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OPERATORS FACTORING THROUGH BANACH LATTICES AND IDEAL NORMS

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Abstract. A new, unified presentation of the ideal norms of factorization of operators through Banach lattices and related ideal norms is given.

1. Introduction. This paper aims at presenting in a unified way two ideal norms of operators between Banach spaces E and F.

One of these norms is the norm $\mu_{p,q}$ of factorization of a bounded linear operator U in the form

(1.1)
$$\begin{tikzcd} E \arrow[r, "U"] \arrow[dr, "A"'] & F \arrow[r, "j_F"] & F'' \\ & L \arrow[ur, "B"'] & \end{tikzcd}$$

where L is a Banach lattice, A is a p-convex operator, B is a q-concave operator and j_F is the canonical embedding of F into its bidual F''. The norm $\mu_{p,q}(U)$ is defined as $\inf K^{(p)}(A)K_{(q)}(B)$ over these factorizations of U.

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The other norm is the 'p, q-G.L.-constant' of U (G.L. stands for Gordon-Lewis), which is the infimum $\gamma_{p,q}(U)$ of the constants C such that for every q'-absolutely summing operator V $(1/q+1/q' = 1)$ defined on F, the adjoint operator $(VU)'$ is p'-integral and satisfies

$$i_{p'}((VU)') \leq C\pi_{q'}(V). \tag{1.2}$$

In the extreme case, $p = 1$, $q = \infty$, $\mu_{1,\infty}(U)$ is the so called *G.L.-l.u.st constant* of U (l.u.st stands for 'local unconditional structure' [6]), and $\gamma_{1,\infty}(U)$ is the *G.L. constant* of U. A Banach space E is said to have G.L.-l.u.st (or to have the G.L.-property) if $\mu_{1,\infty}(I_E) < \infty$ (or $\gamma_{1,\infty}(I_E) < \infty$).

The ideal $[M_{p,q}, \mu_{p,q}]$ of operators which have a factorization (1.1) was treated extensively in [16]. It was shown there that

$$\gamma_{p,q}(U) \leq \mu_{p,q}(U) \tag{1.3}$$

that is

$$[\Gamma_{p,q}, \gamma_{p,q}] \subset [M_{p,q}, \mu_{p,q}] \tag{1.4}$$

in the terminology of the theory of operator ideals.

In the case $1 \leq p < q \leq \infty$ inverse inclusion in (1.4) does not hold. In particular, there exist examples (e.g. [2], [3], [4]) of spaces which have the G.L.-property but fail l.u.st.

The case $p \geq q$ is completely different. In fact, $M_{p,p}$ is the ideal Γ_p of operators factoring through L_p. Kwapien [5] proved that this ideal is characterized by (1.2) holding with $\inf C = \gamma_p(U)$. For $p > q$, both $[M_{p,q}, \mu_{p,q}]$ and $[\Gamma_{p,q}, \gamma_{p,q}]$ are identical with the ideal $[I_{p,q}, i_{p,q}]$ introduced in [8], of operators U which factor in the form

(1.5)
$$\begin{tikzcd} E \arrow[r, "U"] \arrow[d, "A"'] & F \arrow[r, "j_F"] & F'' \\ L_p(\mu) \arrow[rr, "D_g"] & & L_q(\mu) \arrow[u, "B"'] \end{tikzcd}$$

where D_g is an operator of multiplication by a function $g \in L_s$, $1/s = 1/q - 1/p$. The fact that $[I_{p,q}, i_{p,q}] = [M_{p,q}, \mu_{p,q}]$ is proved in [16]. The equality $[I_{p,q}, i_{p,q}] = [\Gamma_{p,q}, \gamma_{p,q}]$, that is, the fact that operators of $I_{p,q}$ are characterized by 'converting' q'-summing operators into operators with p'-integral adjoints, was 'almost' proved in [8], the detail which was missing to prove it completely, namely the fact that $[I_{p,q}, i_{p,q}]$ is a perfect ideal, was completed by Lapreste [9]. This fact follows also from the identity $[I_{p,q}, i_{p,q}] = [M_{p,q}, \mu_{p,q}]$ and the fact, proved in [16] that $M_{p,q}$ is perfect.

We bring in this paper a new presentation of the norm $\gamma_{p,q}$ in the case of finite dimensional Banach spaces. This presentation, when compared with the formula giving the tensor norm $\eta_{p,q}$ of [16], which on finite dimensional Banach spaces is identical with $\mu_{p,q}$, clarifies in a way the connection between the norms and the reason for different behaviour between the cases $p < q$ and $p \geq q$. In fact the roots of the difference lie in the possibility of exchanging 'inf sup' for 'sup inf' in a certain function of two variables, basically by using Ky-Fan's minimax theorem.

