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ON UNIFORMLY CONVEX AND UNIFORMLY KADEC-KLEE RENORMINGS

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Abstract. We give a new construction of uniformly convex norms with a power type modulus on super-reflexive spaces based on the notion of dentability index. Furthermore, we prove that if the Szlenk index of a Banach space is less than or equal to ω (first infinite ordinal) then there is an equivalent weak* lower semi-continuous positively homogeneous functional on X^* satisfying the uniform Kadec-Klee Property for the weak*-topology (UKK*). Then we solve the UKK or UKK* renorming problems for $L^p(X)$ spaces and $\mathcal{C}(K)$ spaces for K scattered compact space.

1. Introduction–notations. Throughout this paper, X will denote a real Banach space, B_X its unit ball and X^* its dual. We will first define the three slicing indices associated to X that we will study in this paper.

Dentability index, $\delta(X)$: Let C be a closed bounded subset of X. We call a slice of C any set S of the form $S = \{x \in C : \ x^*(x) > \alpha\}$, where x^* belongs to X^* and α is real.
For $\varepsilon > 0$, $C'_\varepsilon = \{x \in C$ such that any slice of C containing x is of diameter $> \varepsilon\}$.
For an ordinal α, F_ε^α is defined inductively by:

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$F_\varepsilon^0 = B_X$
$F_\varepsilon^{\alpha+1} = (F_\varepsilon^\alpha)'_\varepsilon$
$F_\varepsilon^\alpha = \bigcap\limits_{\beta<\alpha} F_\varepsilon^\beta$, if α is a limit ordinal.

Then

$$\delta(X,\varepsilon) = \begin{cases} \inf\{\alpha : F_\varepsilon^\alpha = \emptyset\} & \text{if it exists} \\ \infty & \text{otherwise} \end{cases}$$

And $\delta(X) = \sup\limits_{\varepsilon>0} \delta(X,\varepsilon)$.

Weak-Szlenk index, $\boldsymbol{Sz_w(X)}$: Let C be a closed bounded subset of X. For $\varepsilon > 0$, $C_\varepsilon^{\langle\prime\rangle} = \{x \in C$ such that any weak neighborhood of x in C is of diameter $> \varepsilon\}$. For an ordinal α, $F_\varepsilon^{\langle\alpha\rangle}$ is defined inductively by:
$F_\varepsilon^{\langle 0\rangle} = B_X$
$F_\varepsilon^{\langle\alpha+1\rangle} = (F_\varepsilon^{\langle\alpha\rangle})_\varepsilon^{\langle\prime\rangle}$
$F_\varepsilon^{\langle\alpha\rangle} = \bigcap\limits_{\beta<\alpha} F_\varepsilon^{\langle\beta\rangle}$, if α is a limit ordinal.

Then

$$Sz_w(X,\varepsilon) = \begin{cases} \inf\{\alpha : F_\varepsilon^{\langle\alpha\rangle} = \emptyset\} & \text{if it exists} \\ \infty & \text{otherwise} \end{cases}$$

And $Sz_w(X) = \sup\limits_{\varepsilon>0} Sz_w(X,\varepsilon)$.

Szlenk index, $\boldsymbol{Sz(X)}$: Let C be a closed bounded subset of X^*. For $\varepsilon > 0$, $C_\varepsilon^{[\prime]} = \{x^* \in C$ such that for any weak*-neighborhood V of $x^*, \mathrm{diam}(V \cap C) > \varepsilon\}$.
We denote:
$K_\varepsilon^{[0]} = B_{X^*}$
$K_\varepsilon^{[\alpha+1]} = (K_\varepsilon^{[\alpha]})_\varepsilon^{[\prime]}$
$K_\varepsilon^{[\alpha]} = \bigcap\limits_{\beta<\alpha} K_\varepsilon^{[\beta]}$, if α is a limit ordinal.

$$Sz(X,\varepsilon) = \begin{cases} \inf\{\alpha : K_\varepsilon^{[\alpha]} = \emptyset\} & \text{if it exists} \\ \infty & \text{otherwise} \end{cases}$$

$Sz(X) = \sup\limits_{\varepsilon>0} Sz(X,\varepsilon)$.

In [L1] and [L2] it is shown that if $\delta(X)$ is countable then X admits an equivalent locally uniformly convex norm and that if $Sz(X)$ is countable then X admits an equivalent norm whose dual norm is locally uniformly convex. In this paper we are interested in the Banach spaces for which these slicings proceed even faster, namely when they stop before ω (the first infinite ordinal). More precisely, we try to know if these conditions imply the existence of equivalent norms enjoying some uniform properties of convexity.

In section 2 we notice that the renorming theorem of P. Enflo for super-reflexive spaces ([E]) implies that the condition $\delta(X) \leq \omega$ is equivalent to X super-reflexive. Then we show how the geometrical construction introduced in [L1] provides us with a uniformly convex norm, when $\delta(X) \leq \omega$. And we prove that the norm built this way has a modulus of convexity bounded below by a power function. By doing so we obtain Pisier's renorming result ([Pi]).

In section 3, we study the links between the condition $Sz(X) \leq \omega$ and the existence of an equivalent norm on X whose dual norm has the uniform Kadec-Klee property for the weak*-topology (UKK*), a property that has been essentially introduced by R. Huff in [Hu]. After noticing that the existence of such a norm implies $Sz(X) \leq \omega$, we prove a partial result for the general converse problem : if X is a separable Banach space with $Sz(X) \leq \omega$, then there is an equivalent weak* lower semi-continuous positively homogeneous functional on X^* with the UKK* property. Next we show that the situation is particularly simple for $L^p(X)$ spaces. Indeed we obtain that if $1 < p < +\infty$, $L^p([0,1], X)$ has an equivalent UKK norm if and only if $L^p([0,1], X)$ has an equivalent norm whose dual norm is UKK* if and only if X is super-reflexive. Then we solve this problem in the case of $\mathcal{C}(K)$ spaces, for K scattered compact space, by showing that $\mathcal{C}(K)$ has an equivalent norm whose dual norm is UKK* if and only if the ω^{th} Cantor derived set $K^{(\omega)}$ is empty if and only if $Sz(\mathcal{C}(K)) \leq \omega$.

2. Dentability index and uniform convexity. For the definitions and for a survey of the renorming results concerning the super-reflexive spaces, we refer the reader to the book of R. Deville, G. Godefroy and V. Zizler ([D-G-Z]).

