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SOME COMMENTS ON θ-IRRESOLUTE AND QUASI-IRRESOLUTE FUNCTIONS

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Abstract. The aim of this paper is to continue the study of θ-irresolute and quasi-irresolute functions as well as to give an example of a function which is θ-irresolute but neither quasi-irresolute nor an R-map and thus give an answer to a question posed by Ganster, Noiri and Reilly. We prove that RS-compactness is preserved under open, quasi-irresolute surjections.

1. Introduction. In 1963, Levine [20] introduced the concept of a semi-open set in a topological space as a set laying between an open set and its closure. Since then this concept has been used to study various forms of generalized continuous functions between topological spaces. In particular, the class of θ-irresolute functions [19] and the class of quasi-irresolute functions [9] are defined in terms of semi-open sets. The aim of this paper is to continue the study of these two classes of functions.

Let A be a subset of a topological space X. We denote the interior of A, the closure of A and the boundary of A with respect to X by $\text{int}A$, $\overline{A}$ and $\text{bd}A$ respectively. Throughout the paper a space X will always mean a topological space (X, τ).

A subset S of X is called *semi-open* (resp. *regular closed*) if $S \subseteq \overline{\text{int}S}$ (resp. $S = \overline{\text{int}S}$). The complement of a semi-open set is called a *semi-closed* set and the complement of a regular closed set is called regular open. The semi-closure of S is the

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smallest semi-closed set containing S and is denoted by scl$S$. It is well-known that scl$S = S \cup \text{int}\overline{S}$.

A subset S of X is called a *regular semi-open* [4] if for some regular open set V we have $V \subseteq A \subseteq \overline{V}$. Regular semi-open sets are sometimes called *semi-regular* [8]. A set $A \subseteq X$ is called *θ-semi-closed* [18] if S is the intersection of regular open sets. The complement of a θ-semi-closed set is called *θ-semi-open* i.e., S is θ-semi-open if S is the union of regular closed sets.

We will denote the family of all semi-open (resp. regular closed, semi-regular, θ-semi-open, clopen) subsets of X by SO(X) (resp. RC(X), SR(X), θ-SO(X), CO(X)). The set of all real numbers is denoted by $\mathbb{R}$.

Definition 1. A function $f\colon X \to Y$ between spaces (X, τ) and (Y, σ) is called:
(i) an R-map [5] if for each $V \in \text{RC}(Y)$ we have $f^{-1}(V) \in \text{RC}(X)$,
(ii) θ-irresolute [14, 19] if $f^{-1}(V) \in \theta\text{-SO}(X)$ for each $V \in \text{RC}(Y)$,
(iii) quasi-irresolute [9] if $f^{-1}(V) \in \text{SR}(X)$ for each $V \in \text{SR}(Y)$,
(iv) irresolute [6] if $f^{-1}(V) \in \text{SO}(X)$ for each $V \in \text{SO}(Y)$.

Note that R-maps are called *rc-continuous* and *regular irresolute* in [17] and in [22] respectively. A detailed study of the concepts of θ- and quasi-irresoluteness can be found in [9, 14, 19].

Definition 2. A space X is called:

(1) *extremally disconnected* (= e.d.) if the closure of every open subset of X is open,

(2) *strongly s-regular* [13] if for any closed subset $A \subseteq X$ and any point $x \in X \setminus A$ there is an $F \in \text{RC}(X)$ with $x \in F$ and $F \cap A = \emptyset$.

(3) *semi-space* [11] if every semi-open subset of X is open.

The following remark contains some observations which will be used throughout the sequel:

Remark 1.1.
(a) $\theta\text{-SO}(X) \subseteq \text{SO}(X)$,
(b) $A \in \text{SR}(X)$ iff A is both semi-open and semi-closed,
(c) If A, $X \setminus A \in \theta\text{-SO}(X)$, then $A \in \text{SR}(X)$,
(d) If X is e.d., then $\text{SR}(X) = \text{RC}(X) = \text{RO}(X) = \text{CO}(X)$,
(e) If $f\colon X \to Y$ is an R-map, then f is θ-irresolute but not vice versa,

(f) Let X be a space in which we can find a semi-regular set A which is not θ-semi-open. Let $f\colon X \to \mathbb{R}$ be the characteristic function with respect to A:

$$f(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Then f is quasi-irresolute but not θ-irresolute.