This presentation is introduced in Section 2 (Theorem 2.1) and it is applied to derive a result on the G.L.-constant of unconditional sums of Banach spaces (Theorem 2.6.)

Terminology and notations. The notations and definitions which come from the general theory of Banach spaces are mainly those of the books [10] and [11] which may serve as a standard reference. Specifically, for Banach spaces E and F we denote by $L(E,F)$ the space of bounded linear operators from E into F, $B(E)$ denotes the closed unit ball of E and $S(E)$ its unit sphere. The dual space of E is denoted by E'. For $x \in E$ and $x' \in E'$ the notations $x'(x)$, $\langle x, x'\rangle$, or $\langle x', x\rangle$ mean the same thing. The adjoint operator of $T \in L(E,F)$ is T'.

In contrast to the relative uniformity of notations used in the general theory of Banach spaces, there is less uniformity of notations in the theory of operator ideals. An extensive reference to the theory is Pietsch's book [12] we shall mainly use notations which are used in [8] and also in [15] and [16]. This paper is in a strong way a continuation of these last two papers.

From [15] we recall in particular the following construction: Let $[A,a]$ and $[B,b]$ be normed operator ideals. The ideal $\dfrac{A}{B}$ is defined by

$$\frac{A}{B}(E,F) = \{T \in L(E,F)\,;\, \forall G, \forall U \in B(F,G)\ UT \in A(F,G)\}$$

$$\frac{a}{b}(T) = \inf_G \inf \left\{ \frac{a(UT)}{b(U)}\,;\, 0 \neq U \in B(F,G) \right\}.$$

The following facts, A) and B) were proved in [15] (Propositions (2.7) and (2.8)):

A) If $[A,a]$ is a perfect ideal (which means that $[A,a] = [A^{**}, a^{**}]$) and b is a semi-tensorial norm (a property satisfied, for example, by the p-absolutely summing norms π_p), then $\dfrac{A}{B}$ is a perfect normed ideal.

B) Under appropriate conditions on the ideals A and B, the adjoint ideal norm $\left(\dfrac{a}{b}\right)^*$ on operators $T \in L(E,F)$ between finite dimensional Banach spaces is given by the formula

$$\left(\frac{a}{b}\right)^* (T) = \inf \sum_{i=1}^{n} a^*(U_i) b(V_i)$$

the infimum being taken over all representations of T of the form $T = \sum_{i=1}^{n} U_i V_i$ with $V_i \in L(E, G_i)$, $U_i \in L(G_i, F)$ and G_i finite dimensional Banach spaces.

The tensor norm $\eta_{p,q}$ was introduced in [16] it is defined for $U \in E' \otimes F$ (that is, $U \in L(E, F)$ a finite rank operator) by

$$\eta_{p,q}(U) = \inf_{U=\sum_{k=1}^{n} x'_k \otimes y_k} \sup_{\substack{\|(x_i)_i\|_{\ell_p(E)} \le 1 \\ \|(y'_i)_i\|_{\ell_{q'}(F')} \le 1}} \sum_{k=1}^{n} \|(x'_k(x_i))_i\|_{\ell_p} \|(y'_i(y_k))_i\|_{\ell_{q'}} \tag{1.6}$$

here, as throughout this paper q' is such that $1/q + 1/q' = 1$. The infimum above is taken over all representations of U as a finite sum of elementary tensors.

It is shown in [16] (Proposition 3) that

$$\eta_{p,q}(U) = \inf K^{(p)}(A) K_{(q)}(B) \tag{1.7}$$

the infimum being taken over all finite dimensional spaces L with a 1-unconditional basis and factorizations of U of the form

$$\begin{tikzcd} E \arrow[rr, "U"] \arrow[dr, "A"'] & & F \\ & L \arrow[ur, "B"'] & \end{tikzcd} \tag{1.8}$$

$K^{(p)}(A)$ is the p-convexity constant of the operator A and $K_{(q)}(B)$ is the q-concavity constant of B.

