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INVOLUTIVITY AND SIMPLE WAVES IN $\mathbb{R}^2$

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Abstract. A strictly hyperbolic quasi-linear 2×2 system in two independent variables with C^2 coefficients is considered. The existence of a simple wave solution in the sense that the solution is a 2-dimensional vector-valued function of the so called Riemann invariant is discussed. It is shown, through a purely geometrical approach, that there always exists simple wave solution for the general system when the coefficients are arbitrary C^2 functions depending on both, dependent and independent variables.

1. Introduction. We consider the quasi-linear system

$$
(1) \qquad \left|\begin{array}{l} \partial_1 u^1 = a_1^1(x,u)\partial_2 u^1 + a_2^1(x,u)\partial_2 u^2 \\ \partial_1 u^2 = a_1^2(x,u)\partial_2 u^1 + a_2^2(x,u)\partial_2 u^2 \ , \end{array}\right.
$$

where $u \equiv {}^t(u^1, u^2)$ is an unknown vector valued function; $x \equiv {}^t(x^1, x^2)$ is an independent variable; $\partial_k \equiv \partial / \partial x^k$; the coefficients $a_j^i(x,u)$ $(i,j = 1,2)$ are C^2 functions of x and u.

The problem for the existence of symple wave solutions for similar systems in $\mathbb{R}^n$ $(n \geq 2)$ is considered by Z. Peradzinski in [3] but he gives neither methods

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for finding them nor conditions for their existence. Some geometrical properties of such systems are discussed by J. Tabov in [5] and [6]. He proposes an idea of finding simple wave solutions and gives necessary and sufficient conditions for their existence. However, these conditions are in such a general form, that *a priori* it is not clear whether in the hyperbolic case simple waves do always exist, as in the case when a^i_j $(i,j=1,2)$ depend only on u and do not depend on x (see e.g. A. Jeffrey [1]). We show that, for arbitrary coefficients a^i_j $(i,j=1,2)$ which are C^2 functions of (x,u) the system (1) possesses always simple wave solution.

The paper consists of four sections. The basic definitions and results are given in Section 2. In Section 3 we derive conditions for the solvability of the system (1). In Section 4 we show the main result (Theorem 4), that the general hyperbolic system (1) possesses at least one simple wave solution.

2. Basic definitions and results. We begin this section by quoting an important definition.

Definition (M. Burnat [2]). *We say that a solution $u(x)$ of the system* (1) *is constructed by means of Riemann invariants if it is of the form*

$$u^i(x) = v^i(R(x)) \ \ (i=1,2) \tag{2}$$

where $v^i(z)$ $(i=1,2)$ are functions of a single variable and $R(x)$ is a suitable function, called Riemann invariant.

If the system (1) is homogeneous, then the solution (2) is called *a simple wave*, and in the non-homogeneous case it is called *a simple state*.

Following the Pfaff theory the system (1) can be written in differential forms. For that purpose we let $x^3 \equiv \partial_2 u^2$, $x^4 \equiv u^1$, $x^5 \equiv u^2$, $x \equiv {}^t(x^1,\ldots,x^5)$, $dx \equiv {}^t(dx^1,\ldots,dx^5)$. Then the desired Pfaffian differential system takes the form

$$w^1(dx) = 0\ , \qquad w^2(dx) = 0\ , \tag{3}$$

where w^i $(i=1,2)$ are differential forms and dx is a five-dimensional vector field. For (3) there are three linearly independent vector fields ξ_1, ξ_2, ξ_3, annihilating the forms w^i $(i=1,2)$, i.e. $w^i(\xi_k)=0$ $(i=1,2;\ k=1,2,3)$; so we determine a distribution $\theta(x)$ as a linear hull of ξ_1, ξ_2, ξ_3. Thus, in accordance with the theory any pair of linearly independent vector fields η_{01}, $\eta_{02} \in \theta(x)$ determines a two-dimensional subdistribution $\theta_1(x) \subset \theta(x)$ as a linear hull of these two fields. If the commutator of η_{01} and η_{02}

$$[\eta_{01}, \eta_{02}] = \{\eta_{01}^j(\partial_j \eta_{02}^k) - \eta_{02}^j(\partial_j \eta_{01}^k)\}\partial_k \ \ (j, k = 1, \dots, 5), \tag{4}$$

belongs to $\theta_1(x)$, then $\theta_1(x)$ is called *an involutive subdistribution*. Taking into account the Fröbenius theorem, provided that $\theta_1(x)$ is completely integrable, it follows that the system

$$\eta_{01}\Phi(x) = 0, \quad \eta_{02}\Phi(x) = 0, \tag{5}$$

($\Phi(x)$ is the unknown function) possesses three functionally independent solutions

$$\Phi_i(x) = c_i \ \ (c_i \equiv \text{const}\,; \quad i = 1, 2, 3). \tag{6}$$

By means of the implicit function theorem it can be found the so called simple wave solution for the system (3) in the form of set of three functions, namely $x^j = x^j(x^1, x^2)$ $(j = 3, 4, 5)$.

