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TRIPLES OF POSITIVE INTEGERS WITH THE SAME SUM AND THE SAME PRODUCT

A. Schinzel

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Abstract. It is proved that for every k there exist k triples of positive integers with the same sum and the same product.

In this paper we solve the problem D.16 from the book [1] by proving the following

Theorem. *For every k there exist infinitely many primitive sets of k triples of positive integers with the same sum and the same product.*

(A set S of triples is called primitive if the greatest common divisor of all elements of all triples of S is 1.)

Lemma. *The system of equations*

$$x_1 + x_2 + x_3 = x_1 x_2 x_3 = 6 \tag{1}$$

has infinitely many solutions in rational numbers $x_j > 0$.

Proof. The equation $f(x) = x^3 - 9x + 9 = y^2$ has the solution $\langle x, y\rangle = \langle 7, 17\rangle$, which does not satisfy Nagell's condition $y^2|\Delta$, where $\Delta = 3^6$ is the discriminant of f. Hence (see [2], Chap. V, p. 78, Satz 12a) the equation has infinitely many rational solutions and in virtue of the theroem of Poincaré and Hurwitz (see ibid. Satz 11) it has infinitely many rational solutions in every neighbourhood of any one of them. Since the solution $\langle x, y\rangle = \langle 0, 3\rangle$ satisfies the inequality

$$|y| < 6 - 3x$$

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there are infinitely many rational solutions of $f(x) = y^2$ satisfying this inequality; hence also $x < 2$. Put such solutions

$$x_1 = \frac{6}{3-x}, \quad x_2 = \frac{6-3x+y}{3-x}, \quad x_3 = \frac{6-3x-y}{3-x}.$$

We have $x_j > 0$, moreover

$$x_1 + x_2 + x_3 = 6$$

$$x_1x_2x_3 = \frac{6((6-3x)^2 - y^2)}{(3-x)^3} = \frac{6((6-3x)^2 - f(x))}{(3-x)^3} = 6.$$

To different solutions $\langle x, y\rangle$ correspond different (ordered) triples $\langle x_1, x_2, x_3\rangle$, which proves the lemma. □

P r o o f o f t h e t h e o r e m. Take any k solutions $\langle x_{i1}, x_{i2}, x_{i3}\rangle$, where $x_{i1} \le x_{i2} \le x_{i3}$ of the system (1) in rational numbers $x_j > 0$ and let d be the least common denominator of all the numbers x_{ij} $(i \le k,\ j \le 3)$. Thus

$$x_{ij} = \frac{a_{ij}}{d}, \quad a_{ij} \in \mathbb{N}, \quad \left(\operatorname*{g.c.d.}_{i,j} a_{ij}, d\right) = 1.$$

We have

$$\sum_{j=1}^{3} a_{ij} = 6d, \quad \prod_{j=1}^{3} a_{ij} 6d^3 \ (i \le k), \tag{2}$$

hence $\operatorname*{g.c.d.}_{i,j} a_{ij} = 1$.

If for two sets of solutions $\{\langle x_{i1}, x_{i2}, x_{i3}\rangle : 1 \le i \le k\}$ and $\langle x'_{i1}, x'_{i2}, x'_{i3}\rangle : 1 \le i \le k\}$ the sets of triples $\langle a_{i1}, a_{i2}, a_{i3}\rangle : 1 \le i \le k\}$ and $\langle a'_{i1}, a'_{i2}, a'_{i3}\rangle : 1 \le i \le k\}$ coincide, we have by (2) $d = d'$, hence the sets of solutions themselves coincide. Since there are infinitely many choices of k elements from an infinite set the theorem follows. □

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[2] Th. Skolem. Diophantische Gleichungen, Chelsea, 1950.

Andrzej Schinzel
Institute of Mathematics PAN
ul. Sniadeckich 8
00-950 Warszawa, Poland
e-mail: schinzel@impan.gov.pl

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