that $t_1 \vee t_2 < \tilde{t} < t^*$.

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[1]For every $a, b \in \mathbb{R}$, we denote $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

Therefore, in force of the monotonicity of g_1 and g_2, we get

$$x_2(t^*) = x_2(\tilde{t}) + \int_{\tilde{t}}^{t^*} g_2(s, x_2)\ ds > x_1(\tilde{t}) + \int_{\tilde{t}}^{t^*} g_1(s, x_1)\ ds = x_1(t^*),$$

which proves a contradiction.

The following lemma is a continuous dependence result for a particular sequence of Cauchy problems and it will be used in the general continuous dependence Theorem 6.3.

Lemma 6.2. *Suppose $\mathcal{W}$ be an open set of $E \times C(E)$. Let $g : \mathcal{W} \to \mathbb{R}$ be a function satisfying (v_α) and (c).*

Fixed $p_0 \in \mathbb{R}^+$ such that $[t_0, t_0 + p_0] \subset \Pi_E(\mathcal{W})$, let $\phi : \tilde{\alpha}(t_0) \to \mathbb{R}$ be a continuous function. Let $(\gamma_h)_h$ be a sequence of positive numbers converging to $\gamma_0 = 0$.

If y_M^0 is, for $h = 0$, the maximal solution of the Cauchy problem

$$(\tilde{\theta}_h) \qquad \begin{cases} \dot{x}(t) = g(t, x) + \gamma_h, & a.e. \ in \quad [t_0, t_0 + p_0]; \\ x(t) = \phi(t) + \gamma_h, & in \quad \tilde{\alpha}(t_0), \end{cases}$$

property $(+)$ holds for the problems $(\theta_h)_h, h \geq 0$, and property $(++)$ holds for each pair of triples $(t_0, \phi, g), (t_0, \phi + \gamma_h, g + \gamma_h)$ for $h > 0$, then there exists an integer h_0 and for $h > h_0$ there exists the maximal solution y_M^h of problem $(\tilde{\theta}_h)$ in $[t_0, t_0 + p_0]$, such that $\mathcal{X}(\Gamma(y_M^h, \tilde{\alpha}(t_0 + p_0)))$ converges to $\mathcal{X}(\Gamma(y_M^0, \tilde{\alpha}(t_0 + p_0)))$ in $C(E)$.

Proof. Given $\mathcal{X} \in \mathcal{Z}$, since $\mathcal{X} y_M^0 = \mathcal{X}(\Gamma(y_M^0, \tilde{\alpha}(t_0 + p_0)))$ is a solution of $(\tilde{\theta}_0)$ in $[t_0, t_0 + p_0]$, the compact set $Q = \cup_{t \in [t_0, t_0 + p_0]} (t, \mathcal{X}(\Gamma(\mathcal{X} y_M^0, \tilde{\alpha}(t))))$ is contained in $\mathcal{W}$.

By Lemma 2 of [10], there exists a positive number q and a summable function $\tilde{m}(\cdot)$ such that $B(Q, q) \subset \mathcal{W}$ and

$$(6.1) \qquad |g(s, z)| \leq \tilde{m}(s) \qquad \text{for all } \ (s, z) \in B(Q, q).$$

Moreover there is an h^* such that

$$(6.2) \qquad \mathcal{X}(\Gamma(\phi + \gamma_h, \tilde{\alpha}(t_0))) \in B(\mathcal{X}\left(\Gamma(\phi, \tilde{\alpha}(t_0))); \frac{q}{2}\right)$$

for every $h \geq h^*$.

By virtue of Theorem 5.3 (replacing $m(\cdot)$ with $\tilde{m}(\cdot) + 1$), there is $p \in \mathbb{R}^+$, $p < \min\left\{\frac{q}{2}, p_0\right\}$ and there exists the maximal solution $y_{M_1}^h$ of $(\tilde{\theta}_h)$ in $[t_0, t_0 + p]$ such that $(t_0 + p, \mathcal{X}(\Gamma(y_{M_1}^h, \tilde{\alpha}(t_0 + p)))) \in B(Q, q)$ for every $h \geq h^*$.

