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EQUISUMMABILITY THEOREMS FOR LAGUERRE SERIES

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ABSTRACT. Here we prove results about Riesz summability of classical Laguerre series, locally uniformly or on the Lebesgue set of the function f such that $(\int_0^\infty (1+x)^{mp} |f(x)|^p dx)^{1/p} < \infty$, for some p and m satisfying $1 \leq p \leq \infty$, $-\infty < m < \infty$.

1. Introduction and statement of the main results. Consider the Laguerre series in the form

$$f(y) \sim \sum_{k=0}^{\infty} f_k \Phi_k^\delta(y), \quad f_k = \int_0^\infty f(y) \Phi_k^\delta(y) dy, \quad \delta \geq -\frac{1}{2},$$

and the corresponding partial sum

$$E_\lambda f(y) = \int_0^\infty e(\lambda, x, y) f(x) dx,$$

where

$$e(\lambda, x, y) = \sum_{\mu_k < \lambda} \Phi_k^\delta(x) \Phi_k^\delta(y),$$

and

$$\mu_k = 4k + 4, \quad \Phi_k^\delta(x) = \left[\frac{\Gamma(k+1)}{\Gamma(k+\delta+1)} \right]^{1/2} e^{-x^2/2} \sqrt{2} x^{\delta+1/2} L_k^\delta(x^2)$$

are the eigenvalues and orthonormalized eigenfunctions of the operator

$$A = -\frac{d^2}{dx^2} + x^2 + \left(\delta^2 - \frac{1}{4}\right)x^{-2} + 2 - 2\delta \quad \text{in } L^2(0, \infty).$$

Here $L_k^\delta(x) = (k!)^{-1} e^x x^{-\delta} \left(\frac{d}{dx}\right)^k (e^{-x} x^{k+\delta})$ are the Laguerre polynomials and $e(\lambda, x, y)$ is called the spectral function of A .

The Laguerre series are investigated in the classical Szegő book [7], where sufficient conditions are given on the behaviour of the function f at infinity so that the following equiconvergence result holds:

$$E_\lambda f(y) - \int_{y-\varepsilon}^{y+\varepsilon} e^0(\lambda, x, y) f(x) dx \rightarrow 0, \quad 0 < \varepsilon < y,$$

locally uniformly on $(0, \infty)$. Here $e^0(\lambda, x, y)$ is the spectral function of the main part $-\frac{d^2}{dx^2}$.

These conditions are significantly improved in [3], where the method of the spectral function is applied. To enlarge further the classes of functions we can consider the Riesz summability method. For other results see, for example [4], [5], [9] and the bibliography in [8].

Let

$$E_\lambda^\alpha f(y) = \sum \left(1 - \frac{\mu_k}{\lambda}\right)^\alpha f_k \Phi_k^\delta(y), \quad \mu_k < \lambda$$

be the Riesz means of order α . Then

$$(1.1) \quad E_\lambda^\alpha f(y) = \int_0^\infty I^\alpha e(\lambda, x, y) f(x) dx,$$

where

$$I^\alpha e(\lambda, x, y) = \int_0^\lambda \left(1 - \frac{\mu}{\lambda}\right)^\alpha d e(\mu, x, y)$$

is the Riesz kernel of order α .

The main results proved in this paper are concerned with:

a) Equisummability locally uniformly for the functions f from the space L_m^p with a norm

$$\|f\|_{m,p} = \left(\int_0^\infty (1+x)^{mp} |f(x)|^p dx \right)^{\frac{1}{p}}, \quad 1 \leq p \leq \infty.$$

b) Summability on the Lebesgue set of the functions from the space L_m^p for $1 \leq p < \infty$, $m \geq -m_0(\alpha, p)$ if $p \neq 4/3$ and $m > -m_0(\alpha, 4/3)$ if $p = 4/3$.

Here

$$(1.2) \quad m_0(\alpha, p) = 2\alpha + \min\left(\frac{1}{p}, 1 - \frac{1}{3p}\right), \quad \alpha > 0.$$

Note that in [9] a related result is proved for $\alpha > \frac{1}{6}$ and for the case $m = 0$.

c) Summability locally uniformly for the functions f with the properties: $f(x)$ and $f'(x)$ are $O(x^\beta)$ as $x \rightarrow \infty$ for $\beta < 2\alpha + 1$. The case $\alpha = 0$ is considered in [3].

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We start with theorems about equisummability locally uniformly, which means that as $\lambda \rightarrow \infty$

$$(1.3) \quad R_\lambda^\alpha f(y) \stackrel{\text{def}}{=} E_\lambda^\alpha f(y) - \int_{y-\varepsilon}^{y+\varepsilon} f(x) I^\alpha e(\lambda, x, y) dx \rightarrow 0,$$

uniformly with respect to $y \in [c, d]$ for any compact interval $[c, d] \subset (0, \infty)$, where $\varepsilon \in (0, c)$.

Theorem 1 (equisummability locally uniformly). *If $\alpha > 0$ then the convergence (1.3) is fulfilled in the following cases:*

(a) $f \in L_m^p$, $1 \leq p < \infty$, $p \neq \frac{4}{3}$ if $m \geq -m_0(\alpha, p)$

(b) $f \in L_m^{4/3}$ if $m > -m_0\left(\alpha, \frac{4}{3}\right)$

(c) $f \in L_m^\infty$ if $m > -m_0(\alpha, \infty)$

(d) $f \in C_m$ if $m \geq -m_0(\alpha, \infty)$.

Here C_m is the subspace of L_m^∞ consisting of all continuous functions f such that $x^m f(x) \rightarrow 0$ as $x \rightarrow \infty$ and $f(x) \rightarrow 0$ as $x \rightarrow 0$.

Theorem 2. *Let $f \in L_{loc}^1[0, \infty)$ and the derivative $f'(x)$ exists for $x > A_f$. If $f(x)$ and $f'(x)$ are $O(x^\beta)$, $x \rightarrow \infty$ for $\beta < 2\alpha + 1$, $\alpha > 0$, then the convergence (1.3) is true.*

Corollary 1 (equisummability on the Lebesgue set). *Under the conditions of Theorems 1 or 2 we have*

$$(1.4) \quad E_\lambda^\alpha f(y) - \int_{y-\varepsilon}^{y+\varepsilon} f(x) I^\alpha e^0(\lambda, x, y) dx \rightarrow 0,$$

where $y \in (0, \infty)$ is on the Lebesgue set of the function f and $0 < \varepsilon < y$. Here

$$e^0(\lambda, x, y) = \frac{1}{\pi} \cdot \frac{\sin \sqrt{\lambda}(x-y)}{x-y}$$

and

$$(1.5) \quad I^\alpha e^0(\lambda, x, y) = \lambda^{1/2} F_\alpha(\sqrt{\lambda}|x-y|),$$

where

$$(1.6) \quad F_\alpha(s) = d_\alpha s^{-\frac{1}{2}-\alpha} J_{1/2+\alpha}(s), \quad d_\alpha = 2^\alpha (2\pi)^{-\frac{1}{2}} \Gamma(\alpha+1).$$

Corollary 2 (summability on the Lebesgue set). *Under the conditions of Theorem 1 or 2, $E_\lambda^\alpha f(y) \rightarrow f(y)$ on the Lebesgue set of the function f .*

Corollary 3 (summability in L_{loc}^q). *Under the conditions of Theorem 1 or 2, $E_\lambda^\alpha f \rightarrow f$ in L_{loc}^q $1 \leq q < \infty$ if in addition $f \in L_{loc}^q(0, \infty)$.*

Corollary 4 (summability locally uniformly). *Under the conditions of Theorem 1 or 2, $E_\lambda^\alpha f(y) \rightarrow f(y)$ locally uniformly if in addition f is continuous.*

Corollary 5 (localization principle). *Let $y > 0$, $\varepsilon > 0$ be fixed. Then under the conditions of Theorem 1 or 2, $E_\lambda^\alpha f \rightarrow 0$ if $f(x) = 0$ for $|x-y| < \varepsilon$.*

2. Asymptotics of Riesz kernels. In proving the main results, stated in § 1, we shall apply the method of the spectral function as in [3] and especially [4], where this method was used to find the uniform asymptotics of the Riesz kernels of order α in the case of Hermite series.

