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ON THE ISOENERGETICAL NON-DEGENERACY OF THE PROBLEM OF TWO CENTERS OF GRAVITATION

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Abstract. For the system describing the motion of a moss point under the action of two static gravity centers (with equal masses), we find a subset of the set of the regular values of the energy and momentum, where the condition of isoenergetical non-degeneracy is fulfilled.

1. Introduction. The problem of determining the motion of a free particle in the plane attracted by two fixed Newtonian centers is one of the classical examples of mechanics and a special case of the problem of the three bodies. It has applications in celestial and quantum mechanics, molecular physics and etc. Its integrability was discovered by Euler [1, 2]. Here we consider only the case when masses of the centers are equal. Let us have two fixed bodies in the plane (static centers), they act on a mass point according to the Newton's law. Let the distance between the centers O_1 and O_2 be $2c$ and the distances between O_1 and O_2 and the moving mass be r_1, r_2 respectively. Let us introduce new coordinates – so called elliptic coordinates

$$\xi = \frac{(r_1 + r_2)}{2}, \eta = \frac{(r_1 - r_2)}{2}.$$

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In these coordinates the variables are separated and following [3] we have the two first integrals of the system to be:

$$H = \frac{1}{2}(p_\xi^2 \frac{\xi^2 - c^2}{\xi^2 - \eta^2} + p_\eta^2 \frac{c^2 - \eta^2}{\xi^2 - \eta^2}) - \frac{2k\xi}{\xi^2 - \eta^2}$$

$$G = \frac{p_\xi^2(\xi^2 - c^2)}{2} - 2k\xi - h\xi^2$$

where $h = \text{const}$.

In this note we shall state results concerning the isoenergetical non-degeneracy of the above problem. Before we present them we shall give a brief summary on the structure of the generic integrable Hamiltonian system with n degrees of freedom and the KAM theory conditions (see [3, 4] for more details).

According to the Liouville – Arnold theorem the phase space of such system is foliated into invariant manifolds the typical fibre being n dimensional torus on which the motion is quasiperiodic and there exist cannonical coordinates (I, φ) – (action–angle variables) such that I maps a neighbourhood of fixed torus into an open subset of R^n and φ are the coordinates on any of the nearby tori and the first integrals of the system can be expressed as functions only of I. Classical results of Poincaré show that most of the Hamiltonian systems are not integarble so he defined the main problem of mechanics to be the study of Hamiltonian systems that are close to integrable ones [5]. The most powerful approach to do this is KAM theory. The KAM theory gives conditions on the Hamiltonian of an integrable system, which guarantee the survival of most of the invariant tori under small perturbation. There are two independent conditions – of non-degeneracy (Kolmogorov's condition) and of isoenergetical non-degeneracy (Arnold - Moser condition). For simplisity we shall give here only the analytical formulations of these conditions (see [3], app. 8 for more details). Let Hamiltonian of the integrable system H be written in action variables $I = I_1, \dots, I_n$, then the system is non-degenerate if

$$D_1 = det(\frac{\partial^2 H}{\partial I_j \partial I_k}) \neq 0 \tag{1}$$

it is isoenergetically non-degenerate if

$$D_2 = det \begin{pmatrix} \partial^2 H/\partial I^2 & \partial H/\partial I \\ \partial H/\partial I & 0 \end{pmatrix} \neq 0. \tag{2}$$

2. Statement of the main result. Before we state our main theorem we need the set of regular values (h, g) of the first integrals H and G and shall introduce action variables for our problem. Throughout the rest of the text we will assume $k = 1$ and $c = 1$ (these reductions are possible without loss of generality). We have