Concerning semi-regular sets as a concept on the base of which quasi-irresoluteness is defined note that semi-regularity of sets can be characterized via different topological notions. Moreover some classes of topological spaces can be characterized via semi-regular sets. The proofs of next two theorems are not difficult and hence omitted. Recall that a set $A \subseteq X$ is called *semi-preopen* [1] if $A \subseteq \overline{\mathrm{int}\overline{A}}$, *NDB-set* [12] if has nowhere dense boundary and *semi-θ-closed* [9] if $A = \mathrm{scl}_\theta A$. Note [7] that $\mathrm{scl}_\theta A = \{x \in X\colon \mathrm{scl}U \cap A \neq \emptyset$ and $U \in \mathrm{SO}(X)\}$. A set $A \subseteq X$ is called *semi-θ-open* (resp. *semi-preclosed*) if its complement is semi-θ-closed (resp. semi-preopen). Complements of NDB-sets are NDB-sets.

Theorem 1.2. *For a subset $A \subseteq X$ the following conditions are equivalent:*
(1) *A is semi-regular.*
(2) *A is semi-preopen and semi-closed.*
(3) *A is semi-preopen, semi-preclosed and an NDB-set.*
(4) *A is semi-θ-open and semi-θ-closed.* $\square$

Theorem 1.3. *For a space X the following conditions are valid:*
(i) *X is e.d. iff* $\mathrm{SR}(X) \subset \mathrm{CO}(X)$.
(ii) *X is hyperconnected iff* $\mathrm{SR}(X) = \{\emptyset, X\}$. $\square$

2. θ-irresolute and quasi-irresolute functions.

Lemma 2.1. *Let X be a first countable regular space. Let $U \subseteq X$ be open and let $x \in \overline{U}$. Then for some open set $G \subseteq X$ we have $x \in \overline{G} \subseteq U \cup \{x\}$.*

Proof. If $x \in U$ we are done by regularity of X.

So let $x \in \overline{U} \setminus U$ and let (V_n) be a local base at x with $\overline{V_{n+1}} \subseteq V_n$ for each index $n \in \mathbb{N}$.

We can find for each n an open set G_n such that $G_n \neq \emptyset$ and $\overline{G_n} \subseteq V_n \cap U$. Set $G = \cup_{n\in\mathbb{N}} G_n$.

(i) Clearly $x \in \overline{G}$. Otherwise for some $m \in \mathbb{N}$ we have $V_m \cap G = \emptyset$ and hence $G_m = G_m \cap G = \emptyset$, which is a contradiction.

(ii) Let $y \in \overline{G}$, where $y \neq x$. Then for some $m \in \mathbb{N}$ we have $y \notin \overline{V_m}$. Since $G_n \subseteq V_n$ for every n we have

$$y \notin \overline{\cup_{n\geq m} G_n}.$$

So $y \in \overline{G_1} \cup \ldots \cup \overline{G_{m-1}} \cup \overline{\cup_{n\geq m} G_n}$ and hence for some i we have $y \in \overline{G_i} \subseteq U$.

So we have shown that $x \in \overline{G} \subseteq U \cup \{x\}$. $\square$

As a consequence of the lemma above we have the following result. The proof is easy and hence omitted.

Corollary 2.2.

(i) *If X is a first countable regular space, then* $\mathrm{SO}(X) \subseteq \theta\text{-}\mathrm{SO}(X)$ *and* $\mathrm{SR}(X) \subseteq \theta\text{-}\mathrm{SO}(X)$,

(ii) *If $f\colon X \to Y$ is quasi irresolute and X is first countable and regular, then f is θ-irresolute,*

(iii) *If* $f\colon X \to Y$ *is quasi-irresolute and* X *is a metric space, then* f *is* θ*-irresolute.* $\square$

The next theorem gives additional necessary conditions under which quasi-irresoluteness implies θ-irresoluteness. Recall that a function $f\colon X \to Y$ is called *almost continuous* [16] if for each $x \in X$ and each neighborhood V of $f(x)$, $\overline{f^{-1}(V)}$ is a neighborhood of x. For real valued function on Euclidean spaces the notion of almost continuity was studied by Blumberg in 1922 [2]. He used the term *densely approached.* Almost continuity is often called precontinuity or near continuity. We will need the following lemma:

Lemma 2.3 [23, Theorem 6]. *For any function* $f\colon X \to Y$ *the following are equivalent:*

(1) f *is almost continuous.*

(2) $f(\overline{U}) \subseteq \overline{f(U)}$ *for each open subset* U *of* X. $\square$

Theorem 2.4.

(i) *If* $f\colon X \to Y$ *is quasi-irresolute and* X *is e.d., then* f *is an R-map and hence* θ*-irresolute.*

(ii) *If* $f\colon X \to Y$ *is quasi-irresolute and almost continuous, then* f *is an R-map and hence* θ*-irresolute.*

Proof. (1) Follows from Remark 1.1. (d).