First we state the uniform asymptotics of the Riesz kernels (1.1) which we need. It is convenient to consider also the functions

$$(2.1) \quad e_\alpha(\lambda, x, y) = \lambda^\alpha I^\alpha e(\lambda, x, y), \quad E_\alpha(\lambda, x, y) = e_\alpha(\lambda, \sqrt{\lambda}x, \sqrt{\lambda}y).$$

For our purposes it is sufficient to consider only the cases: $0 < a \leq x < \infty$, $0 < c \leq y \leq d < \infty$. It is convenient to split the interval $[a, \infty)$ into the intervals $[a, b]$, $[A, (1-\varepsilon)\sqrt{\lambda}]$, $[(1-\varepsilon)\sqrt{\lambda}, (1+\varepsilon)\sqrt{\lambda}]$, $[(1+\varepsilon)\sqrt{\lambda}, \infty)$.

Theorem 3. *Let $0 < a \leq x \leq b$ and $0 < c \leq y \leq d$. Then,*

$$(2.2) \quad |I^\alpha e(\lambda, x, y) - (I^\alpha e^0(\lambda, x, y) + C_\delta I^\alpha e^0(\lambda, x, -y))| \leq C(1 + \sqrt{\lambda}|x-y|)^{-\alpha-1},$$

where $\alpha > 0$, $I^\alpha e^0(\lambda, x, y)$ is given by (1.5) and C_δ is a constant.

Here and later on, C is a positive constant, not depending on λ, x, y .

Theorem 4. Let $\frac{A}{\sqrt{\lambda}} \leq x \leq 1 - \varepsilon$, $\frac{c}{\sqrt{\lambda}} \leq y \leq \frac{d}{\sqrt{\lambda}}$. Then for $A > d$, $\varepsilon > 0$ we have the uniform asymptotics

$$(2.3) \quad E_\alpha(\lambda, x, y) = F_\alpha(\lambda, x, y) + C_\delta F_\alpha(\lambda, x, -y),$$

where

$$(2.4) \quad F_\alpha(\lambda, x, y) = \lambda^{-1/2} \sum_{k=1}^4 b_k(\lambda, x, y) e^{i\lambda\psi_k} + x^{-1-\alpha} O(\lambda^{-1})$$

and

$$(2.5) \quad |b_k| \leq Cx^{-1-\alpha}, \quad |\partial_x b_k| \leq Cx^{-2-\alpha}$$

$$(2.6) \quad |\partial_x \psi_k|^2 = 1 - x^2, \quad |\partial_x^2 \psi_k|^2 \leq C(1 - x^2)^{-1}.$$

Theorem 5. If $1 - \varepsilon \leq x \leq 1 + \varepsilon$, $\frac{c}{\sqrt{\lambda}} \leq y \leq \frac{d}{\sqrt{\lambda}}$, then there exists a positive number $\varepsilon > 0$ such that the uniform asymptotics (2.3) is satisfied, where

$$(2.7) \quad F_\alpha(\lambda, x, y) = \sum_{k=0}^{\infty} (a_{1k}(\lambda, x, y) \lambda^{-k-1/3} + b_{1k}(\lambda, x, y) \lambda^{-k-2/3})$$

and

$$a_{1k} = (a_k e^{\lambda A} + b_k e^{\lambda \bar{A}}) Ai(\lambda^{2/3} B), \quad b_{1k} = (c_k e^{\lambda A} + d_k e^{\lambda \bar{A}}) Ai'(\lambda^{2/3} B).$$

The functions $\lambda \rightarrow a_k, b_k, d_k, c_k$ or their derivatives with respect to x are bounded. Here $Ai(s) = \frac{1}{2\pi} \int e^{i(st+t^3/3)}$ is the Airy function, $A = A(x, y)$, $B = B(x, y)$ are smooth, $Re A = 0$ and $B(x, y) \sim C(y)(x^2 - 1)$ as $x^2 \rightarrow 1$, $c(y) > 0$.

Analogously to Theorem 6 [3] we have

Theorem 6. Let $x > 1 + \varepsilon$ for some $\varepsilon > 0$. Then

$$|E_\alpha(\lambda, x, y)| \leq C(x^2 - 1)^{-1/4} \lambda^{-1/2} \exp(-C\varepsilon(x^2 - 1)^{1/2} \lambda), \quad C > 0.$$

As a consequence of Theorems 5 and 6 it follows

Corollary 6. *If $x^2 > \lambda + \lambda^{1/3+\varepsilon}$, $\varepsilon > 0$ then*

$$|I^\alpha e(\lambda, x, y)| \leq C\lambda^{-\alpha-1/3} \exp\left(-C\lambda^{1/3} \left(\frac{x^2}{\lambda} - 1\right)^{1/2}\right).$$

From Theorem 5 and the asymptotics of the Airy function it follows

Corollary 7. *If $1 - \varepsilon_1 < x^2 < 1 - \lambda^{-2/3+\varepsilon}$, $\varepsilon > 0$ and $\frac{c}{\sqrt{\lambda}} \leq y \leq \frac{d}{\sqrt{\lambda}}$, then we have the uniform asymptotics (2.3), where*

$$F_\alpha(\lambda, x, y) = \lambda^{-1/2} \sum_{k=1}^4 \left(a_k(1-x^2)^{-1/4} + b_k(1-x^2)^{1/4} \right) e^{i\lambda\psi_k} + (1-x^2)^{-1} O(\lambda^{-1}),$$

the functions $\lambda \rightarrow a_k(\lambda, x, y), b_k(\lambda, x, y)$ or their derivatives over x are bounded, and ψ_k satisfy (2.6).

3. Proof of Theorem 1. First, according to [7], we have $|e(\lambda, x, y)| \leq c$ if $0 < x, y < c$, $|x - y| \geq \varepsilon > 0$, and consequently

$$|I^\alpha e(\lambda, x, y)| \leq c \quad \text{if} \quad 0 < x, y < c, \quad |x - y| > \varepsilon > 0.$$

Since

$$R_\lambda^\alpha f(y) = \left(\int_0^{y-\varepsilon} + \int_{y+\varepsilon}^\infty \right) f(x) I^\alpha e(\lambda, x, y) dx$$

we can write

$$(3.1) \quad |R_\lambda^\alpha f(y)| \leq c \left[\int_0^A |f(x)| dx + |K(\lambda, y)| \right],$$

where $c \leq y \leq d$ and for some large $A > 0$,

$$(3.2) \quad K(\lambda, y) = \int_A^\infty f(x) I^\alpha e(\lambda, x, y) dx.$$

Now let $K_i(\lambda, y) = \int a_i(\lambda, x) f(x) I^\alpha e(\lambda, x, y) dx$, where $a_i(\lambda, x)$ is the characteristic function of the set A_i and

$$\begin{aligned} A_1 &= \{x \in R_+, A^2 < x^2 < (1-\varepsilon)\lambda\}, & A_2 &= \{x \in R_+, (1-\varepsilon)\lambda < x^2 < \lambda - \lambda^{1/3}\}, \\ A_3 &= \{x \in R_+, |x^2 - \lambda| < \lambda^{1/3}\}, & A_4 &= \{x \in R_+, \lambda + \lambda^{1/3} < x^2 < \lambda + \lambda^{1/3+\varepsilon}\}, \\ A_5 &= \{x \in R_+, x^2 > \lambda + \lambda^{1/3+\varepsilon}\} \end{aligned}$$

The estimates below are uniform with respect to $y \in [c, d]$ and the number A is large enough, say $A > d + 1$.

a) Estimate of $K_1(\lambda, y)$. Using Theorem 4 we have

$$|I^\alpha e(\lambda, x, y)| \leq C\lambda^{-\alpha/2}x^{-1-\alpha}, \quad x \in A_1.$$

Hence by the Hölder inequality,

$$\begin{aligned} |K_1(\lambda, y)| &\leq c\lambda^{-\alpha/2} \int a_1(\lambda, x)|f(x)|x^{-1-\alpha}dx \\ &\leq c\lambda^{-\alpha/2}\|f\|_{m,p}J(\lambda), \end{aligned}$$

where

$$J(\lambda) = \left(\int_1^{\sqrt{\lambda}} \sigma^{-(1+\alpha+m)p'} d\sigma \right)^{\frac{1}{p'}}, \quad \frac{1}{p'} + \frac{1}{p} = 1.$$

