Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **22** (1996), 471-496

Serdica
Mathematical Journal

Institute of Mathematics
and Informatics, Bulgarian
Academy of Sciences

SUMS OF A RANDOM NUMBER OF RANDOM VARIABLES AND THEIR APPROXIMATIONS WITH ν-ACCOMPANYING INFINITELY DIVISIBLE LAWS*

Lev B. Klebanov, Svetlozar T. Rachev

Communicated by J.-P. Dion

Abstract. In this paper a general theory of a random number of random variables is constructed. A description of all random variables ν admitting an analog of the Gaussian distribution under ν-summation, that is, the summation of a random number ν of random terms, is given. The ν-infinitely divisible distributions are described for these ν-summations and finite estimates of the approximation of ν-sum distributions with the help of ν-accompanying infinitely divisible distributions are given. The results include, in particular, the description of geometrically infinitely divisible and geometrically stable distributions as well as their domains of attraction.

1. Introduction. The study of sums of a random number of variables was launched in Robbins's pioneering paper [27]. The next essential steps in the development of the theory of limiting behavior of sums of a random number of random terms were Dobrushin's paper [3] and the series of papers by Gnedenko and his students [6], [8], [7]. A partial summary of this direction is given in [19].

Recall that in the classical scheme infinitely divisible distributions can be defined in two ways. In the *first definition* a random variable (r.v.) Y is called infinitely

1991 *Mathematics Subject Classification:* 60F05, 60E07, 60E10

Key words: infinitely divisible laws, geometric sums, rate of convergence, probability metrics

* Research supported by NATO GRANT CRG 900 798 and by Humboldt Award for U.S. Scientists

divisible if for any integer $n \geq 2$ there exist independent identically distributed (iid) r.v.'s $X_1^{(n)}, \dots, X_n^{(n)}$ for which $Y \stackrel{d}{=} X_1^{(n)} + \cdots + X_n^{(n)}$ (here $\stackrel{d}{=}$ denotes equality of distributions). The *second definition* consists in the fact that only infinitely divisible distributions are limits of increasing sums of independent r.v.'s in triangular arrays provided that the terms are infinitely small.

In the classical scheme both definitions are equivalent. However, for sums of a random number of r.v.'s this is no longer so. The results of Robbins, Dobrushin, Gnedenko, and many others are generalizations of the second definition, see the review in [23], [24].

Klebanov et al. [14] generalized the first definition to the case where the number of terms has geometric infinitely divisible and gave the definition of geometrically stable distributions. These concepts proved to be sufficiently productive, see [17], [21], [22], [18] for more than 50 references on applications of random summation schemes in queueing theory, reliability, branching processes, mathematical finance, environmental processes and others.

An attempt to investigate more general summations than the geometric ones was made in [15], [16]. It was based on some limit theorems for a random number of terms.

In this paper we construct *a general theory of summation of a random number of random variables generalizing the first definition of infinite divisibility.* We describe all random variables ν admitting an analog of the Gaussian distribution under the summation of ν random terms. For these summations we describe all the ν-infinitely divisible distributions (i.e., infinitely divisible in the sense of an analog of the first definition). This allows us to introduce the concept of ν-accompanying infinitely divisible distributions as well as obtain finite estimates of the rate of approximation of the distributions of ν-sums using the ν-accompanying infinitely divisible distributions. Furthermore, a description of geometrically infinitely divisible and geometrically stable distributions, sharp estimates of their approximation, and their domains of attraction are obtained.

2. ν-Gaussian random variables. Let $X_1, X_2, \dots$ be a sequence of iid r.v.'s. Assume that $\{\nu_p,\ p \in \Delta\}$, $\Delta \subset (0,1)$ is a family of nonnegative integer-valued r.v.'s independent of $\{X_j,\ j \geq 1\}$. It is then assumed that there exists $E\nu_p$ and that $E\nu_p = 1/p$ for all $p \in \Delta$. We study the distributions of sums $S_p = \sum_{j=1}^{\nu_p} X_j$, $j \in \Delta$.

Definition 1. *A r.v. Y is called* ν-infinitely divisible *if for any $p \in \Delta$ there exists a sequence of iid r.v.'s $\{X_j^{(p)}, j \geq 1\}$ independent of ν_p such that*

$$Y \stackrel{d}{=} \sum_{j=1}^{\nu_p} X_j^{(p)}. \tag{2.1}$$

Definition 2. *A r.v. X is called ν*-strictly Gaussian *if $EX = 0$, $EX^2 < \infty$ and for all $p \in \Delta$*

$$X \overset{d}{=} p^{1/2} \sum_{j=1}^{\nu_p} X_j, \tag{2.2}$$

where $\{X_j,\ j \geq 1\}$ is a sequence of iid r.v.'s independent of $\{\nu_p,\ p \in \Delta\}$, and $X \overset{d}{=} X_1$.

Distributions of ν-infinitely divisible (ν-strictly stable, ν-strictly Gaussian) r.v.'s are called ν-infinitely divisible (correspondingly, ν-strictly stable and ν-strictly Gaussian) distributions.

The first question of interest to us is how to describe families $\{\nu_p,\ p \in \Delta\}$ for which ν-strictly Gaussian r.v.'s exist.

Recall first that if $P^{(1)}$ and $P^{(2)}$ are the generating functions (gf) of two r.v.'s taking natural values, then their superposition $P^{(1)} \circ P^{(2)}(z) := P^{(1)}(P^{(2)}(z))$ is also a gf of some r.v. that takes natural values.

Denote by P_p the gf of r.v. ν_p and by $\mathbb{P}$ a semigroup with operation of superposition $\circ$ generated by the family $\{P_p, p \in \Delta\}$.

Theorem 1. *For a ν-strictly Gaussian random variable X to exist it is necessary and sufficient that semigroup $\mathbb{P}$ be commutative.*

Proof. Let $f(t)$ be the characteristic function (ch.f.) of X. Then (2.2) is equivalent to the system of equalities

$$f(t) = P_p(f(p^{1/2}t)),\ p \in \Delta \tag{2.3}$$

fulfilled for all real t.

Consider (2.3) only for $t \geq 0$. Let $\varphi(t) = f(\sqrt{t})$. It is easy to see that if $f(t)$ satisfies (2.3), then

$$\varphi(t) = P_p(\varphi(pt)), \quad p \in \Delta \tag{2.4}$$

for $t \geq 0$ and, conversely, if $\varphi(t)$ satisfies (2.4), then $f(t) = \varphi(t^2)$ satisfies (2.3). This implies that if $f(t)$ does exist then it is symmetric.

Let $p_0 \in \Delta$. Denote the gf of ν_{p_0} by $P(z)$. From (2.4)

$$\varphi(t) = P(\varphi(p_0 t)). \tag{2.5}$$

Equation (2.5) is Poincaré equation (see [25]). Poincaré was interested in the existence and uniqueness of the analytic solutions of (2.5). It also occurs in the theory of branching processes (see, for example, [10]). It is well-known that (2.5) has a unique and differentiable solution with initial values $\varphi(0) = 1$, $\varphi'(0) = -a$, where $a \geq 0$ is an arbitrary constant. This solution is the Laplace transform of a distribution $A(x)$ concentrated on R_+. Thus

$$\varphi(t) = \int_0^\infty e^{-tx} dA(x) \tag{2.6}$$

and $\varphi(t)$ is determined to within a scale parameter. Clearly, if $\varphi'(0) = -a \neq 0$, then A is not degenerate at zero.

It is clear that the solution of *overdetermined* system (2.4) (if it exists) must satisfy (2.5), that is, it must coincide with (2.6). Of course, for (2.4) to have a solution, it is necessary and sufficient that the solution of (2.5) be independent of the choice $p_0 \in \Delta$; i.e. for every fixed $p \in \Delta$, $p \neq p_0$ equations

$$\varphi_p(t) = P_p(\varphi_p(pt)) \tag{2.7}$$

and (2.5) must have the same solutions with initial values $\varphi(0) = \varphi_p(0) = 1$, $\varphi'(0) = \varphi_p'(0) = -a$, $a > 0$.

Let us show that (2.5) and (2.7) have the same solution if and only if

$$P_p \circ P = P \circ P_p. \tag{2.8}$$

Suppose first that (2.8) holds. Let $\varphi(t)$ be a solution of (2.5) with the desired initial values. Then $P_p(\varphi(tp)) = P_p(P(\varphi(tp_0p))) = P(P_p(\varphi(tp_0p)))$ satisfies (2.5). In addition, $P_p(\varphi(tp))_{|t=0} = 1$ and $\frac{d}{dt}P_p(\varphi(tp))_{|t=0} = -a$.

Consequently, under (2.8) equations (2.5) and (2.7) have the same solution.

Assume now that (2.5) and (2.7) have the same solution. Then $P_p(P(\varphi(pp_0t))) = P_p(\varphi(pt)) = \varphi(t)$ is the Laplace transform of distribution function $A(x)$ which is not degenerate at zero, that is, the values of $\varphi(t)$ for $t > 0$ fill interval $(0, 1]$. Consequently, $P(P_p(z)) = P_p(P(z))$ for $z \in (0, 1]$, which implies (2.8).