(2) Let $F \in \mathrm{RC}(Y)$. Then $f^{-1}(F) \in \mathrm{SR}(X)$, i.e. for some $U \in \mathrm{RO}(X)$ we have $U \subseteq f^{-1}(F) \subseteq \overline{U}$. If $x \in \overline{U}$, then due to Lemma 2.3. $f(x) \in f(\overline{U}) \subseteq \overline{f(U)} \subseteq F$. Thus $x \in f^{-1}(F)$, i.e. $f^{-1}(F) = \overline{U} \in \mathrm{RC}(X)$. $\square$

The next result gives a condition under which θ-irresoluteness implies quasi-irresoluteness.

Theorem 2.5. *If* $f\colon X \to Y$ *is* θ*-irresolute (or in particular an R-map) and if* Y *is e.d., then* f *is quasi-irresolute.*

Proof. Let $T \in \mathrm{SR}(Y)$. By Remark 1.1. (d) T and $Y \setminus T \in \mathrm{RC}(Y)$. By Remark 1.1. (a) $f^{-1}(T)$ and $f^{-1}(Y \setminus T) = X \setminus f^{-1}(T)$ are semi-open in X, i.e. $f^{-1}(T) \in \mathrm{SR}(X)$. $\square$

In the same manner one can prove the next result. Recall that a function $f\colon X \to Y$ is called *semi-continuous* [20] if $f^{-1}(V) \in \mathrm{SO}(X)$ for each open subset V of Y.

Theorem 2.6. *If* $f\colon X \to Y$ *is semi-continuous and* Y *is e.d., then* f *is quasi-irresolute.* $\square$

Recall that a space X is called *hyperconnected* [24] if the intersection of any two non-void open sets is non-void. In [3] hyperconnected spaces are called *irreducible.* Note that a space is hyperconnected iff it is a connected e.d. space.

Theorem 2.7. *If Y is hyperconnected, then every function $f: X \to Y$ is quasi-irresolute and θ-irresolute.*

Proof. It is easily observed (and well-known) that a space is hyperconnected iff the only regular closed (or equivalently the only semi-regular - Theorem 1.3. (ii)) subsets of the space are the trivial ones. □

Theorem 2.8. *If a function $f: X \to Y$ is open and continuous, then f is an R-map (hence θ-irresolute) and quasi-irresolute.*

Proof. (i) We show first that f is an R-map. Let $F \in RC(Y)$, i.e. $F = \overline{V}$, where $V \subseteq Y$ is open. One easily checks that $f^{-1}(F) = \overline{f^{-1}(V)}$.

(ii) We show next that f is quasi-irresolute. Let $T \in \mathrm{SR}(Y)$, i.e. for some $V \in \mathrm{RO}(Y)$ we have $V \subseteq T \subseteq \overline{V}$. By (i) $f^{-1}(V) \in RO(X)$. Then $f^{-1}(V) \subseteq f^{-1}(T) \subseteq f^{-1}(\overline{V})$. Since $f^{-1}(\overline{V}) = \overline{f^{-1}(V)}$, then $f^{-1}(T) \in \mathrm{SR}(X)$. □

Recall [14] that a function $f: X \to Y$ is called *weakly θ-irresolute* (resp. *strongly θ-irresolute*) if for each $x \in X$ and each $V \in \mathrm{SO}(Y, f(x))$ there exists $U \in \mathrm{SO}(X, x)$ such that $f(U) \subseteq \overline{V}$ (resp. $f(\overline{U}) \subseteq V$).

Lemma 2.9 [13, Theorem 1]. *For a space X the following are equivalent:*

(1) *X is strongly s-regular.*

(2) *Every open set is θ-semi-open.* □

Theorem 2.10. *Let X be strongly s-regular semi-space. For a function $f: X \to Y$ the following conditions are equivalent:*

(1) *f is θ-irresolute.*

(2) *f is weakly θ-irresolute.*

Proof. (1) $\Rightarrow$ (2) is always true.

(2) $\Rightarrow$ (1) Let $B \in \mathrm{RC}(Y)$. Since f is weakly θ-irresolute, then according to Theorem 1.2 in [14] $f^{-1}(B) \in \mathrm{SO}(X)$. Since X is a semi-space, then $f^{-1}(B)$ is open in X and thus by Lemma 2.9 θ-semi-open, since X is strongly s-regular. □

In the same manner one can prove:

Theorem 2.11. *Let X be strongly s-regular semi-space. For a function $f: X \to Y$ the following conditions are equivalent:*

(1) *f is strongly θ-irresolute.*

(2) *f is irresolute.* □

A space X is a *semi-irreducible space* [25] (= FCC-space) if every disjoint family of non-void open subsets of X is finite or equivalently if X has only a finite amount of regular open sets.