Therefore

$$(3.3) \quad |K_1(\lambda, y)| \leq c\|f\|_{m,p} \quad \text{if } m \geq -2\alpha - 1/p, \quad 1 \leq p < \infty$$

$$(3.4) \quad |K_1(\lambda, y)| \leq \lambda^{-\gamma}\|f\|_{m,\infty} \quad \text{if } m \geq -2\alpha, \quad \alpha > 0 \quad \text{for some } \gamma > 0.$$

b) Estimate of $K_2(\lambda, y)$. Using Theorem 5 and the estimates $|Ai(s)| \leq c|s|^{-1/4}$, $|Ai'(s)| \leq c(1 + |s|)^{1/4}$, we have

$$|I^\alpha e(\lambda, x, y)| \leq c\lambda^{-\alpha-1/2} \left(1 - \frac{x^2}{\lambda} \right)^{-1/4}, \quad x \in A_2.$$

Therefore

$$|K_2(\lambda, y)| \leq c\lambda^{-\alpha-1/2-m/2}\|f\|_{m,p}J(\lambda),$$

where

$$J(\lambda) = j^{\frac{1}{2p'}} \left(\int_{\lambda^{-2/3}}^1 \sigma^{-p'/4} d\sigma \right)^{\frac{1}{p'}},$$

hence

$$(3.5) \quad |K_2(\lambda, y)| \leq c\lambda^{-(m+m_0)/2}\|f\|_{m,p}, \quad p \neq 4/3$$

$$(3.6) \quad |K_2(\lambda, y)| \leq c\lambda^{-(m+m_0)/2}(\log \lambda)^{1/4}\|f\|_{m,p}, \quad p = 4/3.$$

Here m_0 is given by (1.2).

c) Estimate of $K_3(\lambda, y)$. According to Theorem 5 we have

$$|I^\alpha e(\lambda, x, y)| \leq \lambda^{-\alpha-1/3}, \quad x \in A_3.$$

Hence

$$|K_3(\lambda, y)| \leq c\lambda^{-\alpha-1/3}\|f\|_{m,p}J(\lambda),$$

where

$$J(\lambda) = \left(\int a_3(\lambda, x)x^{-mp'} dx \right)^{\frac{1}{p'}} \leq c\lambda^{-\frac{m}{2}-\frac{1}{6p'}}.$$

Therefore

$$(3.7) \quad |K_3(\lambda, y)| \leq c\lambda^{-(m+m_0)/2}\|f\|_{m,p}.$$

d) Estimate of $K_4(\lambda, y)$. Theorem 6 implies

$$|I^\alpha e(\lambda, x, y)| \leq c\lambda^{-\alpha-1/2} \left(\frac{x^2}{\lambda} - 1 \right)^{-\frac{1}{4}}, \quad x \in A_4.$$

Hence

$$|K_4(\lambda, y)| \leq c\lambda^{-\alpha-\frac{1}{2p}-\frac{m}{2}}\|f\|_{m,p}J(\lambda),$$

where

$$J(\lambda) = \left(\int_{\lambda^{-2/3}}^{\lambda^{-2/3+\varepsilon}} \sigma^{-p'/4} d\sigma \right)^{\frac{1}{p'}}.$$

Therefore

$$(3.8) \quad |K_4(\lambda, y)| \leq c\lambda^{-(m+m_0)/2}\|f\|_{m,p}, \quad 1 \leq p < \frac{4}{3}$$

$$(3.9) \quad |K_4(\lambda, y)| \leq c\lambda^{-(m+m_0)/2}(\log \lambda)^{1/4}\|f\|_{m,p}, \quad p = \frac{4}{3}$$

$$(3.10) \quad |K_4(\lambda, y)| \leq c\lambda^{-(m+m_0)/2-\gamma}\|f\|_{m,p}, \quad \text{if } p > \frac{4}{3} \text{ for some } \gamma > 0.$$

f) Estimate of $K_5(\lambda, y)$. Corollary 6 gives

$$|I^\alpha e(\lambda, x, y)| \leq c\lambda^{-\alpha-1/3} \exp(-c\lambda^{\varepsilon/2}), \quad \text{if } x \in A_5, \quad x < \lambda$$

$$|I^\alpha e(\lambda, x, y)| \leq c\lambda^{-\alpha-1/3} \exp(-c\sqrt{x}), \quad \text{if } x > \lambda, \quad c > 0.$$

Hence we obtain

$$(3.11) \quad |K_5(\lambda, y)| \leq c\lambda^{-\gamma} \|f\|_{m,p} \quad \text{for some } \gamma > 0.$$

Thus the estimates (3.3)–(3.11) give

$$(3.12) \quad |R_\lambda^\alpha f(y)| \leq c\|f\|_{m,p}, \quad \text{if } m \geq -m_0, \quad 1 \leq p < \infty, \quad p \neq 4/3$$

$$R_\lambda^\alpha f(y) \rightarrow 0 \quad \text{if } m > -m_0(\alpha, p) \quad \text{and } p = 4/3 \quad \text{or } p = \infty.$$

On the other hand it is not hard to see that

$$(3.13) \quad R_\lambda^\alpha f \rightarrow 0 \quad \text{uniformly on } [c, d] \quad \text{if } f \in C_0^\infty(0, \infty).$$

Finally, if $f \in L_m^p$, $1 \leq p < \infty$ or $f \in C_m$, then we can find $g \in C_0^\infty$ such that $\|f - g\|_{m,p} < \varepsilon$. Then (3.12) implies $|R_\lambda^\alpha f| \leq c\varepsilon + |R_\lambda^\alpha g|$, whence (3.13) gives $R_\lambda^\alpha f \rightarrow 0$ locally uniformly.

4. Proof of Theorem 2. We start with (3.1) and (3.2), where $1 \leq i \leq 4$, $a_i(\lambda, x)$ is the characteristic function of the set B_i and $B_1 = A_1$,

$$B_2 = \left\{ x : (1 - \varepsilon)\lambda < x^2 < \lambda - \lambda^{\frac{1}{3} + \varepsilon} \right\}, \quad B_3 = \left\{ x : |x^2 - \lambda| < \lambda^{\frac{1}{3} + \varepsilon} \right\}, \quad B_4 = A_5.$$

Now, let $B_i(\lambda, y) = K_i(\lambda, \sqrt{\lambda}y)$, $i = 1, 2$. Then

$$B_i(\lambda, y) = \lambda^{1/2-\alpha} \int_0^\infty a_i(\lambda, \sqrt{\lambda}x) f(\sqrt{\lambda}x) E_\alpha(\lambda, x, y) dx.$$

a) Estimate of $K_1(\lambda, y)$. Using Theorem 4 we can write

$$B_1(\lambda, y) = I(\lambda, y) + C_\delta I(\lambda, -y),$$

where

$$(4.1) \quad I(\lambda, y) = \lambda^{1/2-\alpha} \int a_1(\lambda, \sqrt{\lambda}x) f(\sqrt{\lambda}x) F_\alpha(\lambda, x, y) dx$$

and $F_\alpha(\lambda, x, y)$ is given by (2.4). It is enough to find the asymptotics of $I(\lambda, y)$. We have by (2.4),

$$(4.2) \quad I(\lambda, y) = \lambda^{-\alpha} \sum_{k=1}^4 \int_0^\infty a_1(\lambda, \sqrt{\lambda}x) b_k(\lambda, x, y) e^{i\lambda\psi_k} f(\sqrt{\lambda}x) dx + R_1 O(\lambda^{-\alpha-1/2})$$

where, using $f(x) = O(x^\beta)$, $x \rightarrow \infty$,

$$(4.3) \quad R_1 = \int a_1(\lambda, \sqrt{\lambda x}) |f(\sqrt{\lambda x})| x^{-1-\alpha} dx \leq C \lambda^{\beta/2} J(\lambda),$$

$$J(\lambda) = \int_{\lambda^{-1/2}}^1 x^{-1-\alpha+\beta} dx \leq c \begin{cases} \lambda^{-\beta/2+\alpha/2}, & \beta < \alpha \\ \log \lambda, & \beta = \alpha \\ 1, & \beta > \alpha \end{cases}.$$

Then integrating by parts and using (2.5), (2.6), we get for $\beta \leq 2\alpha + 1$,

$$|I(\lambda, y)| \leq C \lambda^{-\alpha-1/2+\beta/2} J(\lambda) + C \lambda^{-1/2}.$$