Another version of (1.6) will be useful in the sequel. If E and F are finite dimensional Banach spaces and $(\xi_i)_{i=1}^{M}$, $(\eta_i)_{i=1}^{N}$ (M or N may be infinite) are sequences in $S(E)$ and $S(F')$, whose affine spans are dense in $B(E)$ and $B(F')$ respectively, it is not hard to check that (1.6) may be written in the form

$$\eta_{p,q}(U) = \inf_{U=\sum_{k=1}^{n} x'_k \otimes y_k} \sup_{\substack{\|\mu\|_{\ell_p^M} \le 1 \\ \|\nu\|_{\ell_{q'}^N} \le 1}} \sum_{k=1}^{n} \|(\mu_i x'_k(\xi_i))_i\|_{\ell_p^M} \|(\nu_i \eta_i(y_k))_i\|_{\ell_{q'}^N}. \tag{1.9}$$

We shall use (1.9) in the case that E and F are polyhedral spaces (by which we mean that their unit balls are polytopes) and $(\xi_i)_{i=1}^{M}$, $(\eta_i)_{i=1}^{N}$ are the sets of extreme points of $B(E)$ and $B(F')$. Let us see the proof in this case. Denote by $\zeta_{p,q}(U)$ the expression on the right hand side of (1.9). Clearly $\zeta_{p,q}(U) \le \eta_{p,q}(U)$. In order to show a reverse inequality, it is sufficient to show that for every sequence $(x_j)_{j=1}^{\infty} \in \ell_p(E)$ with

$\|(x_j)\|_{\ell_p(E)} \le 1$ it is possible to find a sequence $(\mu_i)_{i=1}^M$ of non-negative numbers, with $\|(\mu_i)\|_{\ell_p^M} \le 1$ such that for all $x' \in E'$

$$\|(\mu_i x'(\xi_i))_i\|_{\ell_p^M} \ge \|(x'(x_j))_j\|_{\ell_p}$$

(an analogous construction will then work for F' and F). For all j write $x_j = \theta_j z_j$ with $\theta_j \ge 0$ and $\|z_j\| = 1$. Then $(\sum \theta_j^p)^{1/p} \le 1$. For all $1 \le j \le M$ there exist non-negative numbers $\alpha_i(j)$, $i = 1, \dots, M$ such that

$$\sum_{i=1}^M \alpha_i(j) = 1 \quad \text{and} \quad z_j = \sum_{i=1}^M \alpha_i(j)\xi_i.$$

Define

$$\mu_i = \left(\sum_j \theta_j^p \alpha_i(j)\right)^{1/p}.$$

Clearly

$$\left(\sum_{i=1}^M \mu_i^p\right)^{1/p} = \left(\sum_j \theta_j^p\right)^{1/p} \le 1.$$

On the other hand, for $x' \in E'$

$$\left(\sum_{i=1}^M \mu_i^p |x'(\xi_i)|^p\right)^{1/p} = \left[\sum_j \theta_j^p \left(\sum_{i=1}^M \alpha_i(j)|x'(\xi_i)|^p\right)\right]^{1/p}$$

now

$$\left(\sum_{i=1}^M \alpha_i(j)|x'(\xi_i)|^p\right)^{1/p} \ge \sum_{i=1}^M \alpha_i(j)|x'(\xi_i)| \ge |x'(z_j)|.$$

Hence

$$\left(\sum_{i=1}^M \mu_i^p |x'(\xi_i)|^p\right)^{1/p} \ge \left(\sum_j \theta_j^p |x'(z_j)|^p\right)^{1/p} = \|(x'(x_j))_j\|_{\ell_p}.$$

□

2. The operator ideal $\Gamma_{p,q}$. For $1 \le p, q \le \infty$ we define the operator ideal $[\Gamma_{p,q}, \gamma_{p,q}]$ by

$$[\Gamma_{p,q}, \gamma_{p,q}] = \left[\frac{I'_{p'}}{\Pi_{q'}}, \frac{i'_{p'}}{\pi_{q'}}\right]$$

Explicitly: $T \in \Gamma_{p,q}(E,F)$ if and only if for every Banach space G, $U \in \Pi_{q'}(F,G)$ implies $(UT)' \in I_{p'}(G',E')$. A Banach space E has the *p,q-G.L.-property* if $I_E \in \Gamma_{p,q}$.

The following hold:

a) $[\Gamma_{p,q}, \gamma_{p,q}]$ is a perfect normed ideal.

b) $[\Gamma_{1,\infty}, \gamma_{1,\infty}] = [\Gamma, \gamma]$ is the G.L.-ideal.

c) For $p > q$, $[\Gamma_{p,q}, \gamma_{p,q}] = [I_{p,q}, i_{p,q}]$. In particular, for $p > q$ no infinite dimensional Banach space has the p,q-G.L.-property.

d) For $p = q$, $[\Gamma_{p,p}, \gamma_{p,p}] = [\Gamma_p, \gamma_p]$ is the ideal of L_p-factorization.

e) $[M_{p,q}, \mu_{p,q}] \subset [\Gamma_{p,q}, \gamma_{p,q}]$.