However, it is not known *a priori* whether the involutive subdistribution $\theta_1(x) \subset \theta(x)$, will always exist. We discuss this question in Section 4 (Theorem 4).

3. Involutive subdistributions of $\theta(x)$. The following lemma is due to Grundland [4]:

Lemma 1 (Grundland [4]). *The functions $u^i(x)$ $(i = 1, 2)$ can be represented in the form* (2) *if and only if ∇u^1 and ∇u^2 are collinear at each point, i.e.*

$$\partial_1 u^1 \partial_2 u^2 = \partial_2 u^1 \partial_1 u^2. \tag{7}$$

Hence we have:

Theorem 1 (Tabov [6]). *The vector-function ${}^t(u^1, u^2)$ is a solution constructed by Riemann invariants for* (1) *if and only if ${}^t(u^1, u^2)$ is a solution of the following system:*

$$\left|\begin{aligned} &\partial_1 u^1 = a_1^1(x,u)\partial_2 u^1 + a_2^1(x,u)\partial_2 u^2 \\ &\partial_1 u^2 = a_1^2(x,u)\partial_2 u^1 + a_2^2(x,u)\partial_2 u^2 \\ &\partial_1 u^1 \partial_2 u^2 = \partial_2 u^1 \partial_1 u^2. \end{aligned}\right. \tag{8}$$

Further, replacing $\partial_1 u^1$ and $\partial_1 u^2$ in the third equation of (8) by the right-hand sides of the first two equations, respectively, we obtain the following algebraic equation:

$$a_1^2 X^2 - (a_1^1 - a_2^2)XY - a_2^1 Y^2 = 0. \tag{9}$$

Since we are interested in strictly hyperbolic systems, let us assume that $D \equiv (a_1^1 - a_2^2)^2 + 4a_1^2 a_2^1 > 0$. By letting $K \equiv X/Y$ in (9) we obtain a quadratic equation with respect to K whose roots are

$$K_{1,2} = (0.5/a_1^2)(a_1^1 - a_2^2 \pm D^{1/2}). \tag{10}$$

In order to reduce the Pfaffian differential system (3), associated with (8), to simple terms we let $t_1 \equiv a_1^1(x)K(x) + a_2^1(x)$, $t_2 \equiv a_1^2(x)K(x) + a_2^2(x)$, $x \in \mathbb{R}^5$, where K is either K_1 or K_2 and thus

$$\left| \begin{aligned} & w^1(dx) \equiv dx^4 - x^3 t_1 dx^1 - x^3 K dx^2 = 0 \\ & w^2(dx) \equiv dx^5 - x^3 t_2 dx^1 - x^3 dx^2 = 0. \end{aligned} \right. \tag{11}$$

The following three linearly independent vector fields

$$\xi_1 = {}^t(0,0,1,0,0), \quad \xi_2 = {}^t(1,0,0,x^3 t_1, x^3 t_2), \quad \xi_3 = {}^t(0,1,0,x^3 K, x^3) \tag{12}$$

satisfy the system (11), i.e. $w^i(\xi_k) = 0$ ($i = 1,2;\ k = 1,2,3$).

The linear hull of the above three vector fields (12) determines a three-dimensional distribution $\theta(x)$. If we choose a pair of linearly independent vector fields $\eta_{01}, \eta_{02} \in \theta(x)$, then their linear hull determines a two-dimensional subdistribution $\theta_1(x) \subset \theta(x)$. Thus, if the commutator $[\eta_{01}, \eta_{02}] \in \theta_1(x)$, then $\theta_1(x)$ will be involutive and from Fröbenius theorem it will follow that $\theta_1(x)$ is a completely integrable subdistribution of $\theta(x)$. Therefore, the system (5) possesses three functionally independent solutions written like (6). Having in mind the results obtained in [6] our first task is to build a basis by the linearly independent vector fields η_{01}, η_{02}, by which we may find all possible involutive two-dimensional subdistributions $\theta_1(x)$ of $\theta(x)$.