If $\beta < 2\alpha + 1$, we see that $|I(\lambda, y)| \leq \lambda^{-\gamma}$ for some $\gamma > 0$, hence

$$I(\lambda, y) \rightarrow 0 \text{ locally uniformly}$$

or

$$(4.4) \quad K_1(\lambda, y) \rightarrow 0 \text{ locally uniformly.}$$

b) Estimate of $K_2(\lambda, y)$. We shall use Corollary 7. Then analogously to (4.1), (4.2) and (4.3) we see that it suffices to estimate

$$(4.5) \quad B(\lambda, y) = \lambda^{-\alpha} \int a_2(\lambda, \sqrt{\lambda x}) a(\lambda, x, y) (1-x^2)^{-1/4} f(\sqrt{\lambda x}) e^{i\lambda\psi} dx + O(\lambda^{-1/2-\alpha}) R_2$$

where $a(\lambda, x, y) = a_k(\lambda, x, y)$ and

$$(4.6) \quad R_2 = \int a_2(\lambda, \sqrt{\lambda}) |f(\sqrt{\lambda x})| (1-x^2)^{-1} dx \leq c \lambda^{\beta/2} \log \lambda.$$

Let

$$I(\lambda) = \int a_2(\lambda, \sqrt{\lambda}) a(\lambda, x, y) (1-x^2)^{-1/4} f(\sqrt{\lambda x}) e^{i\lambda\psi} dx.$$

Integrating by parts and using (2.6) we get

$$|I(\lambda)| \leq C \lambda^{-1} \int a_2(\lambda, \sqrt{\lambda x}) \left[\lambda^{1/2} |f'(\sqrt{\lambda x})| (1-x^2)^{-3/4} + |f(\sqrt{\lambda x})| (1-x^2)^{-7/4} \right] dx + C \lambda^{-1/2}.$$

Since $1-x^2 > \lambda^{-2/3+\delta}$ we obtain for $\beta > 0$, $\varepsilon > 0$,

$$(4.7) \quad |I(\lambda)| \leq C \lambda^{-1/2+\beta/2}.$$

Thus (4.5), (4.6) and (4.7) imply

$$|B(\lambda, y)| \leq C\lambda^{-\alpha-1/2+\beta/2} + C\lambda^{-\alpha-1/2+\beta/2} \log \lambda \leq C\lambda^{-\gamma} \quad \text{for some } \gamma > 0$$

since $\beta < 2\alpha + 1$. In other words,

$$(4.8) \quad |K_2(\lambda, y)| \leq C\lambda^{-\gamma} \rightarrow 0 \quad \text{locally uniformly.}$$

c) Estimate of $K_3(\lambda, y)$. Theorem 5 gives

$$|K_3(\lambda, y)| \leq C\lambda^{-\alpha-1/3} \int_0^\infty a_3(\lambda, x)|f(x)|dx.$$

Hence

$$(4.9) \quad |K_3(\lambda, y)| \leq C\lambda^{-\alpha+\beta/2-1/2+\varepsilon} \rightarrow 0 \quad \text{if } 0 < \varepsilon < \alpha - \frac{\beta}{2} + \frac{1}{2}.$$

Finally it is easy to prove (see 3.11) that

$$(4.10) \quad K_4(\lambda, y) \rightarrow 0 \quad \text{locally uniformly.}$$

Thus (3.1) and the estimates (4.4), (4.8), (4.9), (4.10) give

$$|R_\lambda^\alpha f(y)| \leq C \int_0^A |f(x)|dx + o(1), \quad \text{locally uniformly.}$$

Now the proof finishes analogously to the proof of Theorem 1. \square

5. Proof of Corollaries 1–4.

Proof of Corollary 1. Let $y \in (0, \infty)$ is on the Lebesgue set of the function f and $0 < \varepsilon < y$. Comparing (1.3), (1.4) we have only to prove

$$(5.1) \quad I(\lambda, y) = \int_{y-\varepsilon}^{y+\varepsilon} f(x)[I^\alpha e(\lambda, x, y) - I^\alpha e^0(\lambda, x, y)]dx \rightarrow 0.$$

Let $\tilde{f}(x) = f(x)\chi(x)$ and let $\chi(x)$ be the characteristic function of the set $(y - \varepsilon, y + \varepsilon)$. According to theorem 3,

$$(5.2) \quad |I^\alpha e(\lambda, x, y) - I^\alpha e^0(\lambda, x, y)| \leq C[\lambda^{-\alpha/2} + H_\alpha(\sqrt{\lambda}|x - y|)], \quad \alpha > 0,$$

$0 < y - \varepsilon \leq x \leq y + \varepsilon$, where $H_\alpha(s) = (1 + s)^{-\alpha-1}$, $s > 0$. Since $\alpha > 0$, then $H_\alpha(s) \in L^1(R)$, hence Theorem 1.25 [6] gives

$$\int \tilde{f}(x)\sqrt{\lambda}H_\alpha(\sqrt{\lambda}|x - y|)dx \rightarrow \tilde{f}(y),$$

or

$$(5.3) \quad \int_{y-\varepsilon}^{y+\varepsilon} f(x)H_\alpha(\sqrt{\lambda}|x-y|)dx \rightarrow 0.$$

Evidently (5.1) follows from (5.2), (5.3). \square

Proof of Corollary 2. According to Corollary 1 we have to prove

$$I(\lambda, y) = \int_{y-\varepsilon}^{y+\varepsilon} f(x)I^\alpha e^0(\lambda, x, y) \rightarrow f(y),$$

where y is on the Lebesgue set of the function f and $0 < \varepsilon < y$. Using (1.5) and \tilde{f} , $F_\alpha(s) \in L^1(R)$ for $\alpha > 0$, we see that Theorem 1.25 [6] implies

$$I(\lambda, y) = \int_{-\infty}^{+\infty} \tilde{f}(x)\lambda^{1/2}F_\alpha(\sqrt{\lambda}|x-y|)dx \rightarrow \tilde{f}(y) = f(y). \quad \square$$

Proof of Corollary 3. First we have according to Theorem 1 or 2 $R_\lambda^\alpha f \rightarrow 0$ in $L_{loc}^q(0, \infty)$. Thus according to (1.3) it is sufficient to prove

$$(5.4) \quad I(\lambda, y) = \int_{y-\varepsilon}^{y+\varepsilon} f(x)I^\alpha e(\lambda, x, y)dx \rightarrow f(y) \quad \text{if } L^q[c, d],$$

$0 < c < d$, $0 < \varepsilon < c$. Let $\tilde{f}(x) = f(x)\chi(x)$, where χ is the characteristic function of $(c - \varepsilon, c + \varepsilon)$. Hence we can write

$$(5.5) \quad I(\lambda, y) = J_1(\lambda, y) - J_2(\lambda, y) \quad \text{for } c \leq y \leq d,$$

where

$$J_1(\lambda, y) = \int_0^\infty \tilde{f}(x)I^\alpha e(\lambda, x, y)dx,$$

$$J_2(\lambda, y) = \int_M \tilde{f}(x)I^\alpha e(\lambda, x, y)dx, \quad M = \{x : |x-y| > \varepsilon\} \cap (c - \varepsilon, d + \varepsilon).$$

According to Theorem 3 we have $|I^\alpha e(\lambda, x, y)| \leq C\lambda^{-\alpha/2}$ if $c \leq y \leq d$, $x \in M$. Since $\alpha > 0$ it follows

$$(5.6) \quad J_2(\lambda, y) \rightarrow 0 \quad \text{uniformly in } c \leq y \leq d.$$

On the other hand, Theorem 1.25 [6] gives

$$\int \tilde{f}(x)\sqrt{\lambda}H_\alpha(\sqrt{\lambda}|x-y|)dx \rightarrow \tilde{f}(y) \quad \text{in } L^q \quad \text{if } 1 \leq q < \infty, H_\alpha \in L^1(R).$$

Therefore using (5.2) and $\alpha > 0$, we get

$$\int_0^\infty \tilde{f}(x)I^\alpha e(\lambda, x, y)dx - \int_{-\infty}^\infty I^\alpha e^0(\lambda, x, y)\tilde{f}(x)dx \rightarrow 0$$

in $L^q(c, d)$, $1 \leq q < \infty$.