We shall start with the following easy fact, already mentioned in [L1]:

Proposition 2.1. *$\delta(X) \leq \omega$ if and only if X admits an equivalent uniformly convex norm (or equivalently X super-reflexive).*

Proof. From the existence of an equivalent uniformly convex norm, it follows easily that for any $\varepsilon > 0$, $\delta(X, \varepsilon) < \omega$.

Let us now assume that X is not super-reflexive. Then X has the finite tree property (see R.C. James [J1]). So there exists $\varepsilon > 0$ such that for any $n \in \mathbb{N}$ there is a dyadic tree $(x_s)_{s \in 2^{\leq n}} \subseteq B_X$ (where $2^{\leq n}$ denotes the set of sequences of 0 and 1 with length $\leq n$) satisfying: for any $s \in 2^{\leq n-1}, ||x_{s^\frown 0} - x_{s^\frown 1}|| \geq 2\varepsilon$ and $x_s =$

$\frac{1}{2}(x_{s^\frown 0} + x_{s^\frown 1})$. It is now easy to see that $(x_s)_{s\in 2^{\leq n-1}} \subseteq F'_\varepsilon$. Indeed for $s \in 2^{\leq n-1}$, any slice containing x_s must contain either $x_{s^\frown 0}$ or $x_{s^\frown 1}$. Therefore, this slice is of diameter $> \varepsilon$. Proceeding inductively we obtain that $F^n_\varepsilon \neq \emptyset$. Thus, for any n, $0 \in F^n_\varepsilon$, because F^n_ε is convex and symmetric. Therefore $0 \in F^\omega_\varepsilon$. So $\delta(X) > \omega$. $\square$

We will now use the techniques developed in [L1] in order to give a new construction of uniformly convex norms with a power type modulus on super-reflexive spaces.

Theorem 2.2. (Pisier) *Let X be a Banach space. If $\delta(X) \leq \omega$, then X admits an equivalent uniformly convex norm $|\ \ |$. Moreover, the modulus of convexity $\delta_{|\ |}(\varepsilon)$ of this norm satisfies:*

$$\exists\, C > 0, \exists\, p \geq 2, \quad \textit{such that:} \quad \forall\, 0 < \varepsilon \leq 2,\ \delta_{|\ |}(\varepsilon) \geq C\varepsilon^p.$$

Proof. For any $\varepsilon > 0$, $\delta(X, \varepsilon) < \omega$. Let us denote $N_k = \delta(X, 2^{-k}) - 1$, and

$$f(x) = \|x\| + \sum_{k=1}^{\infty} \sum_{n=1}^{N_k} \frac{2^{-k}}{N_k} d(x, F^n_{2^{-k}}),$$

where $\|\ \ \|$ denotes the initial norm on X and $d(x, F^n_{2^{-k}})$ the distance from x to $F^n_{2^{-k}}$ for this norm.

Let $|\ \ |$ be the Minkowski functional of the convex symmetric set $C = \{x \in X:\ f(x) \leq 1\}$. Then, for all $x \in X: \|x\| \leq |x| \leq 2\|x\|$. So $|\ \ |$ is an equivalent norm on X.

We will first show that f is uniformly convex and evaluate its modulus of convexity in terms of the index $\delta(X, \varepsilon)$.

Lemma 2.3. *For any $\varepsilon > 0$ and any x, y in X:*
if $f(x) = f(y) = 1$ and $||x - y|| \geq \varepsilon$, then: $f\left(\frac{x+y}{2}\right) \leq 1 - \frac{\varepsilon^2}{32\delta^2(X, \frac{\varepsilon}{8})}$.

Proof. Let $\varepsilon > 0$ and let x and y in X such that $f(x) = f(y) = 1$ and $\|x - y\| \geq \varepsilon$. Let $k \in \mathbb{N}$ such that $\frac{\varepsilon}{8} \leq 2^{-k} < \frac{\varepsilon}{4}$.

Let $n = \mathrm{Max}\{m \geq 0 : x \in F^m_{2^{-k}}$ and $y \in F^m_{2^{-k}}\}$. Assume for instance that $x \in F^n_{2^{-k}} \setminus F^{n+1}_{2^{-k}}$. Remark that, since $\|x-y\| \geq \varepsilon$, we have that $n < N_k$. Finally, put $\gamma = \dfrac{\varepsilon}{4N_k}$.

Claim. There exists l, $1 \leq l \leq N_k - n$, such that:

$$\frac{1}{2}\left(d(x, F^{n+l}_{2^{-k}}) + d(y, F^{n+l}_{2^{-k}})\right) - d\left(\frac{x+y}{2}, F^{n+l}_{2^{-k}}\right) \geq \gamma.$$

P r o o f o f C l a i m. Suppose that for all $1 \leq l \leq N_k - n$:

$$(*) \qquad \frac{1}{2}\left(d(x, F^{n+l}_{2^{-k}}) + d(y, F^{n+l}_{2^{-k}})\right) - d\left(\frac{x+y}{2}, F^{n+l}_{2^{-k}}\right) < \gamma$$

Then we will show by induction that for all $1 \leq l \leq N_k - n$:

$$(\mathcal{P}_l) \qquad \frac{1}{2}\left(d(x, F^{n+l}_{2^{-k}}) + d(y, F^{n+l}_{2^{-k}})\right) < l\gamma$$

For $l = 1$: we have $x, y \in F^n_{2^{-k}}$ and $\|x-y\| \geq \varepsilon$, so $\dfrac{x+y}{2} \in F^{n+1}_{2^{-k}}$. Thus, $(*)$ implies that $\dfrac{1}{2}\left(d(x, F^{n+1}_{2^{-k}}) + d(y, F^{n+1}_{2^{-k}})\right) < \gamma$. So $(\mathcal{P}_1)$ is satisfied.

Assume $(\mathcal{P}_l)$ is verified. Then there exist $x', y' \in F^{n+l}_{2^{-k}}$ such that $\dfrac{1}{2}(\|x-x'\| + \|y-y'\|) < l\gamma$, therefore $\|x'-y'\| > \varepsilon - 2l\gamma \geq \dfrac{\varepsilon}{2}$ and $\dfrac{x'+y'}{2} \in F^{n+l+1}_{2^{-k}}$. But $\left\|\dfrac{x+y}{2} - \dfrac{x'+y'}{2}\right\| < l\gamma$, so $d\left(\dfrac{x+y}{2}, F^{n+l+1}_{2^{-k}}\right) < l\gamma$.
Then property $(*)$ implies that:

$$\frac{1}{2}\left(d(x, F^{n+l+1}_{2^{-k}}) + d(y, F^{n+l+1}_{2^{-k}})\right) < (l+1)\gamma$$

which concludes the inductive proof of $(\mathcal{P}_l)$.