Let $\xi_1(x)$, $\xi_2(x)$, $\xi_3(x)$ be a basis of the distribution $\theta(x)$, then the following theorems give us a way to find a pair of suitable vector fields η_{01}, η_{02}.

Theorem 2 (J. Tabov [6]). *There exists only one (up to a scalar multiplier) vector field* $\eta_{02}(x)$ *satisfying the system*

$$w^i(\eta) = 0 \ (i = 1,2) \ , \quad \partial w^2(\xi_j, \eta) = 0 \ \ (j = 1,2,3). \tag{13}$$

Theorem 3 (J. Tabov [6]). *If the restriction of* ∂w^1 *on* $\theta(x)$ *is non-trivial, then there exists only one (up to a scalar multiplier) vector field* $\eta_{01}(x)$ *satisfying the system*

$$w^i(\eta) = 0 \ (i = 1,2) \ , \quad \partial w^1(\xi_j, \eta) = 0 \ \ (j = 1,2,3). \tag{14}$$

Since it is not clear whether the subdistribution $\theta_1(x)$ (determined as a linear hull of η_{01}, η_{02}) is involutive, we will consider the following two hypotheses:

(i) If the subdistribution $\theta_1(x)$ is involutive, then the system of PDEs (5) has a set of three functionally independent solutions. The following lemma holds.

Lemma 2 (J. Tabov [6]). *If there exists a two-dimensional involutive subdistribution* $\theta_1(x)$ *of* $\theta(x)$*, which is the linear hull of the fields* η_{01}*,* η_{02}*, then the system (1) has a solution determined by the implicit function theorem from any three functionally independent solutions of the system (5). The converse is also true.*

Hence, there exist implicit functions $x^j = x^j(x^1, x^2)$ $(j = 3,4,5)$ determined by the system (6), forming a simple wave solution of (11).

(ii) If $\theta(x)$ is not involutive, then the system (11) has no solution.

4. Existence of a simple wave. The following lemma is true.

Lemma 3. *The vector fields*

$$\eta_{01} = p\xi_1 + K\xi_2 - t_1\xi_3 \ , \quad \eta_{02} = q\xi_1 + \xi_2 - t_2\xi_3, \tag{15}$$

where $K \neq 0$*,* $p \equiv (x^3)^2(K\partial_4 t_1 + \partial_5 t_1 - t_1\partial_4 K - t_2\partial_5 K) + x^3(\partial_2 t_1 - \partial_1 K)$*,* $q \equiv (x^3)^2(K\partial_4 t_2 + \partial_5 t_2) + x^3\partial_2 t_2$ *satisfy the conditions of Theorem 3 and Theorem 2, respectively.*

P r o o f. Replacing $\eta = \eta_{02}(x)$ in the system (13) and $\eta = \eta_{01}(x)$ in (14), respectively we immediately get the statement. □

Let C^j $(j = 1, \ldots, 5)$ denote the commutator components in (4), i.e. $[\eta_{01}, \eta_{02}] = C^j(x)\partial_j$ $(j = 1, \ldots, 5)$; then we have

$$
\begin{aligned}
&C^1 = -\partial_1 K + t_2 \partial_2 K \ , \\
&C^2 = -K\partial_1 t_2 + t_1 \partial_2 t_2 + \partial_1 t_1 - t_2 \partial_2 t_1 \ , \\
&C^3 = K\partial_1 q - t_1 \partial_2 q + p\partial_3 q - \partial_1 p + t_2 \partial_2 p - q\partial_3 p \ , \\
&C^4 = C^5 = 0 \ .
\end{aligned} \tag{16}
$$

Theorem 4. *The subdistribution $\theta_1(x) \subset \theta(x)$, defined as a linear hull of the vectorial fields $\eta_{0i}(x)$ $(i = 1, 2)$ (specified in Lemma 3) is involutive and therefore the system (1) has a simple wave solution.*