The same Theorem 1.25 [6] and (1.5), (1.6) imply

$$\int \tilde{f}(x)I^\alpha e^0(\lambda, x, y)dx \rightarrow \tilde{f}(y) \text{ in } L^q \text{ if } 1 \leq q < \infty.$$

Therefore,

$$(5.7) \quad J_1(\lambda, y) \rightarrow \tilde{f}(y) \text{ in } L^q(c, d).$$

Thus (5.4) follows from (5.5)–(5.7). \square

Proof of Corollary 4. Let $f \in C(0, \infty)$ and $0 < c \leq y \leq d$. Find a function $g \in C_0(R)$ such that $g(y) = f(y)$ for $c \leq y \leq d$. Further we can proceed as in the proof of Corollary 3. Thus we have again (5.4)–(5.7) but now the convergence is uniform for $y \in [c, d]$. \square

6. Proof of Theorem 3.

We shall use the formula

$$(6.1) \quad e_\alpha(\lambda, x, y) = \Gamma(\alpha + 1)(2\pi i)^{-1} \int_S e^{\lambda p} V(p, x, y) H_\alpha(\lambda, p) \chi(p) dp,$$

where $S = \left(\varepsilon - i\frac{\pi}{2}, \varepsilon + i\frac{\pi}{2} \right)$, $\varepsilon > 0$, $\alpha > 0$, $\chi(p) \in C_0^\infty(S)$ and $s \rightarrow H_\alpha(s, p)$ is defined by

$$(6.2) \quad H_\alpha(s, p) = \sum_{k=-\infty}^{+\infty} e^{isk\pi/2} (p + ik\pi/2)^{-\alpha-1}, \quad p \in S, \quad \alpha > 0.$$

For proving (6.1) we notice that

$$e_\alpha(\lambda, x, y) = \lambda^\alpha I^\alpha e(\lambda, x, y) = \lambda_+^\alpha * de(\lambda, x, y)$$

and that the Laplace transform of λ_+^α is $\Gamma(\alpha + 1)p^{-\alpha-1}$. Thus

$$\int_0^\infty e^{-\lambda p} e_\alpha(\lambda, x, y) d\lambda = \Gamma(\alpha + 1)p^{-\alpha-1} V(p, x, y),$$

where

$$V(p, x, y) = \int_0^\infty e^{-\lambda p} de(\lambda, x, y), \quad \text{Re } p > 0.$$

Since

$$V(p, x, y) = \sum e^{-\mu_k p} \phi_k^\delta(x) \phi_k^\delta(y), \quad \mu_k = 4k + 4$$

we have

$$(6.3) \quad V\left(p + ik\frac{\pi}{2}, x, y\right) = V(p, x, y).$$

Further the inverse Laplace transform gives

$$e_\alpha(\lambda, x, y) = b \int_{\varepsilon - i\infty}^{\varepsilon + i\infty} e^{\lambda p} p^{-\alpha-1} V(p, x, y) dp, \quad b = \Gamma(\alpha + 1)(2\pi i)^{-1}$$

or using (6.2), (6.3) we get for $\alpha > 0$,

$$(6.4) \quad e_\alpha(\lambda, x, y) = b \int_{S_1} e^{\lambda p} V(p, x, y) H_\alpha(s, p) dp,$$

where $S_1 = \left(\varepsilon - i\frac{\pi}{4}, \varepsilon + i\frac{\pi}{4}\right)$.

Noticing that $p \rightarrow g(p) = e^{\lambda p} V(p, x, y) H_\alpha(\lambda, p)$ is $i\frac{\pi}{2}$ -periodic function it is not hard to see that (6.4) implies (6.1) for some $\chi \in C_0^\infty(S)$, $\chi = 1$ near $\varepsilon + i0$.

Now, we can write

$$(6.5) \quad e_\alpha(\lambda, x, y) = A_\lambda(x, y) + B_\lambda(x, y),$$

where

$$A_\lambda(x, y) = b \int_S e^{\lambda p} V(p, x, y) p^{-\alpha-1} \chi(p) dp,$$

$$B_\lambda(x, y) = b \int_S e^{\lambda p} V(p, x, y) h_\alpha(\lambda, p) \chi(p) dp$$

and the function $h_\alpha(\lambda, p)$ has no singularities on S . Further let the function $f(x) \in C_0^\infty(0, \infty)$ and consider the formula

$$\int_0^\infty e_\alpha(\lambda, x, y) f(x) dx = b \int e^{\lambda p} H_\alpha(\lambda, p) \chi(p) \left(\int_0^\infty V(p, x, y) f(x) dx \right) dp.$$

We want to take limit as $\varepsilon \rightarrow 0$. To this end we write

$$I(\lambda, y) = \int_0^\infty A_\lambda(x, y) f(x) dx = b \int e^{i\lambda t} V(it, y) (it + 0)^{-\alpha-1} i \chi(t) dt,$$

where $\chi(t) \in C_0^\infty\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\chi(t) = 1$, $|t| < \gamma$ for some $\gamma > 0$ and $V(it, y) = \int_0^\infty V(it, x, y) f(x) dx$ is a smooth function. Since in the sense of distributions

$$\left((it + 0)^{-\alpha-1}, \varphi(t)\right) = \lim_{\varepsilon \rightarrow 0} C_1 \int \int e^{-\varepsilon_1 \eta^2 - it\eta^2} \varphi(t) \eta^{2\alpha} dt d\eta, \quad \eta \in \mathbb{R}^2,$$

where

$$\varphi(t) = ibe^{i\lambda t}V(it, y)\chi(t) \quad \text{and} \quad C_1 = \frac{1}{\pi\Gamma(\alpha + 1)} \quad \text{we get}$$

$$I(\lambda, y) = \lim_{\varepsilon_1 \rightarrow 0} ibC_1 \int e^{i\lambda t}V(it, y)e^{-i\eta^2 t - \varepsilon_1 \eta^2} \chi(t) dt d\eta.$$

To represent $V(it, y)$ we shall use the generating function 1.1 46 [10]:

$$V(p, x, y) = (xy)^{1/2} e^{2p(\delta-1)} (\sinh 2p)^{-1} e^{-\left(\frac{x^2+y^2}{2}\right) \coth 2p} {}_i^{-\delta} J_\delta \left(\frac{ixy}{\sinh 2p} \right).$$

Using the formula (1), p. 74, (6), p. 75 and 3, 4, p. 168 from [10] we can write

$$(6.6) \quad J_\delta(z) = z^{-1/2} (e^{iz} C_\delta^+ f(-z) + e^{-iz} C_\delta^- f(z)) \quad \text{if} \quad \delta \geq -\frac{1}{2},$$

where

$$f(z) = \begin{cases} \frac{1}{2} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{1}{\Gamma(\delta + 1/2)} \int_0^\infty e^{-u} u^{\delta - \frac{1}{2}} \left(1 - \frac{iu}{2z}\right)^{\delta - \frac{1}{2}} du, & \delta > -\frac{1}{2} \\ \frac{1}{2} \left(\frac{2}{\pi}\right)^{\frac{1}{2}}, & \delta = -\frac{1}{2} \end{cases}$$

is a holomorphic function for $\text{Re } z \neq 0$. Here $C_\delta^+ = e^{\mp i \frac{\pi}{2} (\delta + \frac{1}{2})}$.

Note also the property for $f(t, u) = f(1/u \sin 2t)$,

$$(6.7) \quad \left| \partial_t^k f(t, u) \right| \leq C_k \quad \text{uniformly in} \quad u \in (0, c).$$

Therefore

$$(6.8) \quad V(p, x, y) = (\sinh 2p)^{-1/2} e^{-\frac{(x^2+y^2)}{2} \coth 2p} (e^{xy/\sinh 2p} a(p, xy) + C_\delta e^{-xy/\sinh 2p} a(p, -xy)),$$

where $C_\delta = e^{-\frac{i\pi}{2} (\delta + \frac{1}{2})} C_\delta^+$ and $a(p, x, y) = e^{2p(\delta-1)} f\left(\frac{ixy}{\sinh 2p}\right)$.

Now, since $-\frac{1}{2}(x^2 + y^2) \coth 2p + \frac{xy}{\sinh 2p} = -\frac{(x-y)^2}{4p} + s(p, x, y)$, $s(0, x, y) = 0$

and s has no singularities as $|\text{Im } p| < \frac{\pi}{2}$ we get

$$\begin{aligned} V(p, x, y) &= \\ &= (\sinh 2p)^{-1/2} \left(e^{-(x-y)^2/4p} b(p, x, y) + C_\delta e^{-(x+y)^2/4p} b(p, x, -y) \right) \end{aligned}$$

where $b(p, x, y) = e^{s(p, x, y)}a(p, xy)$, $b(0, x, y) = (2\pi)^{-1/2}$.