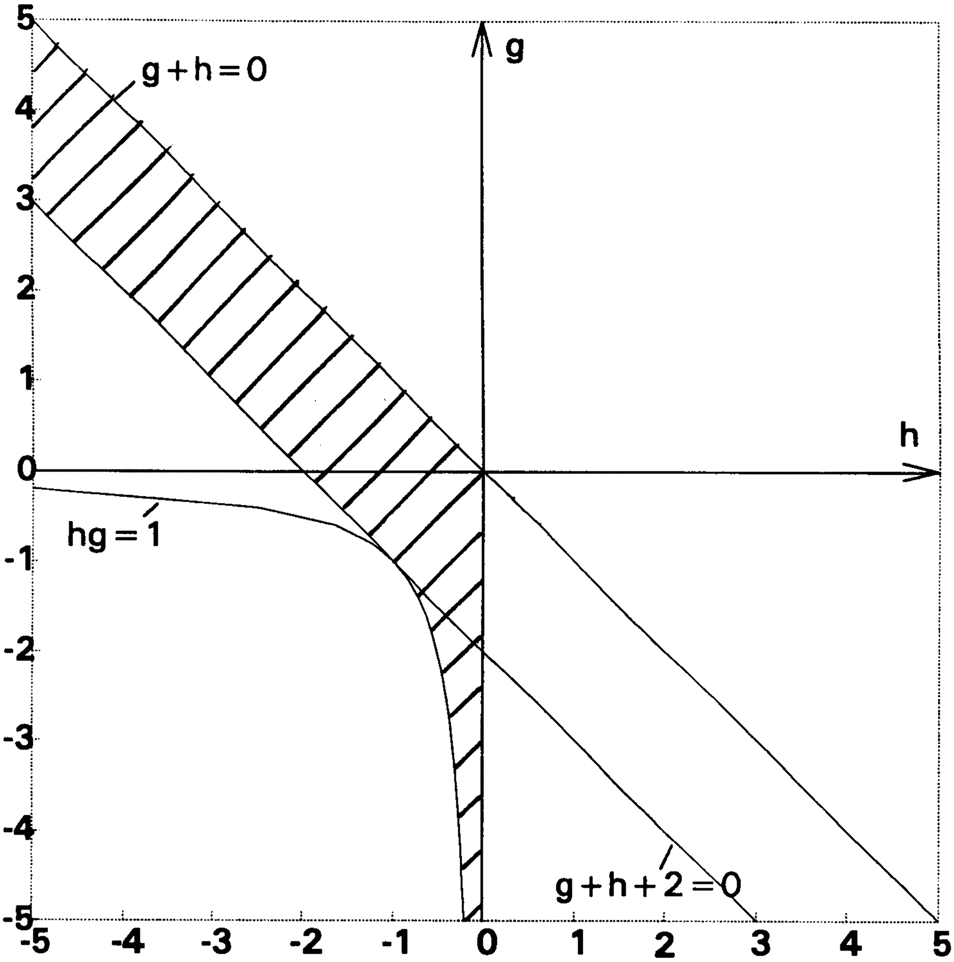

Fig. 1. The set of regular values of h and g.

Lemma 1. *The set of regular values of the first integrals of the system is* $U_r = U_r^1 \cup U_r^2 \cup U_r^3$ *where*

$$\begin{aligned} U_r^1 &= \{h, g \mid g + h < 0, g + h + 2 > 0, g > 0 \\ U_r^2 &= \{h, g \mid g + h + 2 > 0, h < 0, g < 0\} \\ U_r^3 &= \{h, g \mid gh > 1, g + h + 2 < 0, h < 0\} \end{aligned}$$

see fig. 1 shaded region.

The action variables are given by formulae

$$\psi(h,g) \stackrel{\text{def}}{=} I_1 = \int_{\gamma_1} \frac{y d\xi}{\xi^2 - 1}. \tag{3}$$

$$\varphi(h,g) \stackrel{\text{def}}{=} I_2 = \int_{\gamma_2} \frac{z d\eta}{1 - \eta^2}. \tag{4}$$

where γ_1 is the oval of the curve $\Gamma_1 = \{(y,\xi) : y^2 = 2(h\xi^2 + 2\xi + g)(\xi^2 - 1)\}$ and γ_2 that of the curve $\Gamma_2 = \{(z,\eta) : z^2 = -2(h\eta^2 + g)(1 - \eta^2)\}$