Let us return to (2.3). It follows from (2.6) that $f(t)$ must have the form $f(t) = \int\limits_0^\infty e^{-t^2x} dA(x)$. In addition, (2.3) is consistent if and only if (2.4) is consistent, that is, if and only if (2.8) is fulfilled for any $p, p_0 \in \Delta$. The latter is clearly equivalent to the commutativity of $\mathbb{P}$.

Remark 1. If $\mathbb{P}$ is commutative, then the ch.f. of a ν-strictly Gaussian distribution has form $f(t) = \varphi(at^2)$, where $a > 0$ is a parameter and $\varphi(t)$ is a solution of (2.5) with $\varphi(0) = -\varphi'(0) = 1$.

Corollary 1. *$\mathbb{P}$ is commutative if and only if for $z > 0$ the representation $P_p(z) = \varphi\left(\frac{1}{p}\varphi^{-1}(z)\right)$, $p \in \Delta$, holds, where $\varphi(t)$ is a differentiable solution of* (2.5) *provided that $\varphi(0) = -\varphi'(0) = 1$.*

Let us turn to examples of families r.v.'s $\{\nu_p,\ p \in \Delta\}$ admitting ν-strictly Gaussian laws.

Example 1 (The classical scheme of summation)**.** Let $\nu_p = \dfrac{1}{p}$ with probability 1 and $p \in \Delta = \{1/n, n \in \mathbb{N}\}$. Clearly, $P_p(z) = z^{1/p}$. It is also clear that $P_{p_1} \circ P_{p_2}(z) = z^{1/(p_1 p_2)} = P_{p_2} \circ P_{p_1}(z)$. By virtue of Theorem 1, there exists a ν-strictly Gaussian distribution. Of course we are dealing with the classical scheme and ν-strictly Gaussian distributions coincide with ordinary Gaussian ones.

Example 2. Let ν_p be a geometric r.v. with parameter $p : P\{\nu_p = \kappa\} = p(1-p)^{\kappa-1}$, $\kappa \in \mathbb{N}$. We have $P_p(z) = pz/(1-(1-p)z)$. It is easy to see that $P_{p_1} \circ P_{p_2}(z) = \dfrac{p_1 p_2 z}{1-(1-p_1 p_2)z} = P_{p_2} \circ P_{p_1}(z)$. Therefore, there exists a ν-strictly Gaussian distribution. Equation (2.5) has the form (for $p_0 \in (0,1)$)

$$\varphi(t) = \frac{p_0 \varphi(p_0 t)}{1-(1-p_0)\varphi(p_0 t)}. \tag{2.9}$$

As is known (see [2] and [12]) the Laplace transforms of the exponential distribution are solutions of form (2.6) of this equation, that is, $\varphi_a(t) = 1/(1+at)$. In particular, $\varphi(t) = 1/(1+t)$ is the unique solution of (2.9) provided that $\varphi(0) = -\varphi'(0) = 1$. Therefore ν-strictly Gaussian distribution are Laplace distributions with ch.f.

$$f(t) = 1/(1+at^2),\ a > 0. \tag{2.10}$$

Example 3. Let ν be a r.v. taking natural values and having $E\nu > 1$. Denote $p_0 = 1/E\nu$ and let $P(z)$ be the gf ν. Let

$$P_{p_0}(z) = P(z),\ P_{p_0^2}(z) = P(P(z)) = P^{\circ 2}(z), \ldots, P_{p_0^n}(z) = P^{\circ n}(z), \tag{2.11}$$

and suppose that $\Delta = \{p_0^n,\ n \in \mathbb{N}\}$. Assume that $\{\nu_p,\ p \in \Delta\}$ is a family of r.v.'s with gf's (2.11). Clearly $\mathbb{P}$ is commutative since it is a semigroup of degrees in the sense of superposition of function $P(z)$. The system (2.4) has form

$$\varphi(t) = P(\varphi(p_0 t)) = P^{\circ 2}(\varphi(p_0^2 t)) = \cdots = P^{\circ n}(\varphi(p_0^n t)) = \cdots$$

It coincides with (2.5): $\varphi(t) = P(\varphi(p_0 t))$.

Example 3 shows that a summation generated by *only* one r.v. ν is equivalent to a summation related to the family $\{\nu_p,\ p \in \Delta\}$, $\Delta = \{p_0^n,\ n \in \mathbb{N}\}$. Note that this circumstance has a more general character: instead of $\{\nu_p,\ p \in \Delta\}$ (for generic Δ) we can consider a family of all r.v.'s ν whose gf's belong to $\mathbb{P}$. Everywhere below (unless otherwise stipulated) we assume $\{\nu_p,\ p \in \Delta\}$ to be such that $\{P_p,\ p \in \Delta\} = \mathbb{P}$, with $\mathbb{P}$ being commutative semigroup.

A solution of (2.5) given by (2.6), differentiable on $[0,\infty)$, and satisfying the conditions $\varphi(0) = -\varphi'(0) = 1$ will be called *standard*.

3. ν-infinitely divisible random variables. We assume further that 0 is a limit point of $\Delta \subset (0,1)$.

We start with an analog of de Finetti's theorem, (see, for example, [20]).

Theorem 2. *Let φ be a standard solution of* (2.5). *A r.v. Y with ch.f. $g(t)$ is ν-infinitely divisible if and only if*

$$g(t) = \lim_{m\to\infty} \varphi(\alpha_m[1 - g_m(t)]), \tag{3.1}$$

where α_m is a positive constant and $g_m(t)$ is a ch.f.

Proof. (i) We first show that if $h(t)$ is a ch.f. and α a positive constant, then

$$\Psi_\alpha(t) = \varphi(\alpha[1 - h(t)]) \tag{3.2}$$

is a ch.f. Since 0 is a limit point of Δ, then for $\alpha > 0$ and sufficiently small $p \in \Delta$, $h_\alpha(t) = (1-\alpha p) + \alpha p h(t)$ is a ch.f. Consequently, $\varphi\left(\frac{1}{p}\varphi^{-1}\left(h_\alpha(t)\right)\right)$ is a ch.f. (this is the ch.f. of $\sum_{j=1}^{\nu_p} X_j$, where X_j are iid with ch.f. $h_\alpha(t)$). However, $\varphi(\alpha[1-h(t)]) = \lim_{p\to 0}\varphi\left(\frac{1}{p}\varphi^{-1}\left(h_\alpha(t)\right)\right)$, which proves the assertion of part (i).

(ii) For any $\alpha > 0$ and ch.f. $h(t)$, function (3.2) is ν-infinitely divisible. Indeed, for all $p \in \Delta$ function $\Psi_\alpha(t)$ is a ch.f. of $\sum_{j=1}^{\nu_p} X_j$, if $\{X_j,\ j \geq 1\}$ is a sequence of iid variables with ch.f. $\Psi_{\alpha p}(t)$.

(iii) The function (3.1) is ν-infinitely divisible. This follows from (ii) and the fact that the limit of ν-infinitely divisible ch.f.'s is also a ν-infinitely divisible function.

(iv) Any ν-infinitely divisible ch.f. $g(t)$ can be written in the form of (3.1). Indeed, if $g(t)$ is a ν-infinitely divisible ch.f., then for any $p \in \Delta$ there exists a ch.f. $g_p(t)$ such that $g(t) = \varphi\left(\frac{1}{p}\varphi^{-1}g_p(t)\right)$. Hence, $g_p(t) = \varphi(p\varphi^{-1}g(t))$, is a ch.f. By (ii), $\varphi\left(\frac{1}{p}(1 - g_p(t))\right) = \varphi\left(\frac{1}{p}(1 - \varphi(p\varphi^{-1}g(t)))\right)$ is a ν-infinitely divisible ch.f. Finally, $g(t) = \lim_{p\to 0}\varphi\left(\frac{1}{p}[1 - \varphi(p\varphi^{-1}g(t))]\right)$, which proves (iv). $\square$

Theorem 3. *Let φ be a standard solution of* (2.5). *A ch.f. g is ν-infinitely divisible if and only if it is representable in the form of*

$$g(t) = \varphi(-\log f(t)), \tag{3.3}$$

where $f(t)$ is an infinitely divisible ch.f. (in the classical scheme).

The proof of this theorem follows directly from a comparison of Theorem 2 and the Finetti's theorem.

Theorem 3 leads to analogs of the Lévy and Lévy-Khintchine representations for ν-infinitely divisible ch.f.'s, and furthermore, it serves as a basis for the definition of such concepts as ν-strictly stable, ν-stable, and ν-semistable ch.f.'s (r.v.'s), see [20].

Corollary 2 (Analog of the canonical Lévy-Khintchine representation). *A function $g(t)$ is ν-infinitely divisible ch.f. if and only if*

$$g(t) = \varphi\left(ita - \int_{-\infty}^{\infty}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)\frac{1+x^2}{x^2}d\theta(x)\right),$$

where a is a real number, $\theta(x)$ is a nondecreasing bounded function, $\theta(-\infty) = 0$, and φ is the standard solution of (2.5)*. This representation is unique (of course for a fixed family $\{\nu_p,\ p \in \Delta\}$).*

Corollary 3 (Analog of the canonic Lévy representation). *A function $g(t)$ is a ν-infinitely divisible ch.f. if and only if*

$$\begin{aligned} g(t) \quad &= \quad \varphi\left(ita + \frac{\sigma^2 t^2}{2} - \int_{-\infty}^{-0}\left(e^{itx} - 1 - \frac{itx}{1+x^2}\right)dM(x)\right. \\ &\qquad \left. - \int_{+0}^{\infty}\left(e^{ixt} - 1 - \frac{itx}{1+x^2}\right)dN(x)\right), \end{aligned}$$

where φ is a standard solution of (2.5)*, a is a real number, σ is a non-negative number, and functions $M(x)$ and $N(x)$ satisfy the conditions:*

(i) $M(x)$ and $N(x)$ do not decrease on intervals $(-\infty, 0)$ and $(0, +\infty)$, respectively;

(ii) $M(-\infty) = N(0) = 0$;

(iii) integrals $\int\limits_{-\varepsilon}^{0} x^2 dM(x)$ and $\int\limits_{0}^{\varepsilon} x^2 dN(x)$ are finite for any $\varepsilon > 0$.

