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A MEAN VALUE THEOREM FOR NON DIFFERENTIABLE MAPPINGS IN BANACH SPACES

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Abstract. We prove that if f is a real valued lower semicontinuous function on a Banach space X and if there exists a $\mathcal{C}^1$, real valued Lipschitz continuous function on X with bounded support and which is not identically equal to zero, then f is Lipschitz continuous of constant K provided all lower subgradients of f are bounded by K. As an application, we give a regularity result of viscosity supersolutions (or subsolutions) of Hamilton-Jacobi equations in infinite dimensions which satisfy a coercive condition. This last result slightly improves some earlier work by G. Barles and H. Ishii.

Let X be a Banach space and $f : X \to \mathbb{R}$ be an arbitrary function. It has been proved by D. Preiss [12] that if X is an Asplund space and if f is Lipschitz continuous, then the set D of all points of differentiability of f is dense in X and f satisfies the mean value theorem ; that is, for every $x, y \in X$:

$$\|f(x) - f(y)\| \leq L\|x - y\|$$

where

$$L = \sup\{\|f'(x)\|; x \in D\} \qquad (= \text{ Lipschitz constant of } f).$$

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When f is only assumed continuous, it is possible that f is nowhere differentiable. In this case it is natural to look for weaker forms of differentiability. For an arbitrary function $f : X \to \mathbb{R}$, we define the subdifferential of f at x by:

$$D^- f(x) = \{\varphi'(x); \quad \varphi : X \to \mathbb{R} \text{ is } C^1 \text{ and } f - \varphi \text{ has a local minimum at } x\}$$

and the superdifferential of f at x by:

$$D^+ f(x) = \{\varphi'(x); \quad \varphi : X \to \mathbb{R} \text{ is } C^1 \text{ and } f - \varphi \text{ has a local maximum at } x\}.$$

It follows from the Borwein-Preiss smooth variational principle [2] that whenever X has an equivalent Fréchet differentiable norm (away from the origin) and f is lower semi continuous, $D^- f(x) \neq \emptyset$ for all x in a dense subset D of X. It is observed in [4] that to get the above result the assumption on X can be replaced by the following weaker hypothesis:

(*) there exists a C^1 function $b : X \to \mathbb{R}$, Lipschitz continuous, non identically equal to zero with bounded support.

Let us mention here that R. Haydon ([8], [9]) constructed a Banach space satisfying (*) but which does not have any equivalent Fréchet-differentiable norm. The extension of [4] is based on the following smooth variational principle.

Theorem 1. *Let X be a Banach space satisfying (*) and $f : X \to \mathbb{R}$ be lower semicontinuous and bounded below. Then for every $\varepsilon > 0$, there exists a bounded Lipschitz continuous C^1 function $\varphi : X \to \mathbb{R}$ such that:*

a) $f - \varphi$ attains its minimum on X.

b) $\|\varphi\|_\infty = \sup\{|\varphi(x)|; x \in X\} < \varepsilon$ and $\|\varphi'\|_\infty = \sup\{\|\varphi'(x)\|; x \in X\} < \varepsilon$.

Variants of this result to other kind of differentiability are given in [4]. The purpose of this note is to prove the following theorem (we wish to thank the referee to pointing out to us that our result extends recent work of R. Redheffer, W. Walter and V. Weckesser [13], [14]):

Theorem 2. *Let X be a Banach space satisfying (*) and let $f : X \to \mathbb{R}$ be lower semicontinuous. Assume that there exists a constant $K > 0$ such that for all $x \in X$ and for all $p \in D^- f(x)$, $\|p\| \leq K$. Then f is Lipschitz continuous. More precisely, for all $x, y \in X$ $\quad |f(x) - f(y)| \leq K\|x - y\|$.*

Corollary 3. *Under the same assumption on X, if $f : X \to \mathbb{R}$ is continuous, then:*

$$\sup\{\, \|p\| \;; \quad p \in D^- f(x), \ x \in X\} = \sup\{\, \|p\|; \quad p \in D^+ f(x), \ x \in X\}.$$

These quantities are finite if and only if f is Lipschitz continuous on X, and in this case they are equal to the Lipschitz constant of f.