Theorem 2.12. *Let X be a semi-irreducible space. For a function $f: X \to Y$ the following conditions are equivalent:*

(1) *f is an R-map.*

(2) *f is θ-irresolute.*

Proof. (1) $\Rightarrow$ (2) is valid for every function.

(2) $\Rightarrow$ (1) Let $B \in \mathrm{RC}(Y)$. Since f is θ-irresolute, then $f^{-1}(B)$ is θ-semi-open or equivalently $f^{-1}(B)$ is the union of regular closed sets. Since X is semi-irreducible, then X has only a finite amount of regular closed sets and thus $f^{-1}(B)$ is regular closed being the finite union of regular closed sets. This shows that f is an R-map. □

In 1980, Hong [15] introduced the class of RS-compact spaces. He defined a space X to be *RS-compact* if for every cover $(V_i)_{i\in I}$ of X by semi-regular sets, there exists a finite subset $J \subseteq I$ such that $X = \cup_{i\in J}\mathrm{int}V_i$.

Theorem 2.13. *Open, quasi-irresolute images of RS-compact spaces are RS-compact.*

Proof. Assume that $f\colon(X,\tau)\to(Y,\sigma)$ is open, quasi-irresolute and onto as well as that X is RS-compact. Let $(V_i)_{i\in I}$ be a semi-regular cover of Y. Then $(f^{-1}(V_i))_{i\in I}$ is a cover of X such that for every $i \in I$, $f^{-1}(V_i)$ is semi-regular in X due to assumption. Thus, since X is RS-compact for some finite $J \subseteq I$ we have $X = \cup_{i\in J}\mathrm{int}f^{-1}(V_i)$. Then clearly, since f is onto and open, $Y = \cup_{i\in J}f(\mathrm{int}f^{-1}(V_i)) \subseteq \cup_{i\in J}\mathrm{int}f(f^{-1}(V_i)) = \cup_{i\in J}\mathrm{int}V_i$, i.e. Y is RS-compact. □

3. A θ-irresolute function which is not quasi-irresolute.

The following result (due to Ganster) gives an alternative answer to a question posed by Ganster, Noiri and Reilly in [14]. Another example (with finite spaces) can be found in [10]. However the function in [10] is an R-map, while the one below gives an example of a function which is θ-irresolute but neither an R-map nor quasi-irresolute.

Example 3.1. Let $X = \{(x,y)\in\mathbb{R}^2\colon x^2+y^2\leq 1\}$ with the usual topology and let $V = \{(x,y)\in\mathbb{R}^2\colon x^2+y^2<1\}$. Observe that for each $E\subseteq \mathrm{bd}_X V$, $V\cup E$ is a union of regular closed subsets of X.

Let Y be a countably infinite set and let $\{Y_1,Y_2,Y_3\}$ be a partition of Y into pairwise disjoint infinite sets. We now define a topology on Y. A basic neighborhood of $y\in Y_1$ is of the form $\{y\}\cup C_2\cup C_3$ where, for $i=2,3$, C_i is a cofinite subset of Y_i. For $i=2,3$, a basic neighborhood of $y\in Y_i$ is a cofinite subset of Y_i containing y. Then Y_2 and Y_3 are open subspaces of Y and we have $\mathrm{cl}_Y Y_2 = Y_2\cup Y_1$ and $\mathrm{cl}_Y Y_3 = Y_3\cup Y_1$. Moreover, one easily checks that $Y_2\in\mathrm{SR}(Y)$ and $Y_3\in\mathrm{SR}(Y)$ and that $\mathrm{RC}(Y) = \{\emptyset, \mathrm{cl}_Y Y_2, \mathrm{cl}_Y Y_3, Y\}$.

Now let $\mathrm{bd}_X V = E_2\cup E_3$ such that E_2 and E_3 are infinite and disjoint. Define $f\colon X\to Y$ in such a way that f maps E_2 onto Y_2, E_3 onto Y_3 and V onto Y_1. By Remark 3.2 in [14], f is clearly θ-irresolute. On the other hand, $Y_2\in\mathrm{SR}(Y)$ and $E_2 = f^{-1}(Y_2)$ has empty interior, hence $f^{-1}(Y_2)\notin\mathrm{SR}(X)$. This proves that f is not quasi-irresolute.

Remark 3.2. It is pointed out in [9] that every quasi-irresolute function is semi-weakly continuous [21] (but not conversely). It is easily verified that the function f from the example above is not even semi-weakly continuous.

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