Now using the equality

$$\lim_{\varepsilon_2 \rightarrow 0} \int e^{-i\xi^2 t + i(x-y)\xi - \varepsilon_2 \xi^2} \frac{d\xi}{2\pi} = \begin{cases} (4\pi it)^{-1/2} e^{-\frac{i(x-y)^2}{4t}}, & t \neq 0 \\ \delta(x-y), & t = 0 \end{cases}$$

we obtain in $D'(R_+)$

$$V(it, x, y) = \lim_{\varepsilon_2 \rightarrow 0} [G(t, x, y, \varepsilon_2) + C_\delta G(t, x, -y, \varepsilon_2)],$$

where

$$G(t, x, y, \varepsilon_2) = \left(\frac{\sin 2t}{2t}\right)^{-1/2} (2\pi)^{-1/2} b(it, x, y) \int e^{-i\xi^2 t + i(x-y)\xi - \varepsilon_2 \xi^2} d\xi.$$

Hence

$$(6.9) \quad I(\lambda, y) = J(\lambda, y) + C_\delta J(\lambda, -y),$$

where

$$\begin{aligned} J(\lambda, y) &= \lim_{\varepsilon_2 \rightarrow 0} \int e^{-\varepsilon_1 \eta^2 + i\lambda t - i\eta^2 t} g_1(t, \eta, x, y) \times \\ &\quad \times \lim_{\varepsilon_2 \rightarrow 0} \int_0^\infty f(x) \left(\int e^{-i\xi^2 t + i(x-y)\xi - \varepsilon_2 \xi^2} d\xi \right) dx dt d\eta \end{aligned}$$

and

$$g_1(t, \eta, x, y) = b(it, x, y) \left(\frac{\sin 2t}{2t}\right)^{-1/2} (2\pi)^{-1/2} \frac{1}{2\pi^2} \chi(it) \eta^{2\alpha}.$$

Since $f(x) \in C_0^\infty(0, \infty)$ and $h(\xi) = \int_0^\infty f(x) e^{ix\xi} dx$ is rapidly decreasing, the integral $h_{\varepsilon_2}(t) = \int e^{-i\xi^2 t - iy\xi - \varepsilon_2 \xi^2} h(\xi) d\xi$ is absolutely convergent, and $|h_{\varepsilon_2}(t)| \leq \int |h(\xi)| d\xi$. Hence the Lebesgue theorem gives

$$J(\lambda, y) = \lim_{\substack{\varepsilon_1 \rightarrow 0 \\ \varepsilon_2 \rightarrow 0}} G(\lambda, y, \varepsilon_1, \varepsilon_2),$$

where

$$G(\lambda, y, \varepsilon_1, \varepsilon_2) = \int_0^\infty \left[\int e^{i\lambda t - i\xi^2 t - i\eta^2 t + i(x-y)\xi - \varepsilon_1 \eta^2 - \varepsilon_2 \xi^2} g_1(t, \eta, x, y) dt d\xi d\eta \right] f(x) dx$$

or

$$J(\lambda, y) = \lambda^{\alpha+3/2} \left[\int e^{i\lambda\psi} g(\xi, \eta) g_1(t, \eta, x, y) dt d\xi d\eta \right] f(x) dx + \\ + \lim_{\substack{\varepsilon_1 \rightarrow 0 \\ \varepsilon_2 \rightarrow 0}} \int_0^\infty I_{\varepsilon_1 \varepsilon_2}(\lambda, x, y) f(x) dx,$$

where

$$I_{\varepsilon_1 \varepsilon_2}(\lambda, x, y) = \lambda^{\alpha+3/2} \int e^{i\lambda\psi_{\varepsilon_1 \varepsilon_2}} g_1(t, \eta, x, y) (1 - g(\xi, \eta)) dt d\xi d\eta,$$

$$0 < a \leq x \leq b, \quad 0 < c \leq y \leq d, \quad \text{supp } f \subset [a, b],$$

$$\psi_{\varepsilon_1 \varepsilon_2} = (1 - \xi^2 - \eta^2)t + \lambda^{-1/2}(x - y)\xi - \varepsilon_1 \eta^2 - \varepsilon_2 \eta^2, \quad \psi = \psi_{0,0}$$

and $g(\xi, \eta)$ is a cutoff function .

In the last integral we can integrate by parts. Namely, since $|\partial_t \psi_{\varepsilon_1 \varepsilon_2}| > C(\xi^2 + \eta^2)$ for $\xi^2 + \eta^2 > C_1$, then

$$|I_{\varepsilon_1 \varepsilon_2}(\lambda, x, y)| \leq \lambda^{-N+\alpha+3/2} \int |\partial_t^N g_1(t, \eta, x, y)| (\xi^2 + \eta^2)^{-N} (1 - g(\xi, \eta)) dt d\xi d\eta.$$

Using (6.7) we get $|\partial_t^N g_1(t, \eta, x, y)| \leq C_N$. Hence,

$$|I_{\varepsilon_1 \varepsilon_2}(\lambda, x, y)| \leq C_N \lambda^{-N+\alpha+3/2} \quad \text{if } a \leq x \leq b, \quad 0 < c \leq y \leq d.$$

Finally we see that

$$(6.10) \quad J(\lambda, y) = \lambda^{\alpha+3/2} J_1(\lambda, y) + O(\lambda^{-\infty}),$$

where

$$(6.11) \quad J_1(\lambda, y) = \int e^{i\lambda\psi_1} g(\xi, \eta) g_1(t, \eta, x, y) dt d\xi d\eta$$

and

$$(6.12) \quad \psi_1(t, \xi, \eta, x, y) = (1 - \xi^2 - \eta^2)t + \lambda^{-1/2}(x - y)\xi.$$

The second term in (6.5), $B_\lambda(x, y)$, can be treated analogously. Thus (6.5), (6.10) give the basic formula

$$(6.13) \quad e_\alpha(\lambda, x, y) = F_\alpha(\lambda, x, y) + C_\delta F_\alpha(\lambda, x, -y) + O(\lambda^{-\infty}), \quad \delta \geq -\frac{1}{2},$$

$$0 < a \leq x \leq b, \quad 0 < c \leq y \leq d$$

where

$$(6.14) \quad F_\alpha(\lambda, x, y) = \lambda^{\alpha+3/2} J_1(\lambda, y) + \lambda^{1/2} J_2(\lambda, y),$$

$J_1(\lambda, y)$ is given by (6.11), while $J_2(\lambda, y) = \int e^{i\lambda\psi_2} g(\xi) g_2(t, x, y) dt d\xi$ and ψ_1 is the function (6.12), while $\psi_2(t, \xi, x, y) = \psi_1(t, \xi, 0, x, y)$.

Asymptotics of J_1, J_2 . Since in polar coordinates $(\xi, \eta) = \sigma(w, \theta)$, $w \in R^1$, $\sigma > 0$, $w^2 + \theta^2 = 1$, ($w = \cos \varphi$, $\theta_1 = \sin \varphi \cos \varphi_1$, $\theta_2 = \sin \varphi \sin \varphi_1$, $0 < \varphi < \pi$, $0 < \varphi_1 < 2\pi$),

$$I = \int_{w^2+\theta^2=1} e^{i\sqrt{\lambda}(x-y)w\sigma} \theta^{2\alpha} d(w, \theta) = 2\pi \int_{w^2 < 1} e^{i\lambda(x-y)w\sigma} (1-w^2)^\alpha dw,$$

then

$$(6.15) \quad I = C_\alpha (\sqrt{\lambda}|x-y|\sigma)^{-1/2-\alpha} J_{1/2+\alpha}(\sqrt{\lambda}|x-y|\sigma),$$

$$C_\alpha = (2\pi)^{3/2} 2^\alpha \Gamma(\alpha+1).$$

Therefore

$$J_1(\sqrt{\lambda}|x-y|)^{-1/2-\alpha} \int_0^\infty \int e^{i\lambda(1-\sigma^2)t} \sigma^{\alpha+3/2} J_{\alpha+1/2}(\sqrt{\lambda}|x-y|\sigma) q(t, \sigma) dt d\sigma,$$

$$(6.16) \quad q(0, 1) = (2\pi)^{-3/2} 2^{\alpha+1} \Gamma(\alpha+1).$$

For $\sigma \approx 0$ we can integrate by parts with respect to t , and hence we can suppose $q(t, \sigma) \in C_0^\infty(R \times (0, \infty))$. If $\sqrt{\lambda}|x-y| > 1$ we use the formula (6.6). Thus we get

$$(6.17) \quad \begin{aligned} J_{\alpha+1/2}(\sqrt{\lambda}|x-y|\sigma) &= (\sqrt{\lambda}|x-y|\sigma)^{-1/2} \times \\ &\times \left[e^{-i\sqrt{\lambda}(x-y)\sigma} g(\sqrt{\lambda}|x-y|\sigma) + e^{i\sqrt{\lambda}|x-y|\sigma} g(-\sqrt{\lambda}|x-y|\sigma) \right], \end{aligned}$$

where

$$|\partial_\sigma^k g(\sqrt{\lambda}|x-y|\sigma)| \leq C_k \quad \text{if} \quad \sqrt{\lambda}|x-y| > 1, \quad 0 < C_1 \leq \sigma \leq C_2.$$