P r o o f o f c o r o l l a r y 3. Let us denote

$$s^- = \sup\{\, \|p\|; \quad p \in D^- f(x), \;\; x \in X\}$$

and

$$s^+ = \sup\{\, \|p\|; \quad p \in D^+ f(x), \;\; x \in X\}.$$

According to Theorem 2, if s^- or s^+ is finite, then f is Lipschitz continuous with Lipschitz constant less than or equal to $\inf\{s^-, s^+\}$. Conversely, we claim that if f is Lipschitz continuous with constant K, then $s^+ \leq K$ and $s^- \leq K$. Once the claim is proved, we get that either $s^+ = s^- = +\infty$ or $\sup\{s^-, s^+\} \leq K \leq \inf\{s^-, s^+\}$ and so $s^+ = s^- = K$ and this proves the corollary. Let us now prove that $s^+ \leq K$ (the proof of the inequality $s^- \leq K$ is similar). Let $\varepsilon > 0$, $x \in X$ and $p \in D^+ f(x)$ such that $\|p\| > s^+ - \varepsilon$. By definition, there exists $\varphi : X \to \mathbb{R}$ $\mathcal{C}^1$ such that $f - \varphi$ has a local maximum at x and $\varphi'(x) = p$. Let $h \in X$, $\|h\| = 1$ be such that $< p, h > \, > s^+ - \varepsilon$. For t small enough, we have:

$$f(x - th) - \varphi(x - th) \leq f(x) - \varphi(x).$$

So

$$f(x) - f(x - th) \geq \varphi(x) - \varphi(x - th)$$

therefore

$$\liminf_{\substack{t \to 0 \\ t > 0}} \frac{f(x) - f(x - th)}{t} \quad \geq \quad < \varphi'(x), h > \quad > \quad s^+ - \varepsilon.$$

This shows that $K > s^+ - \varepsilon$. Since this is true for all $\varepsilon > 0$, we obtain $s^+ \leq K$. $\square$

P r o o f o f t h e o r e m 2. Let us denote by $B(x, r)$ the closed ball of center $x \in X$ and radius r. Let us fix $x_0 \in X$ and $\varepsilon > 0$. Since f is lower semicontinuous, it is locally bounded below. Therefore there exist $\delta > 0$ and $M > 0$ such that:

$$f(x) \geq f(x_0) - M$$

whenever $\|x - x_0\| \leq 2\delta$. Next, according to a construction of Leduc [11], there exists a Lipschitz C^1 function $d : X \to \mathbb{R}$ which is C^1 on $X \backslash \{0\}$ and which satisfies:

i) $d(\lambda x) = \lambda\, d(x)$ for all $\lambda > 0$

ii) there exists $C > 0$ such that $\|x\| \leq d(x) \leq C\|x\|$ for all $x \in X$.

We will prove that if $x, y \in B(x_0, \delta/4)$, then

$$|f(x) - f(y)| \leq C(K + \varepsilon)\|x - y\|.$$

Indeed, fix $y \in B(x_0, \delta/4)$ and consider a $\mathcal{C}^1$ function $\alpha : [0, +\infty) \to [0, +\infty)$ satisfying:

i) $\alpha(t) = (K + \varepsilon)t$ if $t \leq \delta/2$;

ii) $\alpha'(t) \geq K + \varepsilon$ for all $t > 0$;
iii) $\alpha(\delta) \geq M$.

Finally consider the function F on X defined by:

$$F(x) = \begin{cases} f(x) - f(y) + \alpha(d(x-y)) & \text{if } \|x - y\| \leq \delta \\ +\infty & \text{otherwise.} \end{cases}$$

F is lower semicontinuous and bounded below. We claim that $F \geq 0$. Otherwise, let $m = \inf\ F < 0$. By Theorem 1, there exists a $\mathcal{C}^1$ function $\varphi : X \to \mathbb{R}$ such that:

1) $F - \varphi$ attains its minimum at some point $x_1 \in X$
2) $\|\varphi\|_\infty < -m/2$ and $\|\varphi'\|_\infty < \varepsilon$.

If $x = y$ or $\|x - y\| \geq \delta$, then $F(x) \geq 0$ and so $(F - \varphi)(x) \geq F(x) - \|\varphi\|_\infty > m/2$. On the other hand, $\inf(F - \varphi) \leq \inf F + \|\varphi\|_\infty < m/2$. This proves that $0 < \|x_1 - y\| < \delta$. Thus, if we let:

$$\psi(x) = f(y) - \alpha(d(x-y)) + \varphi(x)$$

$f - \psi$ has a local minimum at x_1 and ψ is $\mathcal{C}^1$ in a neighbourhood of x_1, therefore $p = \psi'(x_1) \in D^- f(x_1)$ and

$$\|p\| \geq \alpha'(d(x_1 - y))\|d'(x_1 - y)\| - \|\varphi'(x_1)\| \geq \alpha'(d(x_1 - y)) - \|\varphi'(x_1)\| > K$$

this contradicts the assumption, and the claim is proved. In particular, we have proved that if $\|x - y\| \leq \delta/2C$, then

$$f(x) \geq f(y) - (K + \varepsilon)d(x - y) \geq f(y) - C(K + \varepsilon)\|x - y\|. \tag{1}$$

Since $\|x - x_0\| \leq \delta/4C$ implies that $\|x - y\| \leq \delta/2C$, inequality (1) is true for all x, y in $B(x_0, \delta/4C)$. Thus

$$|f(x) - f(y)| \leq C(K + \varepsilon)\|x - y\| \tag{2}$$

whenever $x, y \in B(x_0, \delta/4C)$.