This representation is unique.

Remark 2. If a standard solution of (2.5) φ is the Laplace transform of an infinitely divisible distribution, then ν-infinitely divisible distributions are infinitely divisible (in the regular sense), see [4, Section XIII.7].

Definition 3. *A function $g(t)$ is called a ν-*stable *(correspondingly, ν-*strictly stable, ν-semistable*)* characteristic function with exponent α *if it admits representation* (3.3) *in which φ is a standard solution of* (2.5) *and $f(t)$ is the ch.f. of a stable (correspondingly, strictly stable, semistable) law with exponent α.*

However, care should be exercised when using Definition 3. We illustrate this with an example comparing ν-strictly stable and ν-semistable ch.f.'s.

Let $X_1, X_2, \dots$ be a sequence of iid symmetric r.v.'s. Assume that for some $\alpha \in (0,2)$ and for all $p \in \Delta$

$$X_1 \stackrel{d}{=} p^{1/\alpha} \sum_{j=1}^{\nu_p} X_j \tag{3.4}$$

is fulfilled. From (3.4) it apparently follows that X_1 is ν-infinitely divisible. Consequently, we have $g(t) = \varphi(-\log f(t))$, $P_p(z) = \varphi\left(\frac{1}{p}\varphi^{-1}(z)\right)$. Therefore (3.4) takes the form of $\varphi(-\log f(t)) = \varphi\left(\frac{1}{p}\varphi^{-1}[\varphi(-\log f(p^{1/\alpha}t))]\right)$, that is,

$$f(t) = f^{1/p}(p^{1/\alpha}t), \; p \in \Delta. \tag{3.5}$$

If $\Delta \supset \{1/n \;:\; n \in \mathbb{N}\}$, then it clearly follows form (3.5), that $f(t)$ is a strictly stable and, consequently, $g(t)$ is a ν-strictly stable ch.f. in the sense of Definition 3. However, if $\Delta = \{p_0^n \;:\; n \in \mathbb{N}\}$ for a specific $p_0 \in (0,1)$, then $f(t)$ is a semistable ch.f. and, hence $g(t)$ is a ν-semistable ch.f. in the sense of Definition 3.

4. Accompanying laws. Let us now introduce the concept of accompanying ν-infinitely divisible laws (ch.f.'s, r.v.'s). In the classical scheme (i.e., when $\nu_p = 1/p$ a.s., $p \in \{1/n \;:\; n \in \mathbb{N}\}$) the concept of an accompanying infinitely divisible distribution belongs to [5]. It proved to be rather useful when approximating distributions of sums of a large (but not random) number of random terms (see, for example, [1]).

Definition 4. *Let $X_1, \dots, X_n, \dots$ be a sequence of iid r.v.'s with ch.f. $f(t)$. Assume that $\varphi(t)$ is a standard solution of* (2.5). *Let $S_p = \sum_{j=1}^{\nu_p} X_j$. A r.v. Y_p with ch.f.*

$$\Psi_p(t; f) = \varphi\left(\frac{1}{p}(1 - f(t))\right) \tag{4.1}$$

is called an accompanying ν-infinitely divisible r.v. for sum S_p. *Its distribution function (d.f.) is said to be accompanying ν-infinitely divisible for the d.f. of S_p. For the sake of brevity, we speak of a ν-accompanying r.v. (ν-accompanying distribution)*[1]*.*

Let us now investigate whether distributions of sums S_p can be approximated with a ν-accompanying distribution.

---

[1]In the classical case our definition differs somewhat from the one given by Gnedenko in [5] who admits, in addition, the centering and normalization of f, that is, Gnedenko would have had $\Psi_p(t, f; a_p, b_p) = \varphi\left(\frac{1}{p}(1 - f(a_p t)e^{ib_p t})\right)$. Everywhere below we will use the definition corresponding to (4.1).

First let us check whether this is possible in metric χ_0 (for its properties see, [29], [13], [26]); if X, Y are r.v.'s with ch.f.'s $f_X(t)$, $f_Y(t)$, respectively, then

$$\chi_0(X,Y) = \sup_t |f_X(t) - f_Y(t)|. \tag{4.2}$$

Denote by $\mathcal{F}_+$ the class of all r.v.'s with non-negative ch.f.'s.

Theorem 4. *Let $X_1, X_2, \ldots$ be a sequence of iid r.v.'s, $X_1 \in \mathcal{F}_+$. Suppose that $S_p = \sum_{j=1}^{\nu_p} X_j$ and Y is a ν-accompanying r.v. for S_p. Then*

$$\sup_{X_1\in\mathcal{F}_+} \chi_0(S_p, X_p) \to 0 \quad \textit{as} \quad p \to 0,\ p \in \Delta. \tag{4.3}$$

P r o o f. We have

$$\sup_{X_1\in\mathcal{F}_+} \chi_0(S_p, Y_p)$$

$$= \max\left[\sup_{u\in[\sqrt{p},\infty)} \left|\varphi\left(\frac{u}{p}\right) - \varphi\left(\frac{1}{p}(1-\varphi(u))\right)\right|, \sup_{u\in[0,\sqrt{p}]} \left|\varphi\left(\frac{u}{p}\right) - \varphi\left(\frac{1}{p}(1-\varphi(u))\right)\right|\right].$$

For $u \geq \sqrt{p}$, it is clear that as $p \to 0$, $\varphi\left(\frac{u}{p}\right) \to 0$, $\varphi\left(\frac{1}{p}[1-\varphi(u)]\right) \to 0$. Therefore, as $p \to 0$, $\sup_{u\in[\sqrt{p},\infty)} \left|\varphi\left(\frac{u}{p}\right) - \varphi\left(\frac{1}{p}(1-\varphi(u))\right)\right| \to 0$. For $u \leq \sqrt{p}$, denoting $\nu = \frac{u}{p}$, we obtain

$$\sup_{u\in[0,\sqrt{p}]} \left|\varphi\left(\frac{u}{p}\right) - \varphi\left(\frac{1}{p}(1-\varphi(u))\right)\right|$$

$$= \sup_{v\in[0,1/\sqrt{p}]} \left|\varphi(v) - \varphi\left(-\frac{\varphi(pv)-\varphi(0)}{pv}v\right)\right| \to 0, \quad \text{as} \quad p \to 0;$$

in fact, $(\varphi(pv) - \varphi(0))/(pv) \to \varphi'(0) = -1$ as $p \to 0$ uniformly with respect to $v \in [0, 1/\sqrt{p}]$. $\square$

Note that, generally speaking, the constraint $X_1 \in \mathcal{F}_+$ cannot be waived. Namely, if we consider a sequence of iid r.v.'s taking values 1 or -1 with probability 1/2 each a.s. ($p \in \{1/n, n \in \mathbb{N}\}$), then it is easy to calculate that $\chi_0(S_p, Y_p)$ does not vanish as $p \to 0$.

The condition of non-negativity of the ch.f. can be replaced by the condition of non-negativity of r.v. X_1. However, the approximation of S_p by means of Y_p will be attained in terms of a different metric.

Let X, Y be non-negative random variables with Laplace transforms $l_X(u)$, $l_Y(u)$, respectively. Let

$$\chi_{0,l}(X,Y) = \sup_{u\geq 0} |l_X(u) - l_Y(u)|. \tag{4.4}$$

In contrast with the metric (4.2), convergence in metric $\chi_{0,l}$ *is equivalent* to the weak convergence of distributions of r.v.'s, in the class $\mathcal{X}_+$ of all non-negative random variables.

Note that if S_p is the sum of a random number ν_p of iid r.v.'s from $\mathcal{X}_+$ with Laplace transforms $l(u)$, then it is easy to see that the Laplace transform of a ν-accompanying r.v. Y_p is equal to $\Psi_p(u;l) = \varphi\left(\frac{1}{p}[1-l(u)]\right)$.

Theorem 5. *Let $X_1, X_2, \dots$ be a sequence of iid r.v.'s, $X_1 \in \mathcal{X}_+$. Suppose that $S_p = \sum_{j=1}^{\nu_p} X_j$ and Y_p is a ν-accompanying r.v. for S_p. Then*

$$\sup_{X_1\in\mathcal{X}_+} X_{0,l}(S_p, Y_p) \to 0 \quad as \quad p \to 0,\ p \in \Delta.$$

The proof is completely analogous to the proof of Theorem 4 and is, therefore, not provided.