Hence

$$J_1 = (\sqrt{\lambda}|x-y|)^{-\alpha-1/2} [K_1 + K_2 + O(\lambda^{-\infty})],$$

$$K_1(\sqrt{\lambda}|x-y|)^{-1/2} \int e^{i\lambda\psi_i} q_i(t, \sigma, x, y) dt d\sigma,$$

$$\psi_1 = (1-\sigma^2)t - |x-y|\lambda^{-1/2}\sigma, \quad \psi_2 = (1-\sigma^2)t + |x-y|\lambda^{-1/2}\sigma,$$

$$|\partial^k q_i| \leq C_k \quad \text{if} \quad \sqrt{\lambda}|x-y| > 1, \quad (t, \sigma) \rightarrow q_i \in C_0^\infty(R \times (0, \infty)).$$

To find the asymptotics of the integrals K_i we apply the stationary phase method [1].

The critical points $\sigma = 1$, $t_\pm = \pm \frac{1}{2}|x-y|\lambda^{-1/2}$ are nondegenerate and the Taylor formula gives.

$$(6.18) \quad q(t_\pm, 1) = q(0, 1) + |x-y|O(\lambda^{-1/2})$$

Therefore

$$K_1 = (\sqrt{\lambda}|x-y|)^{-1/2} \left[\frac{2\pi}{\lambda} \cdot \frac{1}{2} e^{-i\sqrt{\lambda}|x-y|} g(\sqrt{\lambda}|x-y|) q(t_+, 1) + O(\lambda^{-2}) \right]$$

$$K_2 = (\sqrt{\lambda}|x-y|)^{-1/2} \left[\frac{2\pi}{\lambda} \cdot \frac{1}{2} e^{i\sqrt{\lambda}|x-y|} g(-\sqrt{\lambda}|x-y|) q(t_-, 1) + O(\lambda^{-2}) \right]$$

or according to (6.16), (6.18)

$$(6.19) \quad J_1 = (\sqrt{\lambda}|x-y|)^{-\frac{1}{2}-\alpha} \left[\frac{d_\alpha}{\lambda} J_{\alpha+\frac{1}{2}}(\sqrt{\lambda}|x-y| + O(\lambda^{-7/4})) \right] \quad \text{if} \quad \sqrt{\lambda}|x-y| > 1,$$

where d_α is given by (1.6).

Consider now the case $\sqrt{\lambda}|x-y| < 1$. Then analogously to (6.15) we get

$$J_1 = \int_0^\infty \int e^{i\lambda(1-\sigma^2)t} g(t, \sigma, \lambda) dt d\sigma + O(\lambda^{-\infty}),$$

where

$$g(t, \sigma, \lambda) = \int_{w^2 < 1} (1-w^2)^\alpha e^{i\sqrt{\lambda}(x-y)w\sigma} dw g_1(t, \sigma),$$

$$g_1(t, \sigma) \in C_0^\infty(R \times (0, \infty)), \quad g_1(0, 1) = \frac{1}{2\pi^2}$$

and the method of stationary phase gives

$$(6.20) \quad J_1 = \frac{1}{2\lambda\pi} \int_{w^2 < 1} (1-w^2)^\alpha e^{i\lambda(x-y)w\sigma} dw + O(\lambda^{-2}),$$

$$(6.21) \quad J_1(\sqrt{\lambda}|x-y|)^{-\alpha-1/2} d_\alpha \lambda^{-1} J_{\alpha+1/2}(\sqrt{\lambda}|x-y|) + O(\lambda^{-2}) \quad \text{if} \quad \sqrt{\lambda}|x-y| < 1.$$

By the same method of stationary phase we have

$$(6.22) \quad J_2(\lambda, x, y) = O(\lambda^{-1}).$$

Thus (6.14), (6.19), (6.21) and (6.22) imply

$$F_\alpha(\lambda, x, y) = d_\alpha \lambda^{1/2+\alpha} (\sqrt{\lambda}|x-y|)^{-1/2-\alpha} J_{\alpha+1/2}(\sqrt{\lambda}|x-y|) + R,$$

where

$$|R| \leq \begin{cases} C(\sqrt{\lambda}|x-y|)^{-1/2-\alpha} \lambda^{\alpha-1/4} & \text{if } \sqrt{\lambda}|x-y| > 1 \\ C\lambda^{\alpha-1/2} & \text{if } \sqrt{\lambda}|x-y| < 1. \end{cases}$$

This and (6.13), (2.1) give (2.2). Theorem (1) is completely proved.

7. Proof of Theorem 4. Starting with (6.1) and using (6.2) we can write

$$E_\alpha(\lambda, x, y) = E_1(\lambda, x, y) + E_2(\lambda, x, y),$$

$$(7.1) \quad E_i(\lambda, x, y) = b \int e^{\lambda p} V(p, \sqrt{\lambda}x, \sqrt{\lambda}y) H_\alpha(\lambda, p) K_i(p) dp, \quad i = 1, 2,$$

where for some $\gamma_1 > 0$,

$$K_i \in C_0^\infty(S), \quad \text{supp } K_1(p) \subset \{|\text{Im } p| < \gamma_1\}, \quad K_1(p) = 1 \quad \text{for } |\text{Im } p| < \gamma < \gamma_1.$$

Further, analogously to the proof of Theorem 3,

$$E_1(\lambda, x, y) = A_\lambda(x, y) + B_\lambda(x, y),$$

where

$$A_\lambda(x, y) = b \int e^{\lambda p} V(p, \sqrt{\lambda}x, \sqrt{\lambda}y) p^{-\alpha-1} K_1(p) dp$$

and

$$B_\lambda(x, y) = b \int e^{\lambda p} V(p, \sqrt{\lambda}x, \sqrt{\lambda}y) h_\alpha(\lambda, p) K_1(p) dp.$$

Now instead of (6.8) we shall use

$$V(it, \sqrt{\lambda}x, \sqrt{\lambda}y) = \lim_{\varepsilon_2 \rightarrow 0} \left[G(t, \sqrt{\lambda}x, \sqrt{\lambda}y, \varepsilon_2) + C_\delta G(t, \sqrt{\lambda}x, -\sqrt{\lambda}y, \varepsilon_2) \right]$$

where

$$G(t, x, y, \varepsilon_2) = (4\pi)^{-1/2} e^{-\frac{i(x^2+y^2)}{2} \sin t} \int e^{-i\xi^2 \frac{\sin 2t}{2} + i(x-y)\xi - \varepsilon_2 \xi^2} a(it, x, y) d\xi.$$

Since in the sense of distributions

$$(i \sin 2t + o)^{-\alpha-1} = \lim_{\varepsilon_1 \rightarrow 0} C_\alpha \int e^{-\varepsilon_1 \eta^2 - i\eta^2 \frac{\sin 2t}{2}} \eta^{2\alpha} d\eta, \quad \eta \in \mathbb{R}^2,$$

we obtain, analogously to (6.13), (6.14),

$$E_1(\lambda, x, y) = F_{1\alpha}(\lambda, x, y) + C_\delta F_{1\alpha}(\lambda, x, -y) + O(\lambda^{-\infty}),$$

where

$$(7.2) \quad \begin{aligned} F_{1\alpha}(\lambda, x, y) &= \lambda^{\alpha+3/2} I_1(\lambda, x, y) + \lambda^{1/2} I_2(\lambda, x, y), \\ I_1(\lambda, x, y) &= \int e^{i\lambda\psi(t,x,y,\eta,\xi)} q_1(t, \lambda, x, y) \eta^{2\alpha} g_1(\xi, \eta) dt d\xi d\eta, \\ I_2(\lambda, x, y) &= \int e^{i\lambda\psi(t,x,y,0,\xi)} q_2(t, \lambda, x, y) g_1(\xi) dt d\xi \end{aligned}$$

and

$$q_1(t, \lambda, x, y) = \left(\frac{\sin 2t}{2t} \right)^{-\alpha-1} a(it, \lambda, x, y) (2\pi)^{\alpha+1} \sqrt{\pi} K_1(it),$$

$$q_2(t, \lambda, x, y) = h_\alpha(\lambda, t) a(it, \lambda, x, y) (2\pi)^{-3/2} 2^{-1/2} K_1(it),$$

$$\psi(t, x, y, \eta, \xi) = t - \frac{(\eta^2 + \xi^2)}{2} \sin 2t + (x - y)\xi - \frac{(x^2 + y^2)}{2} \sin t,$$

$g_1(\xi, \eta)$ and $g_1(\xi)$ are cutoff functions.