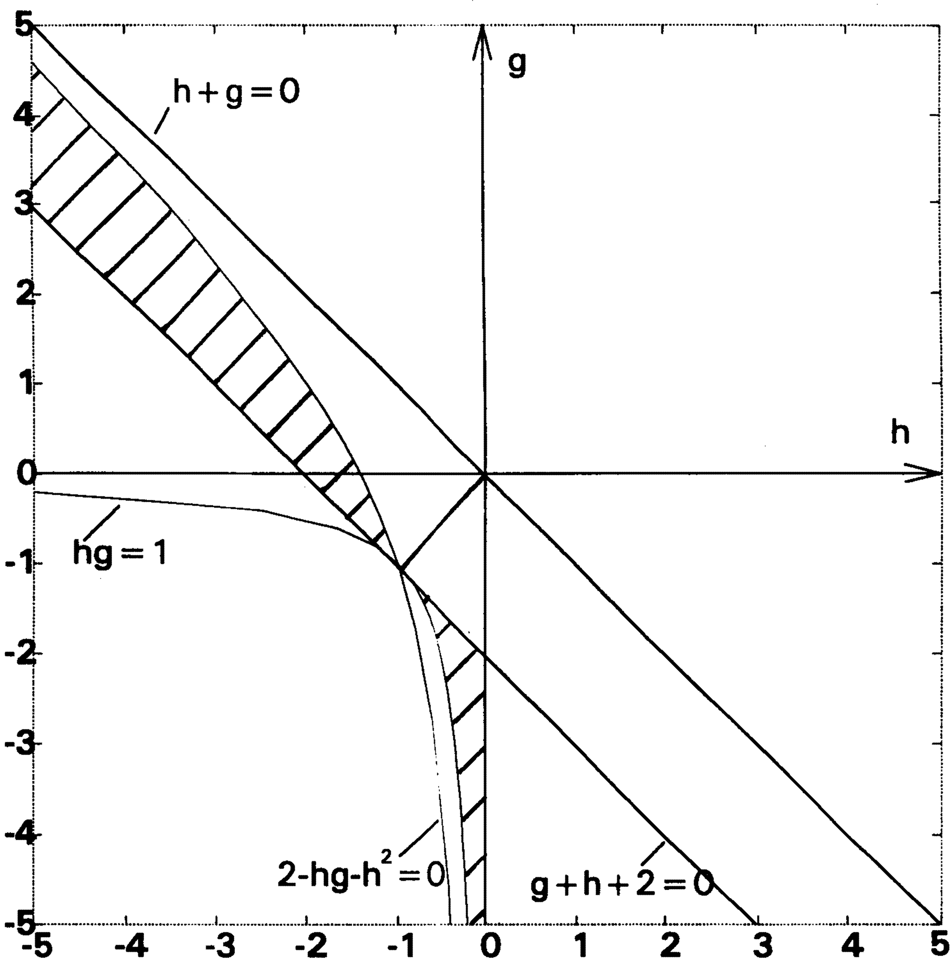

Fig. 2. The subsets where the system is isoenergetically non-degenerate

The following theorem is the central result of this note.

Theorem 1. *The sytem describing the motion of a mass point under the action of two static gravity centers (with equal masses) is isoenergetically non-degenerate for those values of the energy and momentum* (h, g) *which belong the set* $U_r \cap \{h, g \mid 2 - hg - h^2 \leq 0\}$ *(see fig. 2, shaded region).*

For the proof of our theorem we exploit the idea of [6], where it is proved that the spherical pendulum is non-degenerate everywhere out of the bifurcation diagram. In other words we express the determinant (2) via elliptic integrals (3), (4) and their derivatives. We have

$$D_2 = -\frac{1}{J^3}(\varphi_{gg}\psi_g - \psi_{gg}\varphi_g),$$

where

$$J = \psi_h\varphi_g - \varphi_h\psi_g$$

$(J \neq 0, \varphi_g \neq 0, \psi_g \neq 0)$ and then we use Picard - Fuchs and Riccati equations to study them and Picard – Lefshetz theory to compute their asymptotics.

Remark 1. The system is not isoenergetically non-degenerate in the whole U_r because

- $D_2 \mid_{h+g=0}$ changes its sign along $g + h = 0$
- $D_2 \mid_{h+g+2=0} > 0$ and $D_2 \mid_{h=0} < 0$.

Remark 2. Since the number of degrees of freedom of the system is two the condition of isoenergetical non-degeneracy guarantees stability of action variables in the sense that they remain always near their initial values when the perturbations are small (see [3] app. 8).

Detailed proofs of the results presented in this note could be found in [7].

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