Now consider two arbitrary points $x, y \in X$. For each $t \in [0,1]$, let $z_t = x + t(y - x)$. By (2), there exists an open interval I_t containing t such that whenever $s, s' \in I_t$,

$$|f(z_s) - f(z_{s'})| \leq C(K + \varepsilon)\|z_s - z_{s'}\| = C(K + \varepsilon)|s - s'|\ \|y - x\|.$$

By compactness of $[0,1]$, there is a finite number of intervals $I_{t_1}, I_{t_2}, \ldots, I_{t_n}$ such that $\bigcup\limits_k I_{t_k} = [0,1]$. So there is an increasing sequence $s_1 = 0 < s_2 < \cdots < s_p = 1$ such that each interval $[s_i, s_{i+1}]$ is contained in one of the I_{t_k}. By the triangle inequality, we have:

$$\begin{aligned} |f(x) - f(y)| &\leq |f(z_{s_1}) - f(z_{s_2})| + |f(z_{s_2}) - f(z_{s_3})| + \cdots + |f(z_{s_{p-1}}) - f(z_{s_p})| \\ &\leq C(K + \varepsilon)\|y - x\|. \end{aligned}$$

This proves that f is Lipschitz continuous and that the Lipschitz constant of f is less or equal than $C(K+\varepsilon)$. When the norm of X is Fréchet-differentiable on $X\backslash\{0\}$, we can assume that $d(x)=\|x\|$, so $C=1$ and we obtain the result sending ε to zero. In the general case, according to Preiss' theorem,

$$\|f(x)-f(y)\|\leq L\|x-y\|$$

where

$$L=\sup\{\|f'(x)\|; f \ \text{ is differentiable at } \ x\}.$$

We conclude by noting that whenever f is differentiable at some point x and if X admits a $\mathcal{C}^1$ smooth bump function, then $D^-f(x)=\{f'(x)\}$ (see the proof of theorem 1.3 of [6]), so $L\leq K$ and f is Lipschitz continuous of constant less than or equal to K.

Remarks.

a) It is natural to try to extend Theorem 2 in the following way: Let X be a Banach space satisfying (*) and let $f: X\to\mathbb{R}$ be lower semi continuous. Assume that there exists a constant $K>0$ such that for all $x\in X$ such that $D^-f(x)\neq\emptyset$, there exists $p\in D^-f(x)$ of norm less than or equal to K. Does it follow that f is Lipschitz continuous? The answer is no in general: the function of Lebesgue on the unit interval is a counterexample. Indeed this function is continuous but not Lipschitz continuous, and one can check that $D^-f(x)$ is equal to $\{0\}$ if x is not on the Cantor set C, is equal to $[0,+\infty)$ if x is on C but is not an accumulation point of C from the left, and is empty otherwise. We wish to thank J. P. Penot for pointing out to us this remark.

b) The assumption "f lower semicontinuous" in Theorem 2 is necessary. Indeed take:

$$f(x)=\begin{cases} 0 & \text{if } \ x \quad \text{is not rational} \\ \dfrac{1}{q} & \text{if } \ x=\dfrac{p}{q} \quad \text{with} \quad (p,q)=1 \quad \text{and} \quad q>0 \end{cases}$$

f is continuous on a residual set (f is upper semicontinuous but not lower semicontinuous on $\mathbb{R}$), f is not constant but $\sup\{|p|\,;\, p\in D^-f(x),\ \ x\in\mathbb{R}\}=0$.

c) Theorem 2 remains valid if the assumptions on f are replaced by: f is upper semicontinuous and there exists a constant $K>0$ such that for all $x\in X$ and for all $p\in D^+f(x)$, $\|p\|\leq K$ (to see this, apply Theorem 2 to $-f$).

d) A geometric assumption on X is also necessary. For instance, if X admits an equivalent nowhere Fréchet-differentiable differentiable norm $\|\!|\ \|\!|$ (this is possible if and only if X is not an Asplund space) then the function f defined by $f(x)=-\|\!|x\|\!|$ is Lipschitz continuous, $D^-f(x)=\phi$ for all $x\in X$ and so $\sup\{\|p\|\,;\, p\in D^-f(x),\ x\in\mathbb{R}\}\leq 0$ but f is not constant. It would be interesting to know if Theorem 2 holds in every space which admits a $\mathcal{C}^1$ smooth bump function.