f) If a Banach space E has the G.L.-property and is of cotype-$q < \infty$ and type-$p > 1$ then for every $\tilde{q} > q$ and $\tilde{p} < p$, E has the $\tilde{p},\tilde{q}$-G.L.-property.

g) If E and F are finite dimensional Banach spaces and $T \in L(E,F)$ then

i)
$$\gamma^*_{p,q}(T) = \inf \sum_{i=1}^{n} \pi_p(U_i')\pi_{q'}(V_i)$$

the infimum is taken over all representations of T as $T = \sum_{i=1}^{n} U_i V_i$, with $V_i \in L(E,G_i)$, $U_i \in L(G_i,F)$ and G_i finite dimensional Banach spaces.

ii)
$$\gamma^*_{p,q}(T) = \inf \sum_{i=1}^{n} \|\mu_i\|^{1/p} \|\nu_i\|^{1/q'}$$

the infimum is taken over all representations of T as $T = \sum_{i=1}^{n} T_i$ such that there exist positive Radon measures μ_i on $B(E')$ and ν_i on $B(F)$ which satisfy for all $x \in E$, $y' \in F'$:

$$|\langle T_i x, y' \rangle| \le \left(\int_{B(E')} |\langle x, x' \rangle|^p \, d\mu(x') \right)^{1/p} \left(\int_{B(F)} |\langle y, y' \rangle|^{q'} \, d\nu(y) \right)^{1/q'}.$$

h) $[\Gamma'_{p,q}, \gamma'_{p,q}] = [\Gamma_{q',p'}, \gamma_{q',p'}]$.

The claim a) follows from Proposition 2.7 in [15]. b) is obvious. c) is proved in [8] and d) in [5]. e) is proved in [16]. In this paper we provide a unified presentation of the ideals $\Gamma_{p,q}$ and $M_{p,q}$ using norms on tensor products. This approach yields c), d) and e) together. f) follows from results of Pisier ([13] and [14]-cf. [15]). There exist examples of spaces with the G.L.-property and no l.u.st, of arbitrary cotype and type (except, of course, 2, 2) (cf. [2], [3], [4]), hence f) shows that for all $p < 2 < q$, $M_{p,q} \neq \Gamma_{p,q}$.

Claim g) is a consequence of the representation B) in the introduction and is proved exactly like Proposition 3.4 in [15]. h) is a direct consequence of g).

Theorem 2.1. *Let E and F be finite dimensional Banach spaces. We have for $U \in L(E,F)$:*

$$\gamma_{p,q}(U) = \sup_{\substack{\|(x_i)_i\|_{\ell_p(E)}\le 1 \\ \|(y_i')_i\|_{\ell_{q'}(F')}\le 1}} \inf_{U=\sum_{k=1}^n x_k'\otimes y_k} \sum_{k=1}^n \|(x_k'(x_i))_i\|_{\ell_p} \|(y_i'(y_k))_i\|_{\ell_{q'}} \ . \tag{2.1}$$

Remark. Comparing (2.1) with (1.6) we see that for finite dimensional Banach spaces the difference between $M_{p,q}$ and $\Gamma_{p,q}$ is in replacing the 'inf sup' in $M_{p,q}$ with 'sup inf' in $\Gamma_{p,q}$. As these ideals are perfect and therefore determined by the behaviour on finite dimensional spaces, this confirms claim e) above.

Proof. $\gamma_{p,q}(U)$ is the norm of U as a linear functional on $[L(F,E),\gamma^*_{p,q}]$. Therefore, by g) above

$$\gamma_{p,q}(U) = \sup_{0\neq V\in L(F,E)} \sup_{\mu,\nu} \frac{|\operatorname{trace} UV|}{\|\mu\|^{1/p}\|\nu\|^{1/q'}} \tag{2.2}$$

where the second supremum is taken over all positive Radon measures μ on $B(E)$ and ν on $B(F')$ which satisfy

$$|\langle Vy, x'\rangle| \le \mu(|\langle x',\cdot\rangle|^p)^{1/p}\nu(|\langle\cdot,y\rangle|^{q'})^{1/q'} \tag{2.3}$$

for all $y \in F$ and $x' \in E'$.