So in particular: $\dfrac{1}{2}\left(d(x, F^{N_k}_{2^{-k}}) + d(y, F^{N_k}_{2^{-k}})\right) < (N_k - n)\gamma \leq N_k\gamma = \dfrac{\varepsilon}{4}$. Thus there exist $x', y' \in F^{N_k}_{2^{-k}}$ such that $\dfrac{1}{2}(\|x-x'\|+\|y-y'\|) < \dfrac{\varepsilon}{4}$ and therefore $\|x'-y'\| > \dfrac{\varepsilon}{2}$. It follows that $\dfrac{x'+y'}{2} \in F^{N_k+1}_{2^{-k}}$, which is impossible because $F^{N_k+1}_{2^{-k}}$ is empty. $\square$

E n d o f t h e p r o o f o f L e m m a 2.3. The functions $\|\cdot\|$ and $d(\cdot, F^n_{2^{-k}})$ are all convex, so: $\dfrac{1}{2}(f(x)+f(y)) - f\left(\dfrac{x+y}{2}\right) \geq \dfrac{2^{-k}}{N_k} \cdot \dfrac{\varepsilon}{4N_k} \geq \dfrac{\varepsilon^2}{32N_k^2}$.

Therefore $f\left(\frac{x+y}{2}\right) \leq 1 - \frac{\varepsilon^2}{32\delta^2(X, \frac{\varepsilon}{8})}$. □

Let us denote by δ_f the modulus of convexity of the function f. It is not difficult to see that Lemma 2.3. implies that $|\ \ |$ is uniformly convex. More precisely: $\delta_{|\ |}(\varepsilon) \geq \frac{1}{4}\delta_f\left(\frac{\varepsilon}{2}\right)$. Then the conclusion of Theorem 2.2. will follow from the next proposition:

Proposition 2.4. *Let X be a Banach space. If $\delta(X) \leq \omega$, then there exist $q > 1$ and $C' > 0$ such that: for any $0 < \varepsilon \leq 2$, $\delta(X, \varepsilon) \leq \frac{C'}{\varepsilon^q}$.*

We will first prove a similar result for the weak-Szlenk index:

Lemma 2.5. *Let X be a Banach space. If $Sz_w(X) \leq \omega$, then there exist $q > 1$ and $C'' > 0$ such that: for any $0 < \varepsilon \leq 2$, $Sz_w(X, \varepsilon) \leq \frac{C''}{\varepsilon^q}$.*

Proof. First we will show that

$$\forall \varepsilon > 0, \forall \varepsilon' > 0, Sz_w(X, \varepsilon\varepsilon') \leq Sz_w(X, \varepsilon) Sz_w(X, \varepsilon').$$

It is enough to prove by induction that $\forall n \in \mathbb{N}\ \ F_{\varepsilon\varepsilon'}^{\langle n.Sz_w(X,\varepsilon')\rangle} \subseteq F_\varepsilon^{\langle n\rangle}$.
This is clearly true for $n = 0$, so let us assume that $F_{\varepsilon\varepsilon'}^{\langle n.Sz_w(X,\varepsilon')\rangle} \subseteq F_\varepsilon^{\langle n\rangle}$.
Let x such that $x \notin F_\varepsilon^{\langle n+1\rangle}$. We need to show that $x \notin F_{\varepsilon\varepsilon'}^{\langle (n+1).Sz_w(X,\varepsilon')\rangle}$, so we may assume that $x \in F_\varepsilon^{\langle n\rangle}$. Thus there is a weak-open set V containing x and such that $\operatorname{diam}(V \cap F_\varepsilon^{\langle n\rangle}) \leq \varepsilon$. But $(\varepsilon B_X)_{\varepsilon\varepsilon'}^{\langle S_w(X,\varepsilon')\rangle} = \emptyset$, so, for every subset C of diameter $\leq \varepsilon$, $C_{\varepsilon\varepsilon'}^{\langle Sz_w(X,\varepsilon')\rangle} = \emptyset$. Therefore $x \notin F_{\varepsilon\varepsilon'}^{\langle (n+1).Sz_w(X,\varepsilon')\rangle}$.

Now, it follows from the submultiplicativity of the function $Sz_w(X, \cdot)$ that there exists $q > 1$ such that $Sz_w(X, \varepsilon) = O\left(\frac{1}{\varepsilon^q}\right)$ (this argument is classical: see for instance Maurey's argument for Pisier's renorming result appearing in [B] and detailed in [D-G-Z]). □

It seems to us very unlikely that the function $\delta(X, \cdot)$ is submultiplicative. But this difficulty is overcome by the next lemma which enables us to control $\delta(X)$ by $Sz_w(L^2(X))$.

Lemma 2.6. *Let X be a Banach space, $1 < p < +\infty$, $F = B_X$ and $L = B_{L^p([0,1],X)}$. For any $\varepsilon > 0$, any ordinal α and any k in $\mathbb{N}$ we have the following:*

if $x_1, \ldots, x_k$ *belong to* F_ε^α, *then* $\sum_{i=1}^k x_i \mathbb{1}_{[\frac{i-1}{k}, \frac{i}{k}[} \in L_{\varepsilon/2}^{\langle\alpha\rangle}$ *(where* $\mathbb{1}_{[\frac{i-1}{k}, \frac{i}{k}[}$ *is the indicator function of* $\left[\frac{i-1}{k}, \frac{i}{k}\right[$*).*

Consequently $\delta(X, \varepsilon) \leq Sz_w(L^p(X), \varepsilon/2)$ *and* $\delta(X) \leq Sz_w(L^p(X))$.

P r o o f. We will prove this by transfinite induction.

The case $\alpha = 0$ is obvious and the property stated in this lemma passes clearly to limit ordinals.

Assume this property is true for α.

Let $x_1, \ldots, x_k$ in $F_\varepsilon^{\alpha+1}$ and let V be a weakly open subset of $L^p(X)$ containing $\displaystyle\sum_{i=1}^k x_i \mathbb{1}_{[\frac{i-1}{k}, \frac{i}{k}[}$ (by induction hypothesis $\displaystyle\sum_{i=1}^k x_i \mathbb{1}_{[\frac{i-1}{k}, \frac{i}{k}[} \in L_{\varepsilon/2}^{\langle\alpha\rangle}$).