By (6.1), (6.2) the set $\{y^h_{M_1} : h \geq h^*\}$ is relatively compact; let y be a limit point of this set. By using (6.1) and the Lebesgue dominate convergence theorem, it is easy to prove that y is a solution of the problem $(\tilde{\theta}_0)$ in $[t_0, t_0 + p]$. By virtue of $(++)$ we can prove that y is the maximal solution of $(\tilde{\theta}_0)$ in $[t_0, t_0+p]$, since $y^h_{M_1}(t) \geq y^0_M(t)$ for every $h \geq h^*$ and $t \in [t_0, t_0 + p]$.

Now, we proceed by steps of width p. There is an integer $h^*_1 > h^*$ such that for every $h > h^*_1$ we have $\mathcal{X}(\Gamma(y^h_{M_1}, \tilde{\alpha}(t_0 + p))) \in B\left(\mathcal{X}(\Gamma(y^0_M, \tilde{\alpha}(t_0 + p))); \frac{q}{2}\right)$.

As in the first step, for every $h > h^*_1$, we can construct the maximal solution $y^h_{M_2}$ of the problem $(\tilde{\theta}_h)$ in $[t_0+p, t_0+2p]$ with initial value $y^h_{M_1}$ at t_0+p and we can prove that $(\mathcal{X}(\Gamma(y^h_{M_2}, \tilde{\alpha}(t_0+2p))))_h$ converges to $\mathcal{X}(\Gamma(y^0_M, \tilde{\alpha}(t_0+2p)))$ in $C(E)$.

With a finite number of steps we prove the lemma. □

Theorem 6.3 (continuous dependence). *Suppose $\mathcal{W}$ is an open set in $E \times C(E)$. Let $\mathcal{G} = \{g_k : \mathcal{W} \to \mathbb{R}, k \in \mathbb{N}\}$ be a function set satisfying the property (c) and such that*

$$(\gamma) \qquad \lim_{(k,\psi)\to(+\infty,\tau)} g_k(s, \psi) = g_0(s, \tau)$$

for almost all s and for every τ.

Assume that g_k verifies the property (v_α) for every $g_k \in \mathcal{G}$ and

$$(6.3) \qquad g_k(s, \psi) \geq g_0(s, \psi) \quad \text{for every } \psi \text{ and almost all } s.$$

Fixed $p_0 \in \mathbb{R}^+$ such that $[t_0, t_0 + p_0] \subset \Pi_E(\mathcal{W})$, let $(\Gamma(\phi_{t_k}, I_{(t_k,p_0)}))_k$ be a sequence in (G, τ) τ-convergent to $\Gamma(\phi_{t_0}, I_{(t_0,p_0)})$.

Finally, suppose that there is $\mathcal{X}_0 \in \mathcal{Z}_0$, i.e. $\mathcal{X}_0(\Gamma(\phi_0, I_{(t_0,p_0)})) \in \mathcal{W}$, such that

$$(6.4) \qquad \mathcal{X}_0(\Gamma(\phi_{t_k}, I_{(t_k,p_0)}))(t) \geq \mathcal{X}_0(\Gamma(\phi_{t_0}, I_{(t_0,p_0)}))(t) \quad \text{for every } t \in E, k \geq 1;$$

$$(6.5) \qquad \begin{gathered} \text{there is a neighborhood } \mathcal{U} \text{ of } (t_0, \mathcal{X}_0(\Gamma(\phi_0, I_{(t_0,p_0)}))) \text{ such that} \\ g_0(t, x) \leq 0 \text{ for every } (t, x) \in \mathcal{U}; \end{gathered}$$

(6.6)
$g_k(t, x) \geq 0$ *for every $k \geq 1$ and for every $(t, x) \in \mathcal{W}$ with $t \in [t_k, t_0]$ if $t_k < t_0$.*