Corollary 4. *Let $(X_{n,p})_{n\in\mathbb{N}}$ be a sequence of iid r.v.'s taking only two values: 1 with probability α_p or 0 with probability $1-\alpha_p$. Assume that α_p depends only on p so that there exist $\lim_{p\to 0}\frac{\alpha_p}{p} = \lambda \neq 0$. Then the distribution of sum $S_p = \sum_{j=1}^{\nu_p} X_{j,p}$ converges as $p \to 0$ to a distribution with Laplace transform $\varphi(\lambda(1-e^{-u}))$, where φ is a standard solution of* (2.5).

Note that under the condition $P\{p\nu_p < x\} \to A(x)$ as $p \to 0$, the assertion of Corollary 4 follows from Gnedenko's transfer theorem [7].

5. Approximation of random sums. The problems of approximating distributions of sums of a large nonrandom number of random terms with accompanying infinitely divisible distributions have been studied rather at length (see the bibliography in [1].

Below we consider in detail the problem of such an approximation for the case where the number of the terms in S_p has a geometric distribution:

$$P\{\nu_p = \kappa\} = p(1-p)^{\kappa-1}, \kappa \in \mathbb{N}. \tag{5.1}$$

We consider the following metrics in the set of all real-valued r.v.'s on $\mathcal{X}$:

$\chi_0(X,Y)$ — a metric defined on $\mathcal{X}$ by (4.2);

$\rho(X,Y)$ — uniform distance (or Kolmogorov distance) defined as $\rho(X,Y) = \sup_x |F_X(x) - F_Y(x)|$,

where F_X, F_Y are the df's of X, Y, respectively;

$\sigma(X,Y)$ — the total variation distance defined by

$\sigma(X,Y) = \sup_{A\in\mathcal{A}} |P\{X \in A\} - P\{Y \in A\}|$,
where $\mathcal{A}$ is the set of Borel subsets of R (for their properties, see [29], [26]).

These metrics are notable in that they are invariant under linear transformations of random variables. Topologically, metric χ_0 is strictly stronger than uniform distance ρ and strictly weaker than the total variation distance σ.

As was noted in Example 2, the function $\varphi(t) = \dfrac{1}{1+t}$, which is the standard solution of (2.5), corresponds to random variable (5.1). This and (4.1) imply that a ν-accompanying (we call it *geometrically accompanying*) r.v. Y_p has ch.f. $\Psi_p(t; f) = \dfrac{1}{1-(f(t)-1)/p}$. In general, in the case under consideration (i.e. for (5.1)), we speak of *geometrically infinitely divisible, geometrically stable*, and so on, variables (or distributions).

Theorem 6. *Let* $S_p = \sum_{j=1}^{\nu_p} X_j$, *where* $\{X_j,\ j \geq 1\}$, *be a sequence of iid r.v.'s, and suppose that* ν_p *has distribution* (5.1). *Assume that* Y_p *is a geometrically accompanying r.v. Then*

$$\sup_{X_1\in\mathcal{X}} \chi_0(S_p, Y_p) = \frac{p}{1-(p/2)^2}, \quad p \in (0,1).$$

Proof. The ch.f. $f_p(t)$ of S_p has the form $f_p(t) = \dfrac{pf(t)}{1-(1-p)f(t)}$, where $f(t)$ is the ch.f. of X_1. Therefore,

$$\chi_0(S_p, Y_p) \leq p \sup\left\{\left|\frac{(z-2)^2}{1-(1-p)z(1+p-z)}\right| \ :\ z \in \mathbb{C},\ |z| \leq 1\right\} \leq \frac{p}{1-(\frac{p}{2})^2},$$

recalling that the maximum of the modules of a function analytic in $|z| \leq 1$ is attained on $|z| = 1$. Thus,

$$\sup_{X_1\in\mathcal{X}} \chi_0(S_p, Y_p) \leq \frac{p}{1-(\frac{p}{2})^2}. \tag{5.2}$$

To obtain the lower estimate of $\sup_{X_1\in\mathcal{X}} \chi_0(S_p, Y_p)$ we consider the case of degenerate terms $X_j = 1$. Then $f(t) = e^{it}$, $f_p(t) = pe^{it}/(1-(1-p)e^{it})$ and the ν-accompanying ch.f. has form $\Psi_p(t; f) = \dfrac{1}{1-\frac{e^{it}-1}{p}}$. Direct calculations easily lead to $\chi_0(S_p, Y_p) = \dfrac{p}{1-(\frac{p}{2})^2}$. This implies that

$$\sup_{X_1\in\mathcal{X}} \chi_0(S_p, Y_p) \geq \frac{p}{1-(\frac{p}{2})^2}. \tag{5.3}$$

The desired result follows from (5.2) and (5.3). $\square$

Note that in the classical scheme of summation (i.e., $\nu_p = 1/p$ a.s.) a result analogous to Theorem 6 is impossible. This follows, for example, from an investigation of the symmetric binomial distribution (in this case X_j takes the values -1 or 1 with probability 1/2 each).

Theorem 7. *Let ν_p have distribution* (5.1). *Then*

$$\sup_{X_1\in\mathcal{X}} \sigma(S_p, Y_p) = \sup_{X_1\in\mathcal{X}} \rho(S_p, Y_p) = \frac{p}{1+p} + (1+p)^{-(n_0+1)} - (1-p)^{n_0}, \tag{5.4}$$

where

$$n_0 = \left[\left(\ln\frac{1-p}{1+p}\right) / \ln(1-p^2)\right]. \tag{5.5}$$

(Here $[x]$ denotes the integer part of x.)

Proof. For any $X_1 \in \mathcal{X}$

$$\rho(S_p, Y_p) \le \sigma(S_p, Y_p)$$

$$= \sup_{A\in\mathcal{A}} \left| \sum_{n=1}^{\infty} P\left(\sum_{j=1}^{n} X_j \in A\right)(1-p)^{n-1} - (1+p)^{-(n+1)} - \frac{1}{p+1}\right|.$$

It is easy to see that $(1-p)^{n-1} \ge (1+p)^{-(n+1)}$ for all $n \le n_0$ where n_0 is defined by (5.5) while for $n \ge n_0$ the inverse inequality is fulfilled. We then find

$$\sigma(S_p, Y_p) \le$$

$$p\max\left\{\sum_{n=1}^{n_0}\left((1-p)^{n-1} - \frac{1}{(1+p)^{n+1}}\right), \sum_{n=n_0+1}^{\infty}\left(\frac{1}{(1+p)^{n+1}} - (1-p)^{n-1} + \frac{1}{1+p}\right)\right\}$$

$$= \frac{p}{1+p} + \frac{1}{(1+p)^{n_0+1}} - (1-p)^{n_0}.$$

Thus, $\sup_{X_1\in\mathcal{X}} \rho(S_p, X_p) \le \sup_{X_1\in\mathcal{X}} \sigma(S_p, Y_p) \le \frac{p}{1+p} + \frac{1}{(1+p)^{n_0+1}} - (1-p)^{n_0}$.

To see the equality, consider the degenerate case $X_j \equiv 1$, where we have

$$\rho(S_p, Y_p) \ge |P(\nu_p \ge n_0) - P(Y_p \ge n_0)| = \frac{p}{1+p} + \frac{1}{(1+p)^{n_0+1}} - (1-p)^{n_0}. \qquad \square$$

Corollary 5. *Let $(X_{n,p})_{n\in\mathbb{N}}$ be a sequence of iid r.v.'s taking only two values: 1 with probability $\alpha = \lambda p$ or 0 with probability $1-\alpha$, where $\lambda = \text{const}$, and let $P(\mu_\lambda = \kappa) = \dfrac{\lambda^\kappa}{(1+\lambda)^{\kappa+1}}$, $\kappa = 0, 1, \dots$. Then for all $p \in (0,1)$ we have*

$$\chi_0(S_p, \mu_\lambda) \le \frac{p}{1 - (\frac{p}{2})^2}$$

and

$$\sigma(S_p, \mu_\lambda) \le \frac{p}{1+p} + \frac{1}{(1+p)^{n_0+1}} - (1-p)^{n_0},$$

where n_0 is defined by (5.5).

The above corollary follows from Theorems 6 and 7 since μ_λ is a geometrically accompanying variable for $S_p = \sum\limits_{j=1}^{\nu_p} X_{j,p}$.

Theorems 6 and 7 offer the exact estimates of approximations of sums of a geometric number of r.v.'s with geometrically accompanying laws. However, the question can also be posed as to how to approximate with geometrically infinitely divisible distributions. We will show that for χ_0 such an approximation cannot lead to a substantial improvement compared to the result of Theorem 6.