By Hahn Banach theorem, there exists $l \geq 1$ such that

$\forall 1 \leq i \leq k, \exists (x_{i,j})_{j=1}^l \subseteq F_\varepsilon^\alpha$ verifying: $\displaystyle\left\| \frac{1}{l} \sum_{j=1}^l x_{i,j} - x_i \right\| < \gamma$ and for all $1 \leq j \leq l$, $\|x_{i,j} - x_i\| > \dfrac{\varepsilon}{2}$ where γ is a positive real number, small enough to insure that the ball of radius γ and centered at $\sum_{i=1}^k x_i \mathbb{1}_{[\frac{i-1}{k}, \frac{i}{k}[}$ is included in V.

Let $\phi_n = \sum_{i=1}^k \sum_{m=1}^n \sum_{j=1}^l x_{i,j} \mathbb{1}_{[\frac{i-1}{k} + \frac{m-1}{kn} + \frac{j-1}{knl}, \frac{i-1}{k} + \frac{m-1}{kn} + \frac{j}{knl}[}$

We have that $\displaystyle\phi_n \xrightarrow{\omega} \sum_{i=1}^k \left(\frac{1}{l} \sum_{j=1}^l x_{i,j} \right) \mathbb{1}_{[\frac{i-1}{k}, \frac{i}{k}[}$. Therefore there exists $n_0 \geq 1$ such that $\phi_{n_0} \in V$.

But, for all $t \in [0, 1[$, $\displaystyle\left\| \phi_{n_0}(t) - \sum_{i=1}^k x_i \mathbb{1}_{[\frac{i-1}{k}, \frac{i}{k}[}(t) \right\| > \frac{\varepsilon}{2}$, so $\displaystyle\left\| \phi_{n_0} - \sum_{i=1}^k x_i \mathbb{1}_{[\frac{i-1}{k}, \frac{i}{k}[} \right\| > \frac{\varepsilon}{2}$.

But, by induction hypothesis $\phi_{n_0} \in L_{\varepsilon/2}^{\langle\alpha\rangle}$.

Therefore, $\displaystyle\mathrm{diam}(V \cap L_{\varepsilon/2}^{\langle\alpha\rangle}) > \frac{\varepsilon}{2}$ and $\displaystyle\sum_{i=1}^k x_i \mathbb{1}_{[\frac{i-1}{k}, \frac{i}{k}[} \in L_{\varepsilon/2}^{\langle\alpha+1\rangle}$. $\square$

P r o o f o f P r o p o s i t i o n 2.4. Let X be a Banach space such that $\delta(X) \leq \omega$. We already know, by Lemma 2.3, that X has an equivalent uniformly convex norm. So $L^2(X)$ does too (see for instance M.M. Day's proof [Da]). Therefore $Sz_w(L^2(X)) \leq \omega$. Thus there is $q > 1$ so that $Sz_w(L^2(X), \varepsilon) = O(\dfrac{1}{\varepsilon^q})$ (Lemma 2.5). But, Lemma 2.6 implies that $\delta(X, \varepsilon) \leq Sz_w(L^2(X)), \dfrac{\varepsilon}{2})$. So $\delta(X, \varepsilon) = O(\dfrac{1}{\varepsilon^q})$. $\square$

Remark. This can be seen as an alternative proof of Pisier's result, knowing Enflo's theorem. Indeed we are still lacking a direct proof of the fact that X super-reflexive implies $\delta(X) \leq \omega$. However, the main interest of this construction is to give a simple and geometrical procedure for building uniformly convex norms with power type moduli.

3. Szlenk indices and uniform Kadec-Klee Properties. In this section we will study the following notions:

Definition 3.1. *Let X be a Banach space.X has the uniform Kadec-Klee property (denoted UKK), if for any $\varepsilon > 0$, there exists $\Delta > 0$ such that: if for any weak-neighborhood V of x, $\operatorname{diam}(V \cap B_X) > \varepsilon$, then $\|x\| \leq 1 - \Delta$.*

Definition 3.2. *Let X be a Banach space. X^* has the uniform Kadec-Klee property for the weak*-topology (UKK*), if for any $\varepsilon > 0$, there exists $\Delta > 0$ such that: if for any weak*-neighborhood V of x^*, $\operatorname{diam}(V \cap B_{X^*}) > \varepsilon$, then $\|x^*\| \leq 1 - \Delta$.*

These definitions extend the usual ones introduced by R. Huff ([Hu]).

Clearly, if X has the property UKK, then $Sz_w(X) \leq \omega$ and if X^* has the property UKK*, then $Sz(X) \leq \omega$. So it is natural to ask the following questions: let X be a Banach space satisfying $Sz_w(X) \leq \omega$ (respectively $Sz(X) \leq \omega$), does X have an equivalent UKK norm (respectively an equivalent norm whose dual norm is UKK*)? If so, can we construct this norm with a power type modulus $\Delta(\varepsilon)$?

3.1. The general case.

We present now the partial general result that we have obtained in this direction.

Theorem 3.3. *Let X be a separable Banach space.Then $Sz(X) \leq \omega$ if and only if there exists a function $f : X^* \to \mathbb{R}^+$ weak*-lower semi-continuous (ω^*-l.s.c.) verifying:*

i) $\forall x^* \in X^* \quad \frac{1}{2}\|x^*\| \leq f(x^*) \leq \|x^*\|$.

ii) $\forall \lambda \in \mathbb{R} \quad f(\lambda x^*) = |\lambda| f(x^*)$.

iii) $\forall \varepsilon > 0$, $\exists\ \Delta = \Delta_f(\varepsilon) > 0$ *so that, for any sequence* $(x_n^*)_{n\geq 0}$ *in* $\{y^* \in X^* : f(y^*) \leq 1\}$ *and any* x^* *in* X^*: $(x_n^* \xrightarrow{\omega^*} x^*$ *and* $\forall\ n \neq m\ \|x_n^* - x_m^*\| > \varepsilon) \Rightarrow\ f(x^*) \leq 1 - \Delta$.

Moreover, in this case, we can construct f such that there exist $p \geq 1$ and $C > 0$ verifying, for any $0 < \varepsilon \leq 2$, $\Delta(\varepsilon) \geq C\varepsilon^p$.