If x^0_M is, for $k = 0$, the maximal solution of the Cauchy problem

$$(\theta_k) \qquad \begin{aligned} &(\theta_k.1) \quad \dot{x}(t) = g_k(t, x) && a.e. \text{ in } [t_k, t_k + p_0] \\ &(\theta_k.2) \quad x(t) = \phi_k(t) && \text{in } I_{(t_k,p_0)}, \end{aligned}$$

property (+) holds for the Cauchy problems (θ_k) for every $k \geq 1$ and property (++) holds for each pair of triples $(t_k, \phi_k + \gamma, g_k + \gamma), (t_0, \phi_0, g_0)$ with $\gamma > 0$ and $k > 0$, then there exists an integer k_0 and there exists the maximal solution $x_M^k = x_M^k(t_k, \phi_{t_k}, g_k)$ of the problem $(\theta_k), k \geq k_0$, defined in $I_{(t_k,p_0)} \cup [t_k, t_k + p_0]$, such that the sequence $(\Gamma(x_M^k, I_{(t_k,p_0)} \cup [t_k, t_k + p_0]))_k$ τ-converges to $\Gamma(x_M^0, I_{(t_0,p_0)} \cup [t_0, t_0 + p_0])$.

P r o o f. First of all consider the problems

$$(\tilde{\theta}_k) \qquad \begin{array}{lr} \dot{x}(t) = g_k(t, x) & \text{a.e. in } \; [t_k, t_k + p_0] \\ x(t) = \mathcal{X}_0(\Gamma(\phi_{t_k}, I_{(t_k,p_0)}))(t) & \text{in } \; \tilde{\alpha}(t_k). \end{array}$$

Since

$$\Upsilon_{\mathcal{X}_0} x_M^0 = \begin{cases} \mathcal{X}_0(\Gamma(\phi_{t_0}, I_{(t_0,p_0)} \cup [t_0, t_0 + p_0])), & \text{in } \; \tilde{\alpha}(t_0); \\ \mathcal{X}_0(\Gamma(x_M^0, I_{(t_0,p_0)})), & \text{in } \; [t_0, +\infty) \cap E \end{cases}$$

is a solution of $(\tilde{\theta}_0)$ in $[t_0, t_0 + p_0]$, the compact set

$$Q = \bigcup_{t \in [t_0, t_0 + p_0]} (t, \Upsilon_{\mathcal{X}_0}(\Gamma(\mathcal{X}_0 x_M^0, \tilde{\alpha}(t))))$$

is contained in $\mathcal{W}$ by definition. By Lemma 2 of [10], there is a positive number q and a summable function $\tilde{m}(\cdot)$ such that $B(Q, q) \subset \mathcal{W}$ and

$$(6.7) \qquad |g(s, z)| \leq \tilde{m}(s) \text{ for all } (s, z) \in B(Q, q) \text{ and for all } g \in \mathcal{G}.$$

Moreover, as $(\mathcal{X}_0(\Gamma(\phi_{t_k}, I_{(t_k,p_0)})))_k$ converges to $\mathcal{X}_0(\Gamma(\phi_{t_0}, I_{(t_0,p_0)}))$ there is an integer $\overline{k}$ such that

$$(6.8) \qquad t_k \in [t_0 - \frac{q}{2}, t_0 + \frac{q}{2}], \quad \mathcal{X}_0(\Gamma(\phi_{t_k}, I_{(t_k,p_0)})) \in B\left(\mathcal{X}_0(\Gamma(\phi_{t_0}, I_{(t_0,p_0)})), \frac{q}{2}\right)$$

for every $k > \overline{k}$.

By virtue of Theorem 5.3 (replacing $m(\cdot)$ with $\tilde{m}(\cdot) + 1$), there is a $p \in \mathbb{R}^+, p < \min\left\{\frac{q}{2}, p_0\right\}$ and there exists the maximal solution $x_{M_1}^k$ of $(\tilde{\theta}_k)$ on $[t_k, t_k + p]$ such that

$$(t_k + p, \mathcal{X}_0(\Gamma(x_{M_1}^k, \tilde{\alpha}(t_k + p)))) \in B(Q, q) \text{ for every } k \geq \overline{k}.$$

By virtue of (6.7), (6.8), the set $\{x_{M_1}^k\}_{k \geq \overline{k}}$ is relatively compact; we denote by x such a limit point of this set.