Theorem 8. *For the class $\mathcal{G}$ of all geometrically infinitely divisible r.v.'s, the following inequality holds:*

$$\sup_{X_1 \in \mathcal{X}} \chi_0(S_p, \mathcal{G}) \ge \frac{p}{2-p}.$$

P r o o f. Theorem 2 implies that every geometrically infinitely divisible ch.f. $g(t)$ can be represented as

$$g(t) = \lim_{m\to\infty} \frac{1}{1 - \alpha_m(g_m(t) - 1)}, \tag{5.6}$$

where $\{\alpha_m,\ m \ge 1\}$ is a sequence of positive numbers and $\{g_m(t), m \ge 1\}$ is a sequence of ch.f.'s. From (5.6) we can see that $\operatorname{Re} g(t) \ge 0$ for an real t. Suppose that the ch.f.'s of X_1 is $f(t) = \text{const}$. Then the ch.f. $f_p(t)$ of $S_p = \sum\limits_{j=1}^{\nu_p} X_j$ is equal to $f_p(t) = \dfrac{p\cos t}{1-(1-p)\cos t}$. We finally have $\chi_0(S_p, \mathcal{G}) = \inf\limits_{g\in\mathcal{G}} \sup\limits_t |f_p(t) - g(t)| \ge \inf\limits_{g\in\mathcal{G}} |f_p(\pi) - g(\pi)| \ge \dfrac{p}{2-p}$. $\square$

6. Random sums of random vectors. We now turn to the study of the multivariate case. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of iid random vectors in R^s. We consider the same families $\{\nu_p,\ p \in \Delta\}$ as before. Of course, it is assumed that semigroup $\mathbb{P}$ is commutative. Virtually all the results set forth above are also valid (with obvious changes) in this case. This follows from the Cramér-Wold Device for Theorems 1–5, while for the remaining theorems the arguments given in their proofs remain in force. Let us note only certain changes:

(i) The ch.f. of a ν-strictly Gaussian s-variate distribution has the form

$$f(t) = \varphi((At, t)), \tag{6.1}$$

where A is a symmetric, positive definite matrix, and φ is a standard solution of (2.5).

(ii) Let φ be a standard solution of (2.5). A ch.f. $g(t)$ is a ν-infinitely divisible ch.f. of an s-dimensional r.v. if and only if

$$g(t) = \varphi(-\log f(t)), \quad t \in R^s, \tag{6.2}$$

where $f(t)$ is an infinitely divisible ch.f.

(iii) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of iid s-dimensional r.v.'s with ch.f. $f(t)$. Assume that φ is a standard solution of (2.5). Let $S_p = \sum\limits_{j=1}^{\nu_p} X_j$.

An s-dimensional r.v. Y_p with ch.f.

$$\Psi_p(t; f) = \varphi\left(\frac{1}{p}(1 - f(t))\right)$$

is called an accompanying ν-infinitely divisible s-dimensional random vector for S_p. For the sake of brevity we will speak of ν-accompanying random vectors.

Note that the assertions of Theorems 6 and 7 are generally independent of dimension s.

7. Domains of attraction of multivariate geometrically stable laws. Let ν_p ($p \in (0, 1)$) be a geometric r.v. (5.1), and let $(X_n)_{n\in\mathbb{N}}$ be a sequence of iid s-dimensional r.v.'s independent of ν_p. If for some constants $b_p \in R^s$ and non-singular $s \times s$ matrices A_p the d.f.'s of sums

$$\widetilde{S}_p = A_p^{-1} \sum_{j=1}^{\nu_p} (X_j - b_p) \tag{7.1}$$

weakly converge, as $p \to 0$, to some s-dimensional d.f. V, then we say that the d.f. of vector X_1 is weakly geometrically attracted to V. The collection of all d.f.'s weakly geometrically attracted to V is called the domain of geometric attraction of V and is denoted by $\mathrm{reg}_g(V)$.

Below we establish the relationship between domains of geometric attraction and domains of attraction in the classical sense (denoted by $\operatorname{reg} V$).

Theorem 9. *The domain of geometric attraction of a law V with ch.f $h(t)$, $t \in R^s$ coincides with the (classical) domain of attraction of $\widetilde{V}$ with ch.f.*

$$\widetilde{h} = \exp\{1 - 1/h(t)\}, \ t \in R^s. \tag{7.2}$$

P r o o f. Let X_1 be weakly geometrically attracted to V. This means that ch.f. $f_p(t)$ $(t \in R^s)$ of (7.1) converges to $h(t)$, as $p \to 0$. Theorem 6 implies that $\chi_0(\widetilde{S}_p, \widetilde{Y}_p) \to 0$, as $p \to 0$, where $\widetilde{Y}_p$ is a random vector with ch.f.

$$\Psi_p(t; f) = \frac{1}{1 - \dfrac{e^{i(b_p, A_p t)} f(A_p t) - 1}{p}},$$

which tends to $h(t)$, as $p \to 0$. Choosing $p = 1/n$, $n \in \mathbb{N}$, we see that

$$n\left(e^{i(b_{1/n}, A_{1/n} t)} f(A_{1/n} t) - 1\right) \to 1 - \frac{1}{h(t)} \quad \text{as} \quad n \to \infty.$$

This clearly implies that the normalized sum $A_{1/n}^{-1} \sum\limits_{j=1}^{n} (X_j - b_{1/n})$ converges to a distribution with ch.f. (7.2) when $n \to \infty$. Thus, X_1 is weakly attracted to $\widetilde{V}$ and, consequently, $\operatorname{reg}_g(V) \subset \operatorname{reg}(\widetilde{V})$.

Repeating the argument in reverse order, we obtain the inverse inclusion $\operatorname{reg}(V) \supset \operatorname{reg}_g(\widetilde{V})$. $\square$

8. Bounds for random sums. As we have seen in Section 7, the key to proving Theorem 9 is Theorem 6. Therefore, to study domains of ν-attraction we must investigate the possibility of generalizing Theorem 6 to sums $S_p = \sum\limits_{j=1}^{\nu_p} X_j$, where ν_p does not necessarily have geometric distribution (5.1).

As has been noted above, if $\nu_p = 1/p$, $p \in \{1/n : n \in \mathbb{N}\}$, then $\sup_{X_1 \in \mathcal{X}} \chi_0(S_p, Y_p)$ does not tend to zero as $p \to 0$. Therefore, nontrivial estimates of

$$\delta_p = \sup_{X_1 \in \mathcal{X}} \chi_0(S_p, Y_p) \tag{8.1}$$

are far from possible for all families $\{\nu_p,\ p \in \Delta\}$ with commutative $\mathbb{P}$. Below we calculate the estimate of δ_p and establish the conditions under which $\delta_p \to 0$ as $p \to 0$.

Theorem 10. *Let $l_p(t)$ be the ch.f. of a r.v. ν_p and $\varphi(z)$ a standard solution of* (2.5)*. Then* (8.1) *can be calculated in the following way:*

$$\delta_p = \sup_{0\le\theta<2\pi} \left| l_p(\theta) - \varphi\left(\frac{1}{p}[1-e^{i\theta}]\right)\right|. \tag{8.2}$$

Proof. If $(X_n)_{n\in\mathbb{N}}$ is a sequence of iid r.v.'s (or vectors) independent of ν_p and with ch.f. $f(t)$, then $P_p(f(t))$ is the ch.f. of $S_p = \sum_{j=1}^{\nu_p} X_j$. The ν-accompanying variable Y_p has ch.f. $\varphi\left(\frac{1}{p}(1-f(t))\right)$. Since for real t we have $|f(t)|\le 1$, $P_p(z)$ is analytic in $|z|<1$, and $\varphi(u)$ is analytic in the half-plane $\mathrm{Re}\, u > 0$, then

$$\chi_0(S_p,Y_p) \le \sup_{z\in C, |z|\le 1} \left| P_p(z) - \varphi\left(\frac{1}{p}(1-z)\right)\right| = \sup_{\theta\in[0,2\pi)} \left| P_p(e^{i\theta}) - \varphi\left(\frac{1}{p}(1-e^{i\theta})\right)\right|. \tag{8.3}$$

We have used the fact that the maximum of the modules of a function analytic in a domain is attained on its boundary. However, $P_p(e^{i\theta}) = l_p(\theta)$. In addition, it is clear that $\left| P_p(e^{i\theta}) - \varphi\left(\frac{1}{p}(1-e^{i\theta})\right)\right|$ is equal to $\chi_0(S_p,Y_p)$, when the X_j's have a degenerate distribution concentrated at 1. Consequently, an equality is attained in (8.3) and (8.3) coincides with (8.2). □

Theorem 6 shows that in the case of the geometric variable ν_p (c.f. (5.1)) $\delta_p = \dfrac{p}{1-(\frac{p}{2})^2}$ is fulfilled.

Let us now give an estimate for δ_p for the r.v. ν_p from Example 1. Recall that in this case $\nu_p = 1/p$, $p\in\{1/n, n\in\mathbb{N}\}$ and

$$\delta_{1/n} = \sup_{\theta\in[0,2\pi)} |e^{in\theta} - \exp\{-n(1-e^{i\theta})\}| \ge 1 - e^{-2n},$$

and equality is attained for even n. Thus, $\delta_{1/n} \to 1 \ne 0$ as $n\to\infty$.

Unfortunately, we do not know the necessary and sufficient conditions for $\delta_p \to 0$ (cf. (8.2)) as $p\to 0$. However, we can separately give the necessary or the sufficient conditions for such convergence.

Let us assume that

(A) $A(x) = \lim_{p\to 0} P\{p\nu_p < x\}$ is absolutely continuous for $x>0$;

(B) $\frac{1}{p}\sum_{\kappa=1}^{\infty} |P\{\nu_p=\kappa\} - P\{\nu_p = \kappa+1\}| \le C$ for all $p\in\Delta$, where $C>0$ is a constant.

Theorem 11. *Under conditions* (A) *and* (B)

$$\lim_{p\to 0} \delta_p = 0. \tag{8.4}$$

is fulfilled for (8.2).

Proof. Choose $\varepsilon > 0$. Since $A(x)$ is absolutely continuous (see condition (A)), we have $\lim_{t\to\infty} \int_0^\infty e^{itx} dA(x) = 0$. Therefore there exists a $v_0 > 0$ such that $\left|\int_0^\infty e^{itx} dA(x)\right| < \varepsilon$ for all $t \geq v_0$.