Proof. In order to be involutive the subdistribution $\theta_1(x)$ spanned by the pair of vectorial fields $\eta_{01}(x)$, $\eta_{02}(x)$ it is necessary and sufficiently to exist linear dependence between $\eta_{01}(x)$, $\eta_{02}(x)$, $[\eta_{01}(x), \eta_{02}(x)]$, i.e. the rank of the matrix

$$
\mathbf{M} \equiv (\eta_{01}, \eta_{02}, [\eta_{01}, \eta_{02}]) \tag{17}
$$

should be equal to 2. Let us define the functions

$$
\begin{aligned}
&e(x, P(x), Q(x)) \equiv (pt_2 - qt_1)P(x) + (p - Kq)Q(x) \ , \\
&f(x, P(x), Q(x)) \equiv x^3[t_1 P(x) + KQ(x)] \ , \\
&g(x, P(x), Q(x)) \equiv x^3[t_2 P(x) + Q(x)] \ ,
\end{aligned} \tag{18}
$$

where K is determined by (9) and P, Q are some scalar C^2 functions with respect to x. □

Lemma 4. *If $P = C^1(x)$, $Q = C^2(x)$, then $e \equiv 0$, $f \equiv 0$ and $g \equiv 0$.*

Proof. Replacing $P = C^1(x)$, $Q = C^2(x)$ in the right-hand side of the functions e, f, g defined by (18) and making use the obvious identity $t_1 - Kt_2 \equiv 0$ we get the statement. □

Further, having in mind the classical rank theorem, it follows that in order to have $\operatorname{rank} \mathbf{M} = 2$ it is necessary and sufficient each one of the 3×3

determinants constructed by the elements of **M** to be annihilated, i.e.

$$\triangle_1 \equiv \begin{vmatrix} \eta^1_{01}(x) & \eta^2_{01}(x) & \eta^3_{01}(x) \\ \eta^1_{02}(x) & \eta^2_{02}(x) & \eta^3_{02}(x) \\ C^1(x) & C^2(x) & C^3(x) \end{vmatrix} = 0$$

(19) $$\triangle_2 \equiv \begin{vmatrix} \eta^1_{01}(x) & \eta^2_{01}(x) & \eta^4_{01}(x) \\ \eta^1_{02}(x) & \eta^2_{02}(x) & \eta^4_{02}(x) \\ C^1(x) & C^2(x) & C^4(x) \end{vmatrix} = 0$$

$$\triangle_3 \equiv \begin{vmatrix} \eta^1_{01}(x) & \eta^2_{01}(x) & \eta^5_{01}(x) \\ \eta^1_{02}(x) & \eta^2_{02}(x) & \eta^5_{02}(x) \\ C^1(x) & C^2(x) & C^5(x) \end{vmatrix} = 0.$$

Indeed, expanding successively the determinants and having in mind Lemma 4, we get the equalities $\triangle_1 = e \equiv 0$, $\triangle_2 = f \equiv 0$, $\triangle_3 = g \equiv 0$. Further, taking into account Lemma 2, we infer that the system (1) has a simple wave solution obtained by means of the implicit function theorem, namely $u^j = u^j(x^1, x^2)$ $(j = 1, 2)$ satisfying the system (8) as well. □

The last result (Theorem 4) shows, that for arbitrary coefficients $a^i_j(x, u)$ $(i, j = 1, 2)$ which are C^2 functions, the strictly hyperbolic system (1) possesses always a simple wave solution, which can be found following the method sketched in Section 3 (see [6] as well). The above stated result gives the existence only of a simple wave solution. However, for certain hyperbolic systems of type (1) there may exist another type of solutions save the always existing simple wave. By means of both Grundland's Lemma 1 and Theorem 4 it is possible to be clarified what type of solution has been found.

REFERENCES

[1] A. Jeffrey. Equations of Evolution and Waves, Wave Phenomena: Modern Theory and Applications. Nord-Holand Math. Studies vol. **97**, Elsevier, N.Y., 1984, 1-17.

[2] M. Burnat. Riemann invariants. *Fluid Dynamics* **4** (1969), 17-27.

[3] Z. Peradzinski. Riemann invariants for nonplanar k-waves. *Bull. Acad. Polon. Sci.* **19** (1971), 717-732.

[4] A. M. Grundland. Riemann invariants for non-homogeneous systems of quasilinear partial differential equations. *Bull. Acad. Polon. Sci.* **22** (1974), 273-282.

[5] J. Tabov. On the extending of the resolving distributions of Pfaff's systems. *Russ. Math. Surveys* **29** (1974), 243-244.

[6] J. Tabov. Simple waves and simple states in $\mathbb{R}^2$. 1994, unpublished.

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