Using (6.7), (γ), and the Lebesgue dominate convergence theorem, one can verify that x is a solution of $(\tilde{\theta}_0)$ on $[t_0, t_0 + p]$.

Further x is the maximal solution of the Cauchy problem $(\tilde{\theta}_0)$ on $[t_0, t_0 + p_0]$. Indeed, by virtue of properties (6.3), (6.4), (6.5), (6.6), and (++) there is $k^* \geq \overline{k}$ such that for $k > k^*$ the maximal solution $y_M^{(k,h)}$ of the problem

$$(P_h^k) \qquad \begin{cases} \dot{x}(t) = g_k(t,x) + \dfrac{1}{h}, & \text{a.e. in} \quad [t_k, t_k + p]; \\ \\ x(t) = \mathcal{X}_0(\Gamma(\phi_{t_k}, I_{(t_k,p_0)})) + \dfrac{1}{h}, & \text{in} \quad \tilde{\alpha}(t_k), \end{cases}$$

satisfies $y_M^{(k,h)}(t) \geq \Upsilon_{\mathcal{X}_0} x_M^0(t)$ on $\tilde{\alpha}(t_k \wedge t_0 + p)$ for every $h \in \mathbb{N}$. Thus, the assertion follows from Lemma 6.2.

This implies that x and x_M^0 must coincide on $[t_0, t_0 + p]$. Therefore, by Proposition 3.2, $\mathcal{X}_0(\Gamma(x_{M_1}^k, \tilde{\alpha}(t_k + p)))_k$ converges to $\mathcal{X}_0(\Gamma(\Upsilon_{\mathcal{X}_0} x_M^0, \tilde{\alpha}(t_0 + p)))$ in $C(E)$.

Now, we proceed by step of width p.
There is an integer $\overline{k}_1 > k^*$ such that, for every $k > \overline{k}_1$, we have

$$t_k + p \in [t_0, t_0 + 2p], \quad \mathcal{X}_0(\Gamma(x_{M_1}^k, \tilde{\alpha}(t_k + p))) \in B\left(\mathcal{X}_0(\Gamma(\Upsilon_{\mathcal{X}_0} x_M^0, \tilde{\alpha}(t_0 + p))), \frac{q}{2}\right).$$

As in the first step, we can construct the maximal solution $x_{M_2}^k$ of $(\tilde{\theta}_k)$ in $[t_k + p, t_k + 2p]$ with initial value $x_{M_1}^k$ at $t_k + p$.

We can prove analogously that $(\mathcal{X}_0(\Gamma(x_{M_2}^k, \tilde{\alpha}(t_k + 2p))))_k$ converges to $\mathcal{X}_0(\Gamma(\Upsilon_{\mathcal{X}_0} x_M^0, \tilde{\alpha}(t_0 + 2p)))$ in $C(E)$.

With a finite number of steps we prove the existence of a $k_0 \in \mathbb{N}$, such that for every $k \geq k_0$, said x_M^k be the maximal solution of problem $(\tilde{\theta}_k)$ on $[t_k, t_k + p_0]$, we have that $(\mathcal{X}(\Gamma(x_M^k, \tilde{\alpha}(t_k + p_0))))_k$ converges to $\mathcal{X}(\Gamma(\Upsilon_{\mathcal{X}_0} x_M^0, \tilde{\alpha}(t_0 + p_0)))$ in $C(E)$.