Consider the difference $l_p(\theta) - \varphi\left(\frac{1}{p}(1 - e^{i\theta})\right)$ for $\theta \in [0, v_0 p]$. Let $\theta = vp$, $v \in [0, v_0]$. We have: $l_p(vp) = E\exp(iv(p\nu_p))$ is the ch.f. of $p\nu_p$, therefore $l_p(\nu_p) \to \int_0^\infty e^{ivx} dA(x)$ as $p \to 0$, moreover, the convergence is uniform with respect to $\nu \in [0, v_0]$. On the other hand, $\lim_{p\to 0} \frac{1 - e^{ivp}}{p} = -iv$ and, consequently, $\lim_{p\to 0} \varphi = \left(\frac{1 - e^{ivp}}{p}\right) = \varphi(-iv)$. Summarizing,

$$\lim_{p\to 0} \left(l_p(vp) - \varphi\left(\frac{1 - e^{ivp}}{p}\right)\right) = 0 \tag{8.5}$$

uniformly on $v \in [0, v_0]$. The case $\theta \in [2\pi - v_0 p, 2\pi)$ is considered in exactly the same way.

Without loss of generality, we can assume that $2C/v_0 < \varepsilon$, where C is a constant from condition (B).

Suppose now that $\theta \in [v_0 p, 2\pi - v_0 p]$. Consider first $l_p(\theta)$. Denoting $B_m = \sum_{\kappa=1}^{m} e^{i\theta\kappa}$ and applying the Abel transform, we find $l_p(\theta) = \sum_{\kappa=1}^{\infty} P\{\nu_p = \kappa\} e^{i\theta\kappa} = -\sum_{\kappa=1}^{\infty} (P\{\nu_p = \kappa + 1\} - P\{\nu_p = \kappa\}) B_\kappa$. However, $B_m = \sum_{\kappa=1}^{m} e^{i\theta\kappa} = e^{i\theta} \frac{(1 - e^{i\theta m})}{(1 - e^{i\theta})}$. Therefore, $|B_\kappa| \leq \frac{1}{|\sin\left(\frac{\theta}{2}\right)|}$, $k \in \mathbb{N}$. From this and condition (B) we find that $|l_p(\theta)| \leq C \frac{p}{|\sin\left(\frac{\theta}{2}\right)|}$. This way we have for $\theta \in [v_0 p, 2\pi - v_0 p]$

$$|l_p(\theta)| \leq \frac{2C}{v_0} \leq \varepsilon. \tag{8.6}$$

For the same values of θ

$$\left|\int_0^\infty e^{\frac{ix(\sin\theta)}{p}} dA(x)\right| \leq \varepsilon, \tag{8.7}$$

since $\dfrac{|\sin\theta|}{p} \geq v_0$. The desired now follows from (8.5), (8.6) and (8.7). $\square$

Corollary 6. *Suppose that for all $\in \Delta$ and $\kappa \in \mathbb{N}$ we have $P\{\nu_p = \kappa\} \geq P\{\nu_p = \kappa+1\}$. If $A(x)$ from condition (A) has density bounded in some neighborhood of the point $x = 0$, then* (8.4) *is true.*

The necessary conditions (8.4) are given by the following assertion.

Theorem 12. *Let the family $\{\nu_p,\ p \in \Delta\}$ be such that for some $n \geq 1$ and $r \geq 2$ independent of p*

$$\sum_{\kappa=1}^{n} P\{\nu_p = \kappa\} + \sum_{\kappa=1}^{\infty} P\{\nu_p = n + \kappa r\} = 1 \tag{8.8}$$

for all $p \in \Delta$, is fulfilled. Then (8.4) *does not hold.*

Proof. Denote $a_\kappa(p) = P\{\nu_p = \kappa\}$, $\kappa \in \mathbb{N}$. We have $P_p(e^{i\theta}) = \sum\limits_{\kappa=1}^{n} a_\kappa(p)\exp(i\theta\kappa)$ $+ \sum\limits_{\kappa=1}^{\infty} a_{\kappa r+n}(p)\exp(i\theta(\kappa r + n))$. Letting here $\theta = 2\pi/r$, we find

$$P_p\left(\exp\left(\frac{i2\pi}{r}\right)\right) = \exp\left(\frac{i2\pi n}{r}\right) + \sum_{\kappa=1}^{n-1} a_\kappa(p)\left(\exp\left(\frac{i2\pi\kappa}{r}\right) - \exp\left(\frac{i2\pi n}{r}\right)\right).$$

Since $a_\kappa(p) \to 0$ as $p \to 0$ because $p\nu_p$ has a proper limiting distribution, then

$$P_p\left(\exp\left(\frac{i2\pi}{r}\right)\right) \to \exp\left(\frac{2\pi i n}{r}\right), \quad \text{as} \quad p \to 0. \tag{8.9}$$

However, as $p \to 0$

$$\varphi\left(\frac{1 - \exp\left(\frac{2\pi i}{r}\right)}{p}\right) \to 0, \tag{8.10}$$

since $\mathrm{Re}\left(1 - \exp\left(\dfrac{2\pi i}{r}\right)\right) > 0$ and $\varphi(z)$ is the Laplace transform of $A(x)$. From a comparison of (8.9) and (8.10), we see that, as $p \to 0$,

$$\delta_p \geq \left| P_p\left(\exp\left(\frac{2\pi i}{r}\right)\right) - \varphi\left(\frac{1 - \exp\left(\frac{2\pi i}{r}\right)}{p}\right)\right| \to 1,$$

which implies the desired. $\square$

Example 4. Suppose that for every $m \in \mathbb{N}$, $\{\nu_{p,m} : p \in (0,1)\}$ is a family of r.v.'s such that

$$P\{\nu_{p,m} = 1\} = p^{1/m},$$

$$P\{\nu_{p,m} = 1 + \kappa m\} = \left(\prod_{j=0}^{\kappa-1}\left(\frac{1}{m} + j\right)\right) p^{1/m}\frac{(1-p)^\kappa}{\kappa!}, \quad \kappa \in \mathbb{N}. \tag{8.11}$$

It is easy to see that $E\nu_{p,m} = \dfrac{1}{p}$, $P_{p,m}(z) = p^{1/m}\dfrac{z}{(1-(1-p)z^m)^{1/m}}$. In this case the standard solution of (2.5) has the form, $\varphi_m(t) = (1+mt)^{-1/m}$, and $\mathbb{P}$ is commutative. Theorem 12 implies that for $m > 1$ relation (8.4) does not hold. For $m = 1$ Theorem 6 is applicable.

For an even m the r.v. $\nu_{p,m}$ with distribution (8.11) is an example of a case where we may not be able at all to approximate the distributions of sums S_p with ν-infinitely divisible distributions in metric χ_0. Indeed, let $\mathcal{G}_\nu$ be the class of all ν-infinitely divisible r.v.'s. Then

$$\sup_{X_1\in\mathcal{X}} \chi_0(S_p, \mathcal{G}_\nu) \geq 1.$$

Suppose that X_1 has ch.f. $\cos t$. Then the ch.f. of S_p is

$$f_{p,m}(t) = \frac{p^{1/m}\cos t}{(1-(1-p)\cos^m t)^{1/m}}.$$

Clearly, $f_{p,m}(\pi) = -1$. On the other hand, any ν-infinitely divisible ch.f. has form $g(t) = \lim\limits_{\kappa\to\infty}(1 + m(\alpha_\kappa(1-g_\kappa(t)))^{-1/m}$, where $\alpha_k > 0$ and $g_\kappa(t)$ is a ch.f. It is easy to see that $\mathrm{Re}g(t) \geq 0$. Therefore, if U has ch.f. $g(t)$, then

$$\inf_{g\in\mathcal{G}} \sup_{X_1\in\mathcal{X}} \chi_0(S_p, U) \geq |f_{p,m}(\pi) - g(\pi)| \geq 1.$$

We have thus shown the desired estimate.

9. The domain of attraction for ν-stable random vectors. We can now obtain an analog of the result of Section 7 for generic "ν-sums".

Let $\{\nu_p,\ p \in \Delta\}$ be a family of r.v.'s taking natural values and such that $\mathbb{P}$ is commutative. Assume that $(X_n)_{n\in\mathbb{N}}$ is a sequence of iid s-dimensional r.v.'s independent of ν_p. If for some choice of $b_p \in R^s$ and non-singular $s\times s$ matrices A_p the d.f.'s of sums $\widetilde{S}_p = A_p^{-1}\sum\limits_{j=1}^{p}(X_j - b_p)$ weakly converges as $p \to 0$ to some d.f. V, then we say that the d.f. of X_1 is weakly ν-attracted to V. The collection of all the d.f.'s weakly ν-attracted to V is called the domain of ν-attraction of V.

Theorem 13. *Let* (8.4) *be fulfilled. Then the domain of ν-attraction of law V with ch.f. $h(t)$, $t \in R^s$ coincides with the (classic) domain of attraction of law $\widetilde{V}_\nu$ with ch.f. $\widetilde{h}_\nu = \exp\{-\varphi^{-1}(h(t))\}$, where φ is a standard solution of* (2.5).

The proof of this is analogous to the proof of Theorem 9 and is therefore omitted.