We can represent E_2 as follows:

$$E_2(\lambda, x, y) = F_{2\alpha}(\lambda, x, y) + C_\delta F_{2\alpha}(\lambda, x, -y),$$

where

$$F_{2\alpha}(\lambda, x, y) = b \int e^{i\lambda\varphi} q(t, \lambda, x, y) dt,$$

$$(7.3) \quad \varphi(t, y, x) = t + \frac{(x^2 + y^2)}{2} \cot 2t - \frac{xy}{\sin 2t}$$

and

$$q(t, \lambda, x, y) = (i \sin 2t)^{-1/2} H_\alpha(\lambda, it) i b a(it, \lambda xy) K_2(it).$$

Note that $t \rightarrow q \in C_0^\infty \left(0 < |t| < \frac{\pi}{2} \right)$.

To find the uniform asymptotics of the integrals I_i ($i = 1, 2$) in the domain $\left\{ (x, y) \in R^2, \frac{A}{\sqrt{\lambda}} < x < 1 - \varepsilon, 0 < \frac{c}{\sqrt{\lambda}} \leq y < \frac{d}{\sqrt{\lambda}} \right\}$ we shall apply the method of the stationary phase.

Asymptotics of I_1 . Analogously to (6.20) we have

$$I_1 = (\lambda|x - y|)^{-1/2-\alpha} \int_0^\infty \int e^{i\lambda\psi_0} \sigma^{3/2+\alpha} J_{1/2+\alpha}(\lambda|x - y|\sigma) q(t, \sigma) dt d\sigma + O(\lambda^{-\infty}),$$

where

$$\psi_0 = t - \frac{\sigma^2}{2} \sin 2t - \frac{(x^2 + y^2)}{2} \tan t, \quad q \in C_0^\infty(R \times (0, \infty)).$$

Let $s = \lambda|x - y|\sigma$. The asymptotics of $J_{1/2+\alpha}(s)$ gives

$$(7.4) \quad J_{1/2+\alpha}(s) = \sum_{k=0}^2 s^{-1/2-k} (C_k e^{is} = \bar{C}_k e^{-is}) + O(s^{-7/2}).$$

Since $|x - y| \geq c|x| > c\lambda^{-1/2}$, then (7.2) and (7.4) imply

$$(7.5) \quad I_1(\lambda|x - y|)^{-1-\alpha} \sum_{k=0}^2 (\lambda|x - y|)^{-k} M_k + x^{-1-\alpha} O(\lambda^{-5/2-\alpha}),$$

where

$$(7.6) \quad M_k = \int_0^\infty \int e^{i\lambda\psi} \sigma^{1+\alpha-k} q_k(t, \sigma) dt d\sigma, \quad q_k \in C_0^\infty(R \times (0, \infty))$$

and

$$(7.7) \quad \psi = t - \frac{\sigma^2}{2} \sin 2t - \frac{(x^2 + y^2)}{2} \tan t \pm |x - y|\sigma.$$

The critical points (t_i, σ_i) of ψ satisfy

$$(7.8) \quad \det \psi'' = \pm 4d, \quad d = \sqrt{(1 - x^2)(1 - y^2)}$$

$$(7.9) \quad \cos 2t_i = xy + (-1)^{i+1} d \quad (i = 1, 2), \quad t_3 = -t_1, \quad t_4 = -t_2, \quad \sigma_i \sin 2t_i = \pm |x - y|$$

and these critical points are nondegenerate for $x < 1 - \varepsilon$, $y < 1 - \varepsilon$.

Thus the method of the stationary phase gives

$$(7.10) \quad M_0 = \lambda^{-1} \sum_{i=1}^4 e^{\lambda\psi_i} C_i(\lambda, x, y) + O(\lambda^{-2}), \quad |C_i(\lambda, x, y)| \leq C, \\ M_k = O(\lambda^{-1}), \quad k = 1, 2,$$

$$(7.11) \quad \psi_i(x, y) = \psi(t_i, \sigma_i) = \varphi(t_i, x, y).$$

Then (7.5) and (7.10) imply

$$(7.12) \quad I_1 = \lambda^{-2-\alpha} \sum_{i=1}^4 e^{i\lambda\psi_i} \tilde{b}_{1i}(\lambda, x, y) + x^{-1-\alpha} O(\lambda^{-5/2-\alpha}),$$

$$(7.13) \quad \tilde{b}_{1i}(\lambda, x, y) = |x - y|^{-1-\alpha} C_i(\lambda, x, y).$$

Asymptotics of I_2 . The critical points of the phase function $\psi(t, x, y, 0, \xi)$ satisfy (7.9), where $\xi_i \sin 2t_i = x - y$. Therefore the stationary phase method gives

$$(7.14) \quad I_2 = \lambda^{-1} \sum_{i=1}^4 e^{i\lambda\psi_i} \tilde{b}_{2i}(\lambda, x, y) + O(\lambda^{-2}),$$

where

$$(7.15) \quad \psi_i(x, y) = \psi(t_i, x, y, 0, \xi_i)$$

and

$$(7.16) \quad |\tilde{b}_{2i}| \leq C, \quad |\partial_x \tilde{b}_{2i}| \leq C.$$

Now (7.2), (7.12) and (7.14) show that

$$(7.17) \quad F_{1\alpha}(\lambda, x, y) = \lambda^{-1/2} \sum_{i=1}^4 e^{i\lambda\psi} b_{1i}(\lambda, x, y) + x^{-1-\alpha} O(\lambda^{-1}),$$

where ψ_i satisfy (7.11), (7.15) and b_{1i} according to (7.13), (7.16) satisfy (2.5). To find the asymptotics of $F_{2\alpha}$, we first notice that the critical points t_i of the phase function $\varphi(t, x, y)$ given by (7.3) satisfy (7.9) and $\varphi''(t, x, y) = (-1)^{i+1} 4d(\sin 2t_i)^{-1}$, $1 \leq i \leq 4$. Thus the critical points are nondegenerate and

$$(7.18) \quad F_{2\alpha}(\lambda, x, y) = \lambda^{-1/2} \sum_{i=1}^4 e^{i\lambda\psi} b_{2i}(\lambda, x, y) + O(\lambda^{-3/2}),$$

$$|b_{2i}| + |\partial_x b_{2i}| \leq C, \quad \psi_i(x, y) = \varphi(t_i, x, y).$$

Evidently (7.17) and (7.18) give (2.4).

8. Proof of Theorem 5. Starting with (6.4) and (2.1) we get formula (2.3), where

$$F_\alpha(\lambda, x, y) = \int_{S_1} e^{\lambda\varphi} q(p, \lambda) dp,$$

$$\varphi(p) = p - 2^{-1}(x^2 + y^2) \coth p + \frac{xy}{\sinh 2p},$$

$$q(p, \lambda) = b(\sinh 2p)^{-1/2} H_\alpha(\lambda, p) e^{2p(\delta-1)} f\left(\frac{i\lambda xy}{\sinh 2p}\right), \quad S_1 = \left(\varepsilon_1 - i\frac{\pi}{4}, \varepsilon_1 + i\frac{\pi}{4}\right), \quad \varepsilon_1 > 0.$$

Now we can apply the same method as in [3], which gives the asymptotics (2.7).

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