It is easy to verify that x_M^k is also the maximal solution to problem (θ_k), for $k \geq k_0$, and that the sequence $(\Gamma(x_M^k, I_{(t_k,p_0)} \cup [t_k, t_k + p_0]))_k$ τ-converges to $\Gamma(x_M^0, I_{(t_0,p_0)} \cup [t_0, t_0 + p_0])$. $\square$

Remark 6.4. In light of Remark 6.1, conditions (+) and (++) are a consequence of the monotonicity of g_k with respect to the functional argument.

Furthermore, we underline that hypothesis (6.5) can be omitted if $t_k \leq t_0$ definitively; and hypothesis (6.6) can be omitted if $t_k \geq t_0$ definitively.

Now, we present some examples, which state the necessity of hypotheses (6.3), (6.4), (6.5), and (6.6).

Example 6.5. Let $\mathcal{W} =]-2, 2[\times\mathbb{R}, \alpha(t) = \{t\}$, $t_k = 0$ and $\phi_k(0) = 0$, for every $k \in \mathbb{N}$. Consider the problems

$$(\theta_k) \qquad \begin{cases} \dot{x}(t) = g_k(t,x), & \text{a.e. in } [t_k, t_k+1]; \\ x(t_k) = \phi_k(t_k), \end{cases}$$

where for $k > 1$,

$$g_k(t,x) = \begin{cases} \sqrt{x}, & \text{if } x > \dfrac{1}{k}, & \text{for every } t; \\ \sqrt{k}\cdot x, & \text{if } 0 \le x \le \dfrac{1}{k}, & \text{for every } t; \\ 0, & \text{if } x < 0, & \text{for every } t. \end{cases}$$

and

$$g_0(t,x) = \begin{cases} \sqrt{x}, & \text{if } x \ge 0, \text{ for every } t; \\ 0, & \text{if } x < 0, \text{ for every } t. \end{cases}$$

All hypotheses of Theorem 6.3 are satisfied, except (6.3). Each problem (θ_k), for $k \ge 1$ has an unique solution, $x_M^k \equiv 0$ on $[0,1]$. But the maximal solution of problem (θ_0) on $[0,1]$ is $x_M^0(t) = \dfrac{t^2}{4}$.

Example 6.6 Let $\mathcal{W}, \alpha, t_k$ be as in Example 6.5. Consider the problem $(\theta_k), k \in \mathbb{N}$, where for $k \ge 1$

$$g_k(t,x) = \begin{cases} \sqrt{x}, & \text{if } x > \dfrac{1}{k}, & \text{for every } t; \\ \dfrac{k^2}{k-1}\dfrac{1}{\sqrt{k}}x - \dfrac{1}{(k-1)\sqrt{k}}, & \text{if } \dfrac{1}{k^2} \le x \le \dfrac{1}{k}, & \text{for every } t; \\ x - \dfrac{1}{k^2}, & \text{if } 0 \le x \le \dfrac{1}{k^2}, & \text{for every } t; \\ -\dfrac{1}{k^2}, & \text{if } x < 0, & \text{for every } t. \end{cases}$$

$$\phi_k(0) = \frac{1}{k^2}$$

and

$$g_0(t,x) = \begin{cases} \sqrt{x}, & \text{if } x \ge 0, \text{ for every } t; \\ 0, & \text{if } x < 0, \text{ for every } t \end{cases}$$

$$\phi_0(0) = 0.$$

As in Example 6.5, all the hypotheses of Theorem 6.3 are satisfied except (6.3), noticing that (6.4) holds strictly.

The problem (θ_k) has an unique solution, $x_M^k \equiv \dfrac{1}{k^2}$ on $[0,1]$;but the maximal solution of (θ_0) is $x_0(t) = \dfrac{t^2}{4}$.