Proof. The "if" part is clear, so let us assume that $Sz(X) \leq \omega$. The first step of our construction will be to show the following proposition:

Proposition 3.4. *Let X be a separable Banach space.*

If $Sz(X) \leq \omega$, then for any $\varepsilon > 0$, there exists $h_\varepsilon : X^* \to \mathbb{R}^+$ such that:

i) $\forall x^* \in X^* \quad \frac{1}{2}\|x^*\| \leq h_\varepsilon(x^*) \leq \|x^*\|$.

ii) $\forall \lambda \in \mathbb{R} \quad h_\varepsilon(\lambda x^*) = |\lambda| h_\varepsilon(x^*)$.

iii) There exists $\Delta_1(\varepsilon) > 0$ such that for any $x^* \in X^* \setminus \{0\}$ and any $(x_n^*)_{n\geq 0}$ in X^*, if $x_n^* \xrightarrow{\omega^*} x^*$ and $\forall k \neq k'$ $\dfrac{\|x_k^* - x_{k'}^*\|}{\limsup \|x_n^*\|} > \varepsilon$ then $h_\varepsilon(x^*) \leq (1-\Delta_1(\varepsilon)) \liminf h_\varepsilon(x_n^*)$. Moreover, there are $q \geq 1$ and $C' > 0$ so that for all $0 < \varepsilon \leq 2$, $\Delta_1(\varepsilon) \geq C'\varepsilon^q$.

Proof. This proof is inspired by the construction made by P. Enflo in [E] in order to renorm super-reflexive spaces. We will therefore use a similar vocabulary:

Let $x^* \in X^* \setminus \{0\}$, $n \in \mathbb{R}$ and $\varepsilon > 0$.

we call (n, ε)-*partition* of x^* any family $(x_s^*)_{s\in\omega^{\leq n}} \subseteq X^*$ verifying:

a) $x_\emptyset^* = x^*$.

b) $\forall s \in \omega^{\leq n-1}$, $\forall k \neq k'$, $\dfrac{\|x_{s\frown k}^* - x_{s\frown k'}^*\|}{\limsup \|x_{s\frown n}^*\|} > \varepsilon$.

c) $\forall s \in \omega^{\leq n-1}$, $x_{s\frown n}^* \xrightarrow{\omega^*} x_s^*$.

We will begin with the following lemma:

Lemma 3.5. *Let $\varepsilon > 0$ and $n \geq Sz(X, \frac{\varepsilon}{3}) = n(\varepsilon)$. If $(x_s^*)_{s\in\omega^{\leq n}}$ is an (n, ε)-partition of x^* then*

$$\liminf_{i_1} \ldots \liminf_{i_n} \|x_{(i_1,..,i_n)}^*\| \geq 3\|x^*\|.$$

Proof. We may assume $\|x^*\| = 1$. Let $(x_s^*)_{s\in\omega^{\leq n}}$ be an (n, ε)-partition of x^* such that $\liminf_{i_1} \ldots \liminf_{i_n} \|x_{(i_1,\ldots,i_n)}^*\| < 3$. By extracting a subpartition, we may assume that $(x_s^*)_{s\in\omega^{\leq n}} \subseteq 3B_{X^*}$. But since $\|x^*\| = 1$, we may also assume that for all $s \in \omega^{\leq n-1}$, $\limsup \|x_{s\frown n}^*\| \geq 1$. So $\forall k \neq k'$, $\|x_{s\frown k}^* - x_{s\frown k'}^*\| > \varepsilon$. Thus $x^* \in (3B_{X^*})_\varepsilon^{[n]}$. Hence $\frac{1}{3}x^* \in (B_{X^*})_{\varepsilon/3}^{[n]}$ and therefore $n < Sz(X, \frac{\varepsilon}{3})$. $\square$

Remark. By Lemma 2.5. there exists $q \geq 1$ such that $n(\varepsilon) = O(\frac{1}{\varepsilon^q})$.

End of proof of Proposition 3.4. Put $h_\varepsilon(0) = 0$. and for $x^* \neq 0$:

$$h_\varepsilon(x^*) = \inf\{\frac{\liminf_{i_1} \dots \liminf_{i_n} \|x^*_{(i_1,\dots,i_n)}\|}{1+\gamma\sum_{k=1}^{n}\frac{1}{k^2}}$$
$$n \in \mathbb{N}, (x^*_s)_{s\in\omega^{\leq n}}\ (n,\varepsilon)-\ \text{partition of } x^*\},\ \text{where } \gamma = \frac{6}{\pi^2}.$$

since x^* is an (n,ε)-partition of x^*, we have $h_\varepsilon(x^*) \leq \|x^*\|$.
On the other hand, for any (n,ε)-partition of x^*:

$$\frac{\liminf_{i_1} \dots \liminf_{i_n} \|x^*_{(i_1,\dots,i_n)}\|}{1+\gamma\sum_{k=1}^{n}\frac{1}{k^2}} > \frac{\|x^*\|}{1+\gamma\sum_{k=1}^{\infty}\frac{1}{k^2}} = \frac{1}{2}\|x^*\|.$$

So point i) of Proposition 3.4. is satisfied.

It follows clearly from the definition of h_ε that ii) is also satisfied.

Now let $x^* \in X^* \setminus \{0\}$ and $(x^*_n)_{n\geq 0}$ in X^* such that

$$x^*_n \xrightarrow{\omega^*} x^* \text{ and } \forall k \neq k' \frac{\|x^*_k - x^*_{k'}\|}{\limsup \|x^*_n\|} > \varepsilon.$$

Let $0 < \beta < \frac{1}{2}$ and let $(x^*_s(n))_{s\in\omega^{\leq k_n}}$ a (k_n,ε)-partition of x^*_n such that:

$$(1+\beta)h_\varepsilon(x^*_n) > \frac{\liminf_{i_1} \dots \liminf_{i_{k_n}} \|x^*_{(i_1,\dots,i_{k_n})}(n)\|}{1+\gamma\sum_{l=1}^{k_n}\frac{1}{l^2}}.$$

We want to show an inequality of the type $h_\varepsilon(x^*) \leq (1-\Delta_1)\liminf h_\varepsilon(x^*_n)$. So we may assume, by taking a subsequence, that $h_\varepsilon(x^*_n) \longrightarrow \liminf h_\varepsilon(x^*_n)$. Moreover, by Lemma 3.5, we have that for all $n \in \mathbb{N}$, $k_n < n(\varepsilon)$. So we can assume, by taking a new subsequence, that there exists $k < n(\varepsilon)$ such that for all $n \in \mathbb{N}$, $k_n = k$. Then we get that $\{x^*\} \cup \bigcup_{n=0}^{\infty} (x_s^*(n))_{s\in\omega^{\leq k}}$ is a $(k+1,\varepsilon)$-partition of x^*. Therefore

$$\begin{aligned} h_\varepsilon(x^*) &\leq \frac{\liminf_n \liminf_{i_1} \dots \liminf_{i_k} \|x^*_{(i_1,\dots,i_k)}(n)\|}{1+\gamma \displaystyle\sum_{l=1}^{k+1} \frac{1}{l^2}} \\ &\leq \frac{1+\displaystyle\sum_{l=1}^{k} \frac{1}{l^2}}{1+\gamma \displaystyle\sum_{l=1}^{k+1} \frac{1}{l^2}} (1+\beta) \liminf h_\varepsilon(x_n^*). \end{aligned}$$