In the sequel we assume that $A(x) = \lim_{p\to 0} P\{p\nu_p \le x\}$, cf. condition (A) in Theorem 11. Let $a_\kappa(p) = P\{\nu_p = \kappa\}$, $\kappa \in \mathbb{N}$ and $b_\kappa(p) = \frac{1}{\kappa! p^\kappa} \int_0^\infty x^\kappa e^{-x/p} dA(x)$, $\kappa = 0, 1, \ldots$. As before, $S_p = \sum_{j=1}^{\nu_p} X_j$ and Y_p is a ν-accompanying r.v. for S_p.

Theorem 14. *Assume that $a_\kappa(p)$ and $b_\kappa(p)$ are such that for any $p \in \Delta$ there exists a natural number $n_0 = n_0(p)$ such that*

$$a_\kappa(p) - b_\kappa(p) \ge 0, \quad \kappa = 1, \ldots, n_0. \tag{9.1}$$

Let

$$K_p = \{\kappa : \kappa \ge 1, \quad a_\kappa(p) - b_\kappa(p) \ge 0\}. \tag{9.2}$$

Then

$$\sum_{\kappa=1}^{n_0} (a_\kappa(p) - b_\kappa(p)) \ge \sup_{X_1 \in \mathcal{X}} \sigma(S_p, Y_p) \le \sum_{\kappa \in K_p} (a_\kappa(p) - b_\kappa(p)). \tag{9.3}$$

P r o o f. We have

$$\varphi\left(\frac{1-z}{p}\right) = \int_0^\infty \exp\left(-\frac{(1-z)x}{p}\right) dA(x) = \sum_{\kappa=0}^{\infty} b_\kappa(p) z^\kappa.$$

Therefore, $\varphi\Bigl(\frac{1-f(t)}{p}\Bigr) \sum_{\kappa=0}^{\infty} b_\kappa(p) f^\kappa(t)$. This representation implies that for any $X_1 \in \mathcal{X}$

$$\rho(S_p, Y_p) \le \sigma(S_p, Y_p) = \sup_{A \in \mathcal{A}} \left| \sum_{n=1}^{\infty} P\left\{\sum_{j=1}^{n} X_j \in A\right\} (a_n(p) - b_n(p)) - b_0(p) \right|.$$

Taking into account (9.2) and the inequality $P(A) \ge 1$, we find that

$$\sigma(S_p, Y_p) = \max\left(\sum_{\kappa \in K_p} (a_\kappa(p) - b_\kappa(p)), b_0(p) + \sum_{\kappa \notin K_p} (b_\kappa(p) - a_\kappa(p)) \right). \tag{9.4}$$

However, $\sum_{n=1}^{\infty} a_\kappa(p) = \sum_{n=1}^{\infty} b_\kappa(p) = 1$. Therefore, $b_0(p) + \sum_{\kappa \notin K_p} (b_\kappa(p) - a_\kappa(p)) = \sum_{\kappa \in K_p} (a_\kappa(p) - b_\kappa(p))$, and from (9.4) we find $\sup_{X_1 \in \mathcal{X}} \sigma(S_p, Y_p) \le \sum_{\kappa \in K_p} (a_\kappa(p) - b_\kappa(p))$. This proves the upper bound in (9.3) and it now remains to prove the lower bound.

Let us consider the degenerate r.v.'s $X_j \equiv 1$. For these variables

$$\rho(S_p, Y_p) = \sup_{x \in R} |P\{\nu_p \le x\} - P\{1 \le Y_p \le x\}| \ge \sum_{\kappa=1}^{n_0} (a_\kappa(p) - b_\kappa(p)),$$

producing the lower bound in (9.3) and concluding the proof. $\square$

Corollary 7. *Assume that* $K_p = \{1, 2, \ldots, n_0\}$ *or, in other words,* $a_\kappa(p) - b_\kappa(p) \ge 0$ *for* $1 \le \kappa \le n_0$ *and* $a_\kappa(p) - b_\kappa(p) < 0$ *for* $\kappa > n_0$. *Then*

$$\sup_{X_1 \in \mathcal{X}} \rho(S_p, Y_p) = \sup_{X_1 \in \mathcal{X}} \sigma(S_p, Y_p) = \sum_{\kappa=1}^{n_0} (a_\kappa(p) - b_\kappa(p)). \tag{9.5}$$

Example 5. Let $\nu_p = 1/p = n$ with probability 1. In this case $A(u)$ is the function of a degenerate distribution concentrated at $u = 1$. Of course,

$$b_\kappa(p) = b_\kappa\left(\frac{1}{n}\right) = \frac{n^\kappa e^{-n}}{\kappa!}.$$

Clearly, $a_\kappa(p) = a_\kappa(1/n) = 0$ for $\kappa \ne n$ and $a_\kappa(1/n) = 1$ for $\kappa = n$. Therefore we find from Theorem 14 $e^{-n} \sum\limits_{\kappa=1}^{n} \dfrac{n^\kappa}{\kappa!} \le \sup\limits_{X_1 \in \mathcal{X}} \rho(S_{1/n}, Y_{1/n}) \le \sup\limits_{X_1 \in \mathcal{X}} \sigma(S_{1/n}, Y_{1/n}) \le 1 - e^{-n}\dfrac{n^n}{n!}$. If $n \to \infty$ then it is easy to verify that

$$\frac{1}{2} \le \varliminf_{n\to\infty} \sup_{X_1 \in \mathcal{X}} \rho(S_{1/n}, Y_{1/n}) \le \varlimsup_{n\to\infty} \sup_{X_1 \in \mathcal{X}} \rho(S_{1/n}, Y_{1/n}) \le 1.$$

Thus, $\rho(S_{1/n}, Y_{1/n})$ and $\sigma(S_{1/n}, Y_{1/n})$ do not tend to zero as $n \to \infty$[2].

Example 6. Suppose now that ν_p is a geometric r.v. distribution (5.1). In this case $A(u) = 1 - e^{-u}$ $(u \ge 0)$. It is easy to calculate that $b_\kappa(p) = \dfrac{p}{(1+p)^{\kappa+1}}$, $\kappa = 0, 1, \ldots$, and (5.1) implies that $a_\kappa(p) = p(1-p)^{\kappa-1}$, $\kappa \in \mathbb{N}$. Clearly we are under the hypotheses of Corollary 7. The verification of this is, in essence, the content of Theorem 7 which leads to (5.4) and (5.5) coinciding with (9.5).

Corollary 8. *Assume that the hypotheses of Theorem* 14 *are fulfilled and*

$$\lim_{p\to 0} \sum_{\kappa \in K_p} (a_\kappa(p) - b_\kappa(p)) = 0.$$

[2] The difference between the result of Example 5 and Theorem 3.2 from Arak and Zaitsev's book [1] is explained by the difference in the definition of accompanying law. Arak and Zaitsev use centralization and normalization while we do not. Regarding this, see the footnote following Definition 4.

Then the domain of ν-attraction of V with $h(t)$ coincides with the classic domain of attraction $\widetilde{V}_\nu$ with

$$\widetilde{h}(t) = \exp\{-\varphi^{-1}(h(t))\},$$

where, as before, φ is the standard solution of (2.5).

10. Rate of convergence. Let X be an s-dimensional r.v. with a non-singular distribution F (i.e., F is not concentrated on any proper subspace of R^s).

Assume that $r > 0$ and consider $B(r) = \{x : x \in R^s, \|x\| < r\}$. It is known that F is uniquely determined from the probabilities that X occurs in the various shifted balls $B_y(r) = B(r) - y$, $y \in R^s$ (see [28]). Therefore, if Y is another random vector in R^s, then

$$d_r(X, Y) = \sup_{y \in R^s} |P\{X \in B_y(r)\} - P\{Y \in B_y(r)\}|. \tag{10.1}$$

is a metric in the space of d.f.'s.

Suppose that for $t > 0$, $\chi(t) = \int\limits_{\|x\| \le 1/t} d(F(x) * F(-x))$, where $F(x) * F(-x)$ is the symmetrization of F. If $Q_F^0(r)$ is a spheric concentration function, that is, $Q_F^0(r) = \sup\limits_{u \in R^s} P\{X \in B_y(r)\}$, then we know that (see [11])

$$Q^0_{F^{*n}}(r) \le \mathcal{A}(s)(\sup_{u \ge r} u^{-2} \chi(u)^{-s/2} \cdot n^{-s/2}), \tag{10.2}$$

where F^{*n} is an n-fold convolution of F and $\mathcal{A}(s)$ is a positive variable dependent on the dimension of the space (i.e., only on s).

Theorem 15. *Suppose that $A(x) = \lim\limits_{p \to 0} P\{p\nu_p < x\}$, $a_\kappa(p)$, $b_\kappa(p)$ are defined as in Theorem* 14 *and K_p by* (9.2)*. Assume that $(X_n)_{n \in \mathbb{N}}$ is a sequence of iid s-dimensional r.v.'s with non-singular distributions. Then*

$$d_r(S_p, Y_p) \le \mathcal{A}(s)(\sup_{u \ge r} u^{-2} \chi(u))^{-s/2} \times$$

$$\times \max\left\{ \sum_{\kappa \in K_p} (a_\kappa(p) - b_\kappa(p))\kappa^{-s/2}, \sum_{\kappa \notin K_p} (b_\kappa(p) - a_\kappa(p))\kappa^{-s/2} + b_0(p) \right\}, \tag{10.3}$$

where $S_p = \sum\limits_{j=1}^{\nu_p} X_j$ and Y_p is a ν-accompanying r.v.