Example 6.7. Let $\mathcal{W}, \alpha, t_k$ be as in Example 6.5. Consider the problems (θ_k), where for $k \geq 1$

$$g_k(t,x) = \begin{cases} \sqrt{x + \dfrac{1}{k}}, & \text{if } \ x \geq -\dfrac{1}{k}, \quad \text{for every } \ t; \\ 0, & \text{if } \ x < -\dfrac{1}{k}, \quad \text{for every } \ t. \end{cases}$$

$$\phi_k(0) = -\frac{1}{k}$$

and

$$g_0(t,x) = \begin{cases} \sqrt{x}, & \text{if } \ x \geq 0, \quad \text{for every } \ t; \\ 0, & \text{if } \ x < 0, \quad \text{for every } \ t. \end{cases}$$

$$\phi_0(0) = 0.$$

All hypotheses of Theorem 6.3 are satisfied except (6.4). The problem (θ_k) has an unique solution, $x_M^k \equiv -\dfrac{1}{k}$ on $[0,1]$; but the maximal solution of problem (θ_0) on $[0,1]$ is $x_M^0(t) = \dfrac{t^2}{4}$.

Example 6.8. Let $\mathcal{W}, \alpha$ be as in Example 6.5. Consider the problems (θ_k), where for $k \geq 1$

$$g_k(t,x) = \begin{cases} -2, & \text{if } \ x \in \mathbb{R}, & t < -\sqrt{\dfrac{1}{k}}; \\ \sqrt{x}, & \text{if } \ x \geq \dfrac{1}{k}, & t \geq -\sqrt{\dfrac{1}{k}}; \\ \sqrt{\dfrac{1}{k}}, & \text{if } \ 0 \leq x \leq \dfrac{1}{k}, & t \geq -\sqrt{\dfrac{1}{k}}; \\ \dfrac{1}{2}x + \sqrt{\dfrac{1}{k}}, & \text{if } \ -\dfrac{2}{\sqrt{k}} \leq x < 0, & t \geq -\sqrt{\dfrac{1}{k}}; \\ 0, & \text{if } \ x \leq -\dfrac{2}{\sqrt{k}}, & t \geq -\sqrt{\dfrac{1}{k}}; \end{cases}$$

$$\phi_k(-\frac{2}{\sqrt{k}}) = 0$$

and

$$g_0(t,x) = \begin{cases} \sqrt{x}, & \text{if } x \geq 0, \quad t \geq 0; \\ 0, & \text{if } x < 0, \quad t \geq 0; \\ -2, & \text{if } x \in \mathbb{R}, \quad t < 0, \end{cases}$$

$$\phi_0(0) = 0.$$

All the hypotheses of Theorem 6.3 are satisfied except (6.6). The unique solution of the problem (θ_k) in $\left[-\frac{2}{\sqrt{k}}, 1\right]$ is

$$x_M^k = \begin{cases} -2(t + \frac{2}{\sqrt{k}}), & \text{if } -\frac{2}{\sqrt{k}} \leq t \leq -\sqrt{\frac{1}{k}}; \\ -\frac{2}{\sqrt{k}}, & \text{if } t \geq -\sqrt{\frac{1}{k}}, \end{cases}$$

and x_M^k converges to $x(t) \equiv 0$.

But the maximal solution of problem (θ_0) in $[0,1]$ is $x_M^0(t) = \frac{t^2}{4}$.

Example 6.9. Let $\mathcal{W}, \alpha$ be as in Example 6.5. Consider the problem (θ_k), where for $k \geq 1$

$$g_k(t,x) = \begin{cases} 1, & \text{if } x \in \mathbb{R}, \quad t \leq 1; \\ \sqrt{x-1}, & \text{if } x \geq 1 + \frac{1}{k}, \quad t > 1; \\ \frac{1}{\sqrt{k}}, & \text{if } 1 \leq x \leq 1 + \frac{1}{k}, \quad t > 1; \\ \sqrt{k} \cdot x + \frac{1}{\sqrt{k}} - \sqrt{k}, & \text{if } 1 - \frac{1}{k} \leq x \leq 1, \quad t > 1; \\ 0, & \text{if } x \leq 1 - \frac{1}{k}, \quad t > 1, \end{cases}$$

$$\phi_k(\frac{1}{k}) = 0$$

and

$$g_0(t,x) = \begin{cases} 1, & \text{if } x \in \mathbb{R}, \quad t \leq 1; \\ \sqrt{x-1}, & \text{if } x \geq 1, \quad t > 1; \\ 0, & \text{if } x \leq 1, \quad t > 1. \end{cases}$$