Since $k < n(\varepsilon)$,

$$\frac{1+\gamma \displaystyle\sum_{l=1}^{k} \frac{1}{l^2}}{1+\gamma \displaystyle\sum_{l=1}^{k+1} \frac{1}{l^2}} \leq \frac{1+\gamma \displaystyle\sum_{l=1}^{n(\varepsilon)-1} \frac{1}{l^2}}{1+\gamma \displaystyle\sum_{l=1}^{n(\varepsilon)} \frac{1}{l^2}} = 1-\Delta_1(\varepsilon).$$

From the above remark it follows that there exist $q \geq 1$ and $C' > 0$ such that for all $0 < \varepsilon \leq 2$, $\Delta_1(\varepsilon) \geq C'\varepsilon^q$.
Furthermore, for all $0 < \beta < \frac{1}{2}$: $h_\varepsilon(x^*) \leq (1-\Delta_1(\varepsilon))(1+\beta) \liminf h_\varepsilon(x_n^*)$, so $h_\varepsilon(x^*) \leq (1-\Delta_1(\varepsilon)) \liminf h_\varepsilon(x_n^*)$. $\square$

Proof of Theorem 3.3. Let us now denote f_ε the weak*-lower semi-continuous regularization of h_ε, namely

$$f_\varepsilon(x^*) = \sup\{\inf_{y^*\in V} h_\varepsilon(y^*) : V \text{weak}^*\text{-neighborhood of} x^*\}$$

f_ε is ω^*-l.s.c. and keeps clearly the properties i) and ii) of h_ε.

f_ε enjoys also a property similar to iii). More precisely, we have:

Lemma 3.6. *Let $\varepsilon > 0$. For any $x^* \in X^* \setminus \{0\}$ and any sequence $(x_n^*)_{n\geq 0}$ in X^*: if*

$$x_n^* \xrightarrow{\omega^*} x^* \text{ and } \forall\ k \neq k' \ \frac{\|x_k^* - x_{k'}^*\|}{\limsup \|x_n^*\|} > \varepsilon$$

then

$$f_\varepsilon(x^*) \leq (1-\Delta_1(\frac{\varepsilon}{8})) \liminf f_\varepsilon(x_n^*).$$

P r o o f. Since f_ε satisfies ii), it is enough to show that if $(x_n^*)_{n\geq 0} \subseteq \{y^* \in X^* : f_\varepsilon(y^*) < 1\}$, then $f_\varepsilon(x^*) \leq 1 - \Delta_1(\frac{\varepsilon}{8})$.

So let $x^* \neq 0$ and $(x_n^*)_{n\geq 0} \subseteq \{y^* \in X^* : f_\varepsilon(y^*) < 1\}$ satisfying the hypotheses of Lemma 3.6. Let V and V' two weak*-neighborhoods of x^* such that $\overline{V'}^* \subseteq V$ ($\overline{V'}^*$ denotes the weak*-closure of V). By taking a subsequence we may assume that for all $n \in \mathbb{N}$:

$$x_n^* \in V' \text{ and } \forall n \geq 0 \frac{\|x_n^* - x^*\|}{\limsup \|x_n^*\|} > \frac{\varepsilon}{2}.$$

On the other hand, we have that for any $n \in \mathbb{N}$ and any weak*-neighborhood W of x_n^*, there exists $z^* \in W$ such that $h_\varepsilon(z^*) < 1$.

We will now build by induction a sequence $(z_k^*)_{k\geq 0} \subseteq V'$ such that:

$$\forall k \in \mathbb{N},\ \frac{\|z_k^* - x^*\|}{\limsup \|x_n^*\|} > \frac{\varepsilon}{2} \text{ and } h_\varepsilon(z_k^*) < 1 \text{ and } \forall k \neq k', \frac{\|z_k^* - z_{k'}^*\|}{\limsup \|x_n^*\|} > \frac{\varepsilon}{2}.$$

Put $z_0^* = x_0^*$.
Suppose $z_0^*, \ldots, z_k^*$ constructed. Then there is a weak*-neighborhood U of x^* such that:

$$\forall 0 \leq i \leq k, \forall y^* \in U : \frac{\|z_i^* - y^*\|}{\limsup \|x_n^*\|} > \frac{\varepsilon}{2}.$$

Since $x_n^* \xrightarrow{\omega^*} x^*$, there exists N such that $\forall 0 \leq i \leq k$, $\dfrac{\|z_i^* - x_N^*\|}{\limsup \|x_n^*\|} > \dfrac{\varepsilon}{2}$.
On the other hand $\dfrac{\|x^* - x_N^*\|}{\limsup \|x_n^*\|} > \dfrac{\varepsilon}{2}$. So there is a weak*-neighborhood W of x_N^* with $W \subseteq V'$ and such that

$$\forall z^* \in W, \forall 0 \leq i \leq k : \frac{\|z_i^* - z^*\|}{\limsup \|x_n^*\|} > \frac{\varepsilon}{2} \text{ and } \frac{\|x^* - z^*\|}{\limsup \|x_n^*\|} > \frac{\varepsilon}{2}.$$

To conclude this induction we choose $z_{k+1}^* \in W$ such that $h_\varepsilon(z_{k+1}^*) < 1$.

To show that $f_\varepsilon(x^*) \leq 1 - \Delta_1(\frac{\varepsilon}{8})$, we may assume that $\limsup \|x_n^*\| > \frac{1}{2}$.
But $h_\varepsilon(z_k^*) < 1$ implies $\|z_k^*\| < 2$, thus $\|z_k^*\| < 4 \limsup \|x_n^*\|$. Therefore

$$\forall k \neq k', \frac{\|z_k^* - z_{k'}^*\|}{\limsup \|z_n^*\|} > \frac{\varepsilon}{8}.$$

Now, there are a subsequence $(z_{k_i}^*)_{i\geq 0}$ and $z^* \in X^*$ such that $z_{k_i}^* \xrightarrow{\omega^*} z^*$.