P r o o f. As in Theorem 14, we can write

$$d_r(S_p, Y_p) = \sup_{y \in R^s} |P\{S_p \in B_y(r)\} - P\{Y_p \in B_y(r)\}| \le$$

$$\le \sup_{y\in R^s} \max \left\{ \sum_{\kappa\in K_p} (a_\kappa(p) - b_\kappa(p)) P\left\{ \sum_{j=1}^{\kappa} X_j \in B_y(r) \right\}, \right.$$

$$\left. \sum_{\kappa\notin K_p} (b_\kappa(p) - a_\kappa(p)) P\left\{ \sum_{j=1}^{\kappa} X_j \in B_y(r) \right\} + b_0(p) \right\}. \tag{10.4}$$

However (10.2) implies that

$$P\left\{ \sum_{j=1}^{\kappa} X_j \in B_y(r) \right\} \le \mathcal{A}(s)(\sup_{u\ge r} u^{-2}\chi(u))^{-s/2} n^{-s/2}.$$

After substituting the last estimate into (10.4) we arrive at (10.3). □

Corollary 9. *If in the hypotheses of Theorem* 15 $s > 2$, *then*

$$d_r(S_p, Y_p) \le \mathcal{A}(s)(\sup_{u\ge r} u^{-2}\chi(y))^{-s/2} \sum_{\kappa=1}^{\infty} \kappa^{-s/2} \max_{\kappa} |a_\kappa(p) - b_\kappa(;p)|.$$

Example 7. Let ν_p be the degenerate r.v. $\nu_p = 1/p = n$ a.s., $p \in \{1/n : n \in \mathbb{N}\}$. As in Example 6 we have $b_\kappa(p) = \dfrac{n^\kappa e^{-n}}{\kappa!}$, $a_\kappa(p) = 0$ for $k \ne n$ and $a_n(1/n) = 1$. Theorem 15 implies that

$$d_r(S_p, Y_p) \le \max \left\{ \left(1 - \frac{n^n}{n!} e^{-n}\right) n^{-s/2}, e^{-n} + \sum_{\kappa\ne n} \frac{n^\kappa}{\kappa!} e^{-n} \kappa^{-s/2} \right\}.$$

$$\mathcal{A}(s)(\sup_{u\ge r} u^{-2}\chi(u))^{-s/2}.$$

Of course the last inequality can be made cruder into the form

$$d_r(S_p, Y_p) \le \left(\sum_{\kappa=1}^{\infty} \frac{1}{\kappa^{s/2}} \cdot \frac{n^\kappa}{\kappa!} e^{-n} + e^{-n} \right) A(s)(\sup_{u\ge r} u^{-2}\chi(u))^{-s/2}. \tag{10.5}$$

Since

$$e^{-n} \sum_{\kappa=1}^{\infty} \frac{1}{\kappa^{s/2}} \cdot \frac{n^\kappa}{\kappa!} \le \left(\sum_{\kappa=1}^{\infty} \frac{1}{\kappa^{s}} \cdot \frac{n^\kappa}{\kappa!} e^{-n} \right)^{1/2} \le \frac{C}{n^{s/2}},$$

where C is an absolute constant. Thus, (10.5) implies

$$d_r(S_p, Y_p) \le A(s)(\sup_{u\ge r} u^{-2}\chi(u))^{-s/2} C n^{-s/2}. \tag{10.6}$$

Note that (10.6) is remarkable in that the rate of convergence of distributions $S_{1/n}$ and $Y_{1/n}$ grows as dimension s of random vector X_1 increases.

REFERENCES

[1] T. V. Arak, A. Yu. Zaitsev A. Yu. Uniform Limit Theorems for Sums of Independent Random Variables. Proc. Steklov Institute of Math., AMS, Providence, Rhode Island, 1988.

[2] T. A. Azlarov, A. A. Dzhamirzaev, M. M. Sultanov. Characterizing properties of the exponential distribution and their stability. Random Processes and Statistical Inference, 10-19, 2nd ed., Tashkent, 1972, (in Russian).

[3] P. L. Dobrushin. A lemma on the limit of a complex random function, *Uspekhi Math. Nauk* **10**, *2* (1955), 157-159.

[4] V. Feller. Probability Theory and its Applications, vol. **II**, 2nd ed., Wiley, New York, 1971.

[5] B. V. Gnedenko. On convergence of laws of a distribution of sums of independent summands, *Doklady Akad. Nauk SSSR* **18**, *4-5* (1938), 231-234.

[6] B. V. Gnedenko. The link between the theory of summation of independent random variables and the problems of queuing theory and reliability theory. *Rev. Roumaine Math. Pure Appl.* **12**, *9* (1967) 1243-1253.

[7] B. V. Gnedenko. On some stability theorems. Lecture notes in Math. vol. **982**, Berlin, Springer-Verlag, 1983, pp. 24-31.

[8] B. V. Gnedenko, Kh. Fakhim. A transfer theorem. *Dokl. Akad. Nauk SSSR* **187**, *1* (1967), 15-17.

[9] B. V. Gnedenko, B. Fraier. Several remarks on a paper by I.N. Kovalenko, *Litovsk. Math. Sb.* **9**, *1* (1969), 181-187.

[10] T. E. Harris. The Theory of Branching Processes. Berlin, Springer-Verlag, 1963.

[11] W. Hengartner, R. Theodorescu. Concentration Functions. New York – London, Academic Press, 1973.

[12] A. V. Kakosyan, L. B. Klebanov I. A. Melamed. Characterization of Distributions by the Method of Intensively Monotone Operators. Lecture Notes in Math. vol. **1088**, Berlin, Springer-Verlag, 1984.

[13] A. V. Kakosyan, L. B. Klebanov, S. T. Rachev. Quantitative Criteria for the Convergence of Probability Measures. Erevan, Aiastan, 1988, (in Russian).

[14] L. B. Klebanov, G. M. Manija, I. A. Melamed. A problem of V.M. Zolotarev and analogs of infinitely divisible and stable distributions in the scheme for summation of a random number of random variables. *Theory Prob. Appl.* **29** (1984), 791-794.

[15] L. B. Klebanov, G. M. Manija, I. A. Melamed. Analogs of infinitely divisible and stable laws for sums of random number of random variables. Fourth International Vilnius Conference on the Theory of Probability and Mathematical Statistics, vol. **2**, 1985, pp. 40-41, (in Russian).

[16] L. B. Klebanov, G. M. Manija, I. A. Melamed. ν_p-strictly stable laws and their estimation of their parameters. Lecture Notes in Math. vol. **1233**, Berlin, Springer-Verlag, 1987, pp. 21-31.

[17] L. B. Klebanov, I. A. Melamed, S. T. Rachev. Products of a random number of random variables and a problem of mathematical economics, Lecture Notes in Math. vol. **1412**, Berlin, Springer-Verlag, 1989, pp. 103-109.

[18] T. Kozubowski. The Theory of Geometric Stable Distributions With Applications to Mathematical Finance. PhD Dissertation, Department of Statistics and Applied Probability, University of California, 1992.

[19] V. M. Kruglov, V. Yu. Korolev. Limit Theorems for Random Sums. Moscow, Izdat. Moskov. Gos. Univ., 1990, (in Russian).

[20] E. Lukacs. Characteristic Functions. London, Griffin, 1969.

[21] S. Mittnik, S. T. Rachev. Stable distributions for asset returns. *Appl. Math. Lett.* **213** (1989), 301-304.

[22] S. Mittnik, S. T. Rachev. Alternative multivariate stable distributions and their applications to financial modeling. In: Stable Processes and Related Topics, (eds. S. Cambanis et al) Boston, Birkhäuser, 1991, pp. 107-119.

[23] S. Mittnik, S. T. Rachev. Alternative Multivariate Stable Distributions and Their Applications to Financial Modeling, *Econometric Rev.* **12** (1993), 261-330.

[24] S. Mittnik, S. T. Rachev. Modeling Asset Returns with Alternative Stable Distributions. Wiley, Chichester, 1997, (to appear).

[25] H. Poincaré. Sur une classe nouvelle de transcendantes uniformes. *J. Math. Pures Appl. 4^e^ Sér.* **6** (1890), 313-365.

[26] S. T. Rachev. Probability Metrics and the Stability of Stochastic Models. Wiley, New York, 1991.

[27] H. Robbins. The asymptotic distribution of the sum of a random number of random variables. *Bull. Amer. Math. Soc.* **54**, *12* (1948), 1151-1161.

[28] N. A. Sapogov. On a uniqueness problem for finite measures in Euclidean spaces. *Zap. Nauchn. Sem. Leningrad. Otdel. Math. Inst. Steklov.* **41** (1974), 3-13.

[29] V. M. Zolotarev. Contemporary Theory of Summation of Independent Random Variables. Moscow, Nauka, 1986, (in Russian).

Lev B. Klebanov
Department of Mathematics
St. Petersburg Civil Engineering Institute
2 Krasnoarmeiskaya 4
St. Petersburg 198005
Russia

Svetlozar T. Rachev
Department of Statistics and Applied Probability
University of California
Santa Barbara
CA 93106-3110, USA

Received September 8, 1994
Revised December 24, 1995