$$\phi_0(0) = 0.$$

All the hypotheses of Theorem 6.3 are satisfied except (6.5). The unique solution of problem (θ_k) in $\left[\frac{1}{k}, \frac{3}{2}\right]$ is

$$x_M^k(t) = \begin{cases} t - \frac{1}{k}, & \text{if } \frac{1}{k} \le t \le 1; \\ 1 - \frac{1}{k}, & \text{if } 1 \le t \le \frac{3}{2} \end{cases}$$

and x_M^k converges to

$$x(t) = \begin{cases} t, & \text{if } 0 \le t \le 1; \\ 1, & \text{if } 1 \le t \le \frac{3}{2}. \end{cases}$$

But the maximal solution of (θ_0) on $\left[0, \frac{3}{2}\right]$ is

$$x_M^0(t) = \begin{cases} t, & \text{if } 0 \le t \le 1; \\ 1 + \frac{(t-1)^2}{4}, & \text{if } 1 \le t \le \frac{3}{2}. \end{cases}$$

Finally, we study the relation between condition (++), the monotonicity condition with respect to the functional argument (see Remark 6.1), and the continuous dependence of extremal solutions; in short, we give examples which prove that the monotonicity condition with respect to the functional argument is strictly stronger that condition (++), and both of them are not necessary for the continuous dependence of extremal solutions.

Example 6.10. This example shows that condition (++) is strictly weaker than the monotonicity of g_k with respect to the functional argument. Let $\mathcal{W}$ be as in Example 6.5. Let

$$g_k(t,x) = \begin{cases} -\sqrt{|\, x\,|} & \text{if } t \ge 0 \\ 2^k & \text{if } -\frac{1}{2^{2k}} \le t < 0 \\ 0 & \text{if } t < -\frac{1}{2^{2k}}, \end{cases}$$

$$g_0(t,x) = \begin{cases} -\sqrt{|\, x\,|} & \text{if } t \ge 0 \\ 0 & \text{if } t < 0, \end{cases}$$

$$\alpha(t) = t.$$

Consider the Cauchy problems

$$(\theta_k) \qquad \begin{aligned} &\dot{x}(t) = g_k(t, x(\alpha(t))) \quad \text{a.e. in } \left[-\frac{1}{2^{2k}}, 1\right] \\ &x(-\frac{1}{2^{2k}}) = 0 \end{aligned}$$

$k \geq 1$, and

$$(\theta_0) \qquad \begin{aligned} &\dot{x}(t) = g_0(t, x(\alpha(t))) \quad \text{a.e. in } \ [0,1] \\ &x(0) = 0. \end{aligned}$$

It is easy to check that hypotheses (γ), (6.3), (6.4), (6.5), and (6.6) hold, but $(g_k), k \geq 0$ do not satisfy the monotonicity condition with respect to the second argument.

Problem (θ_k) has as unique solution

$$x_M^k(t) = \begin{cases} 2^k \left(t + \dfrac{1}{2^{2k}}\right) & \text{if } \ -\dfrac{1}{2^{2k}} \leq t \leq 0 \\ \dfrac{1}{4}\left(t - \dfrac{1}{2^{\frac{k}{2}-1}}\right)^2 & \text{if } \ 0 \leq t \leq \dfrac{1}{2^{\frac{k}{2}-1}}, \\ 0 & \text{if } \ t \geq \dfrac{1}{2^{\frac{k}{2}-1}} \end{cases}$$

and problem (θ_0) has $x_0(t) = 0, t \in [0,1]$ as maximal solution. Therefore (++) is verified and $(x_M^k)_k$ converges to x_0.