So, by Proposition 3.4, $h_\varepsilon(z^*) \leq 1 - \Delta_1(\frac{\varepsilon}{8})$.
But $(z^*_{k_i})_{i\geq 0} \subseteq V' \subseteq \overline{V'}^* \subseteq V$. Thus $z^* \in V$ and therefore $\inf_{y^*\in V} h_\varepsilon(y^*) \leq 1 - \Delta_1(\frac{\varepsilon}{8})$.
This is true for any weak*-neighborhood V of x^*, so we have indeed $f_\varepsilon(x^*) \leq 1 - \Delta_1(\frac{\varepsilon}{8})$.
□

End of proof of Theorem 3.3. Put $f(x^*) = \sum_{i=1}^{\infty} 2^{-i} f_{2^{-i}}(x^*)$.

f is ω^*-l.s.c. and satisfies properties i) and ii).
Let $\varepsilon > 0$ and $(x^*_n)_{n\geq 0} \subseteq \{y^* \in X^* : f(y^*) \leq 1\}$ such that $x^*_n \xrightarrow{\omega^*} x^*$ and $\forall n \neq m$ $\|x^*_n - x^*_m\| > \varepsilon$ for any $n \geq 0$, $f_\varepsilon(x^*_n) \leq 1$, so $\|x^*_n\| \leq 2$ and therefore

$$\forall k \neq k', \frac{\|x^*_k - x^*_{k'}\|}{\limsup \|x^*_n\|} > \frac{\varepsilon}{2}.$$

Let $i_0 \geq 1$ such that $\frac{\varepsilon}{4} < 2^{-i_0} \leq \frac{\varepsilon}{2}$. By Lemma 3.6:

$$f_{2^{-i_0}}(x^*) \leq \left(1 - \Delta_1(\frac{2^{-i_0}}{8})\right) \liminf f_{2^{-i_0}}(x^*_n).$$

Moreover, for any $i \neq i_0$, $f_{2^{-i}}(x^*) \leq \liminf f_{2^{-i}}(x^*_n)$, because the functions $f_{2^{-i}}$ are ω^*-l.s.c.
So $f(x^*) \leq \sum_{i=1}^{\infty} 2^{-i} \liminf f_{2^{-i}}(x^*_n) - 2^{-i_0}\Delta_1(\frac{2^{-i_0}}{8}) \liminf f_{2^{-i_0}}(x^*_n)$
In order to show iii), we may assume $\|x^*\| > \frac{1}{2}$ and then $\liminf f_{2^{-i_0}}(x^*_n) \geq \frac{1}{4}$. So $f(x^*) \leq \liminf f(x^*_n) - \frac{\varepsilon}{16}\Delta_1(\frac{\varepsilon}{32}) \leq 1 - \frac{\varepsilon}{16}\Delta_1(\frac{\varepsilon}{32})$.

$\Delta_f(\varepsilon) \geq \frac{\varepsilon}{16}\Delta_1(\frac{\varepsilon}{32})$. So by Proposition 3.4, there exist $p \geq 1$ and $C > 0$ such that for any $0 < \varepsilon \leq 2$, $\Delta_f(\varepsilon) \geq C\varepsilon^p$. □

Remark. S. Prus studied in [Pr] the UKK renorming problem in the case of reflexive Banach spaces with a Schauder basis. He proved that such a space has an equivalent UKK norm if and only if there is a sequence of blocks of the original basis satisfying some ℓ_p estimates. Building on this idea, Odell and Knaust recently solved the renorming for spaces with a Szlenk index less than or equal to ω, in the case of reflexive spaces with a finite dimensional decomposition.

3.2. $L^p(X)$ spaces.

In this paragraph we consider the Lebesgue-Bochner space $L^p([0,1], X)$ (denoted $L^p(X)$), for $1 < p < \infty$. In [Pa], J.R. Partington proves that if $L^p(X)$ is reflexive with the UKK property, then X is uniformly convex. We give now an isomorphic version of this result, which follows from Lemma 2.6., as it has been already partly noticed in [D-G-K]. The result is the following:

Theorem 3.7. *Let X be a Banach space and let $1 < p < \infty$. The following assertions are equivalent:*

i) X is super-reflexive.

ii) $L^p(X)$ admits an equivalent UKK norm.

iii) $Sz_w(L^p(X)) \leq \omega$.

iv) $L^p(X)$ admits an equivalent norm whose dual norm is UKK*.

v) $Sz(L^p(X)) \leq \omega$.

Proof. i) implies ii): If X is super-reflexive, then X admits an equivalent uniformly convex norm which induces on $L^p(X)$ an equivalent uniformly convex norm which is therefore UKK.
ii) implies iii) is clear.
iii) implies i): Suppose $Sz_w(L^p(X)) \leq \omega$. Then, by Lemma 2.6 we have that $\delta(X) \leq \omega$. So, by Proposition 2.1, X is super-reflexive.
i) implies iv): If X is super-reflexive, then X admits an equivalent norm whose dual norm is uniformly convex. This norm induces on $(L^p(X))^* = L^q(X^*)$ (where $\frac{1}{p}+\frac{1}{q} = 1$) a dual uniformly convex norm which is therefore UKK*.
iv) implies v) clearly.
v) implies i): let us assume that $Sz(L^p(X) \leq \omega$, and let q be such that $\frac{1}{p} + \frac{1}{q} = 1$. We may consider $L^q(X^*)$ as a closed subspace of $(L^p(X))^*$. Thus $Sz_w(L^q(X^*)) \leq Sz_w((L^p(X))^*) \leq Sz(L^p(X))$. On the other hand, by Lemma 2.6, we have that $\delta(X^*) \leq Sz_w(L^q(X^*))$. So $\delta(X^*) \leq \omega$ and therefore X and X^* are super-reflexive. □

3.3. $\mathcal{C}(K)$ spaces.

The $\mathcal{C}(K)$ spaces, for K scattered compact space, have been in the last few years the source of many results and especially of many counterexamples in renorming theory (see for instance the papers of R. Deville [De], M. Talagrand [T], R. Haydon [H1,2], R. Haydon and C.A. Rogers [H-R]). We are able to give a positive answer to our renorming problem for this class of Banach spaces.

So, let K be a compact space. Let us recall that for a closed subset F of K the Cantor derived set $F^{(')}$ of F is the set of all non isolated points of F. $K^{(\alpha)}$, for α

ordinal, can then be defined inductively in the usual way.