e) If the function f of Theorem 2 is only defined on an open subset Ω of X, easy modifications in the proof show that the conclusion can be replaced by:

$$|f(x) - f(y)| \leq K \; \|x - y\|$$

whenever the line segment $[x, y]$ is contained in Ω.

f) It is possible to prove a mean value theorem for other kinds of differentiability. If β is a bornology on X and φ is a real valued function on X, we say that φ is β-differentiable at $x_0 \in X$ with β-derivative $\varphi'(x_0) = p \in X^*$ if

$$\lim_{t \to 0} \; t^{-1}\Big(f(x_0 + th) - f(x_0) - < p, th >\Big) = 0$$

uniformly for h in the elements of β. We denote by τ_β the topology on X^* of uniform convergence on the elements of β. When β is the class of all bounded subsets (resp. all singletons) of X, the β-differentiability coincides with the usual Fréchet-differentiability (resp. Gâteaux differentiability), and τ_β coincides with the norm (resp. weak*) topology on X^*. Finally, if f is a real valued function on X, the β-subdifferential of f at some point $x_0 \in X$ is the set:

$d_\beta^- f(x_0) = \{\varphi'(x_0); \varphi : X \to \mathbb{R}$ is β-differentiable and $f - \varphi$ has a local minimum at $x_0\}$.

The following result is a straightforward adaptation of Theorem 2.

Proposition 4. *Let X be a Banach space such that there exists a β-differentiable function $b : X \to \mathbb{R}$, Lipschitz continuous, non identically equal to zero with bounded support. Let $f : X \to \mathbb{R}$ be lower semicontinuous. If we denote by $K = \sup\{\|p\|;\; p \in d_\beta^- f(x),\; x \in X\}$, we have for all $x, y \in X$:*

$$|f(x) - f(y)| \leq K\|x - y\|.$$

We conclude this paper with an application to a regularity result of viscosity subsolutions of Hamilton-Jacobi equations which have a "coercive" structure. We are concerned here with a nonlinear first order scalar partial differential equation of the form

$$F(x,\, u,\, Du) = 0 \quad \text{in} \;\; \Omega \tag{1}$$

where Ω is an open subset of a Banach space X, $F : \Omega \times \mathbb{R} \times X^* \to \mathbb{R}$, $u : \Omega \to \mathbb{R}$ is the unknown and Du denotes the gradient of u (i.e. the Fréchet derivative of u). Even when $X = \mathbb{R}$ classical solutions do not exist necessarily. M. G. Crandall and P. L. Lions [3] introduced a notion of solution for which one has existence and uniqueness under some suitable structural conditions on F.

Definition. A function $u : X \to \mathbb{R}$ is called a viscosity supersolution of (1) on Ω if:

i) u is lower semicontinuous;

ii) for every $x \in \Omega$ and for every $p \in D^{-}u(x)$, $F(x, u, p) \leq 0$;

u is called a viscosity subsolution if

i′) u is upper semicontinuous;

ii′) for every $x \in \Omega$ and for every $p \in D^{+}u(x)$, $F(x, u, p) \geq 0$.

Finally u is a viscosity solution of (1) on Ω if u is both a viscosity subsolution and a viscosity supersolution of (1) on Ω.

According to Theorem 2 and Remark b) following its proof, we have:

Proposition 5. *Let u be a viscosity subsolution of (1) on Ω. We assume that X satisfies $(*)$ and that there is a constant K such that $F(x, u, p) > 0$ for all $(x, u) \in \Omega \times \mathbb{R}$ and for all $p \in X^*$ such that $\|p\| > K$. Then for all $x, y \in \Omega$ such that the line segment $[x, y]$ is contained in Ω, $|u(x) - u(y)| \leq K\|x - y\|$.*

Results in the direction have been observed by G. Barles [1] and H. Ishii [10]. Let us observe here that if u_0 is a viscosity supersolution of (1) on Ω and if v_0 is a viscosity subsolution of (1) on Ω, then in order to prove the existence of a viscosity solution lying between u_0 and v_0, one can use Perron method as in [10]: Consider

$$\mathcal{S} = \{\, u : \Omega \to \mathbb{R};\ \ v_0 \leq u \leq u_0 \ \text{ and } \ u \ \text{ is subsolution of (1) on } \ \Omega\}$$

and set $w(x) = \sup\{\, u(x); u \in \mathcal{S}\}$ so that w is Lipschitz continuous of constant K as supremum of Lipschitz continuous functions with constant K, and it can be shown that w is a viscosity solution of (1) on Ω. The point we want to emphasize here is that upper semicontinuous and lower semicontinuous envelopes of the function w are used in [10], but are not necessary if F is coercive as above since then we stay within the setting of Lipschitz continuous functions.

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