If, in Example 6.10, we modify the lag function as follows:

$$\alpha(t) = \begin{cases} t & \text{if } \ t < 0 \\ 0 & \text{if } \ t \geq 0, \end{cases}$$

then problem (θ_k) admits as unique solution

$$x_M^k(t) = \begin{cases} 2^k \left(t + \dfrac{1}{2^{2k}}\right) & \text{if } \ -\dfrac{1}{2^{2k}} \leq t \leq 0 \\ -\dfrac{1}{\sqrt{2^k}}t + \dfrac{1}{2^k} & \text{if } \ t \geq 0, \end{cases}$$

and problem (θ_0) has still $x_0(t) = 0, t \in [0,1]$ as maximal solution. This proves that $(++)$ is not verified since $x_M^k(t) < x_0(t)$ for $t > \dfrac{1}{\sqrt{2^k}}$, but $(x_M^k)_k$ still converges to x_0.

Finally, we give a sequence of Cauchy problems (θ_k) which do not satisfy $(++)$ and for which there is not continuous dependence of extremal solutions.

Example 6.11. Let $g_k : [-2,2] \times C([-2,2]) \to \mathbb{R}, k \geq 0$ and $\alpha : [-2,2] \to [-2,2]$ be defined by

$$g_k(t,x) = \begin{cases} -\sqrt{|x(t) - x(0)|} & \text{if } \ t \geq \dfrac{1}{2} \\ -\sqrt{|x(-t - \dfrac{1}{2}) - x(0)|} & \text{if } \ 0 \leq t < \dfrac{1}{2} \\ 2^k & \text{if } \ -\dfrac{1}{2^{2k}} \leq t < 0 \\ 0 & \text{if } \ t < -\dfrac{1}{2^{2k}}, \end{cases}$$

$$g_0(t,x) = \begin{cases} -\sqrt{|x(t) - x(0)|} & \text{if } \ t \geq \dfrac{1}{2} \\ -\sqrt{|x(-t - \dfrac{1}{2}) - x(0)|} & \text{if } \ 0 \leq t < \dfrac{1}{2} \\ 0 & \text{if } \ t < 0, \end{cases}$$

$$\alpha(t) = \begin{cases} t & \text{if } \ -2 \leq t < 0 \\ \left\{-t - \dfrac{1}{2}\right\} \cup \{0\} & \text{if } \ 0 \leq t < \dfrac{1}{2} \\ \{t\} \cup \{0\} & \text{if } \ \dfrac{1}{2} \leq t \leq 2. \end{cases}$$

Consider the following Cauchy problems,

$$(\theta_k) \qquad \begin{aligned} &\dot{x}(t) = g_k(t,x) && \text{a.e. in } \left[-\frac{1}{2^{2k}}, 2\right] \\ &x(t) = 0 && \text{in } \left[-1, -\frac{1}{2}\right] \cup \left\{-\frac{1}{2^{2k}}\right\} \end{aligned}$$

$k \geq 1$, and

$$\begin{aligned} &\dot{x}(t) = g_0(t,x) && \text{a.e. in } [0,2] \\ &x(t) = 0 && \text{in } \left[-1, -\frac{1}{2}\right] \cup \{0\}. \end{aligned}$$

All the hypotheses (γ), (6.3), (6.4), (6.5), and (6.6) hold; furthermore, problem (θ_k) has as unique solution

$$x_M^k(t) = \begin{cases} 2^k\left(t+\dfrac{1}{2^{2k}}\right) & \text{if } \ -\dfrac{1}{2^{2k}} \le t \le 0 \\ -\dfrac{1}{\sqrt{2^k}}t+\dfrac{1}{2^k} & \text{if } \ 0 \le t < \dfrac{1}{2} \\ \dfrac{1}{2^k} - \dfrac{(t+\frac{\sqrt{2}}{2^{\frac{k}{4}}}-\frac{1}{2})^2}{4} & \text{if } \ \dfrac{1}{2} \le t \le 2 \end{cases}$$

and problem (θ_0) has $x_0(t) = 0, t \in [0,2]$ as maximal solution. Thus, condition $(++)$ is not verified and $(x_M^k)_k$ does not converge to x_0.

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