Theorem 3.8. *Let K be a compact space. The following assertions are equivalent :*

i) $K^{(\omega)} = \emptyset$.

ii) $Sz(\mathcal{C}(K)) \leq \omega$.

iii) $\mathcal{C}(K)$ *admits an equivalent norm whose dual norm is UKK*.*

Proof. iii) $\Rightarrow$ ii) is clear and ii) $\Rightarrow$ i) relies on the fact that if $x \in K^{(\alpha)}$ then the Dirac measure $\delta_x \in (B_{(\mathcal{C}(K))^*})_1^{[\alpha]}$. So let us prove that i) $\Rightarrow$ iii). For that purpose we will adapt to our setting Deville's construction (in [De]) of a norm with a locally uniformly convex dual norm on $\mathcal{C}(K)$ spaces with $K^{(\omega_1)} = \emptyset$.

Let K be a compact space such that $K^{(\omega)} = \emptyset$. Then there exists an integer N for which $K^{(N)}$ is finite. For $\mu \in (\mathcal{C}(K))^*$ we denote $|||\mu||| = \sum_{x\in K} \alpha_x |\mu(x)|$, where α_x is defined by:

$$\text{if } x \in K^{(i)} \setminus K^{(i+1)}, \text{ then } \alpha_x = \frac{1}{2^i}.$$

$|||\cdot|||$ is an equivalent norm on $(\mathcal{C}(K))^*$. The fact that $|||\cdot|||$ is a dual norm needs a proof that can be found in [De]. Let us just point out that this is essentially due to the fact that α_x is a decreasing function of the integer i such that $x \in K^{(i)} \setminus K^{(i+1)}$.

We need to show that $|||\cdot|||$ has the UKK* property. So let $\varepsilon > 0$ and $\mu \in (\mathcal{C}(K))^*$ such that for every weak*-neighborhood V of μ, $|||\cdot||| - \mathrm{diam}(V \cap B_{|||\cdot|||}) > 2\varepsilon$ ($B_{|||\cdot|||}$ is the unit ball of $|||\cdot|||$).

We can find a finite subset F of K such that $K^{(N)} \subseteq F$ and $\mu = \lambda + \sum\limits_{x\in F} \mu(x)\delta_x$ with $\|\lambda\| < \gamma$ ($\|\cdot\|$ is the natural norm on $(\mathcal{C}(K))^*$ and γ is a positive number that we will precise later). Since $K^{(N)} \subseteq F$, we can find $(A_x)_{x\in F}$ a partition of K into clopen sets satisfying : for any x in F, $A_x \cap K^{(i_x)} = \{x\}$, where i_x is the integer i such that $x \in K^{(i)} \setminus K^{(i+1)}$. Thus there is ν such that $|||\nu||| \leq 1$, $|||\nu - \mu||| > \varepsilon$ and

$$\forall x \in F \quad \left| \sum_{y\in A_x} \mu(y) - \sum_{y\in A_x} \nu(y) \right| < \gamma' \ (\gamma' > 0 \text{ to be chosen later}).$$

Let $x \in F$, we have $\displaystyle\sum_{y\in A_x\setminus\{x\}} |\mu(y) - \nu(y)| \ > |\mu(x) - \nu(x)| - \gamma'$.

Since for any y in $A_x \setminus \{x\}$, $\alpha_y \geq 2\alpha_x$, we get

$$\sum_{y\in A_x\setminus\{x\}} \alpha_y(|\mu(y)| + |\nu(y)|) \ > 2\alpha_x|\mu(x) - \nu(x)| - \gamma'.$$

Hence,

$$\sum_{y\in A_x\setminus\{x\}} \alpha_y|\mu(y)| < \sum_{y\in A_x\setminus\{x\}} \alpha_y|\nu(y)| - 2\alpha_x|\mu(x)-\nu(x)| + \gamma' + 2\sum_{y\in A_x\setminus\{x\}} \alpha_y|\mu(y)|.$$

On the other hand $|\mu(x)| \leq |\nu(x)| + |\mu(x)-\nu(x)|$, therefore

$$\sum_{y\in A_x} \alpha_y|\mu(y)| < \sum_{y\in A_x} \alpha_y|\nu(y)| - \alpha_x|\mu(x)-\nu(x)| + \gamma' + 2\sum_{y\in A_x\setminus\{x\}} \alpha_y|\mu(y)|.$$

So $|||\mu||| \leq |||\nu||| - \sum\limits_{x\in F} \alpha_x|\mu(x)-\nu(x)| + |F|\gamma' + 2\gamma \quad (|F|$ is the cardinality of $F)$.

We have now two possibilities:

1) if $\sum\limits_{x\in F} \alpha_x|\mu(x)-\nu(x)| > \dfrac{\varepsilon}{3}$, then a right choice of γ and γ' will insure, by the above inequality, that $|||\mu||| < 1 - \dfrac{\varepsilon}{4}$.

2) if $\sum\limits_{x\in F} \alpha_x|\mu(x)-\nu(x)| \leq \dfrac{\varepsilon}{3}$, then $\sum\limits_{x\notin F} \alpha_x|\mu(x)-\nu(x)| > \dfrac{2\varepsilon}{3}$. So $\sum\limits_{x\notin F} \alpha_x|\nu(x)| > \dfrac{2\varepsilon}{3} - \gamma$, while $\sum\limits_{x\notin F} \alpha_x|\mu(x)| < \gamma$.

Therefore $|||\mu||| \leq |||\nu||| + \dfrac{\varepsilon}{3} - \dfrac{2\varepsilon}{3} + 2\gamma$, which implies again, if γ was chosen small enough, that $|||\mu||| < 1 - \dfrac{\varepsilon}{4}$. $\square$

Remark. It is a well known phenomenon in geometry of Banach spaces that the existence of nicely convex dual or bidual norms implies nice properties of the space, such as being an Asplund space or reflexivity (see for instance the book of R. Deville, G. Godefroy and V. Zizler [D-G-Z]).

The situation in similar for the property UKK*:

If X has an equivalent norm whose dual norm is UKK*, then X is an Asplund space. Indeed, in this case, $Sz(X) \leq \omega$, so X has also an equivalent Fréchet-differentiable norm, by the results in [L2], and therefore X is an Asplund space.

If X has an equivalent norm whose bidual norm is UKK*, then it is easy to prove that X is reflexive.

However, the James space J introduced by R.C. James in [J2] satisfies the following properties: J has an equivalent norm whose dual norm is UKK*, J^* has an equivalent norm whose dual norm is UKK* but J is not reflexive. A detailed proof of this counterexample can be found in [L3].

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