Let (ξ_i) and (η_i) be sequences which are dense, respectively, in $S(E)$ and $S(F')$. Let I and J be the isometric embeddings:

$$I : E' \to \ell_\infty\,; \quad I(x') = (x'(\xi_i))_i$$

$$J : F \to \ell_\infty\,; \quad J(y) = (\eta_i(y))_i.$$

Clearly, (2.2) is equivalent to

$$\gamma_{p,q}(U) = \sup_{0\neq V\in L(F,E)} \sup_{\mu,\nu} \frac{|\operatorname{trace} UV|}{\|\mu\|_{\ell_p}\|\nu\|_{\ell_{q'}}} \tag{2.4}$$

where the second supremum is taken over all the sequences $\mu = (\mu_i) \in \ell_p$ and $\nu = (\nu_i) \in \ell_{q'}$ which satisfy

$$|\langle Vy, x'\rangle| \le \left(\sum_i |x'(\xi_i)\mu_i|^p\right)^{1/p} \left(\sum_i |\eta_i(y)\nu_i|^{q'}\right)^{1/q'} \tag{2.5}$$

for all $y \in F$ and $x' \in E'$.

Let D_μ and D_ν be the diagonal operators associated with the sequences μ and ν. We define subspaces: $S_{q'} = D_\nu J(F) \subset \ell_{q'}$ and $S_p = D_\mu I(E') \subset \ell_p$. The inequality (2.5) holds for all y and x' if and only if there exists a norm-1 operator $Q : S_{q'} \to Q_{p'} = S'_p$ such that

$$V = (D_\mu I : E' \to S_p)' Q (D_\nu J : F \to S_{q'}) \tag{2.6}$$

(compare with the proof of Lemma 3.5 in [15]).

$$\begin{tikzcd}
 & E \arrow[r, "U"] & F \arrow[d] \arrow[dr, "J"] & \\
\ell'_\infty \arrow[ur, "I'"] \arrow[r] & I(E')' \arrow[u] & J(F) \arrow[r] \arrow[d] & \ell_\infty \arrow[d, "D_\nu"] \\
\ell_{p'} \arrow[u, "D'_\mu"] \arrow[r] & Q_{p'} \arrow[u] & S_{q'} \arrow[l, "Q"'] \arrow[r] & \ell_{q'}
\end{tikzcd} \tag{2.7}$$

For given μ and ν, let $M(\mu, \nu)$ be the subset of $L(E, F)$ consisting of operators V admitting a factorization of the form (2.6), without the restriction $\|Q\| = 1$. Interchanging the order of suprema in (2.4) we get

$$\gamma_{p,q}(U) = \sup_{\substack{\|\mu\|_{\ell_p} \le 1 \\ \|\nu\|_{\ell_{q'}} \le 1}} \left[\sup_{0 \ne V \in M(\mu,\nu)} \frac{|\operatorname{trace} UV|}{\|Q\|} \right]. \tag{2.8}$$

The expression inside the square brackets in (2.8) is clearly the norm of the operator

$$U_{\mu,\nu} = (D_\nu J : F \to S_{q'}) U (D_\mu I : E' \to S_p)'$$

as a functional on $L(S_{q'}, Q_{p'})$. This is exactly the nuclear norm $\nu_1(U_{\mu,\nu})$. Assume that U has a representation

$$U = \sum_{k=1}^n x'_k \otimes y_k$$

then

$$U_{\mu,\nu} = \sum_{k=1}^{n} (\mu_i x'_k(\xi_i))_i \otimes (\nu_i \eta_i(y_k))_i \tag{2.9}$$

here $(\mu_i x'_k(\xi_i))_i \in S_p \subset \ell_p$ is taken as a functional on $Q_{p'}$, also $(\nu_i \eta_i(y_k))_i \in S_{q'} \subset \ell_{q'}$.

If, on the other hand, $U_{\mu,\nu}$ has a representation

$$U_{\mu,\nu} = \sum_{k=1}^{n} \varphi_k \otimes \psi_k \in S_p \otimes S_{q'} \tag{2.10}$$

then there exist x'_k and y_k so that (2.10) can be written in the form (2.9). Therefore

$$\nu_1(U_{\mu,\nu}) = \inf_{U=\sum_{k=1}^{n} x'_k \otimes y_k} \sum_k \|(\mu_i x'_k(\xi_i))_i\|_{\ell_p} \|(\nu_i \eta_i(y_k))_i\|_{\ell_{q'}}.$$

The density of (ξ_i) and (η_i) in $S(E)$ and $S(F')$ completes now the proof of the theorem. □

We notice that in the proof of Theorem 2.1. we could restrict the measures μ and ν of (2.2) to be supported on appropriate subsets of $B(E)$ and $B(F')$ and thus get the following version of Theorem 2.1 which is more convenient technically (notice the analogy with (1.9)).

Proposition 2.2. *Let E and F be finite dimensional Banach spaces and let $(\xi_i)_{i=1}^M$ and $(\eta_i)_{i=1}^N$ be subsets of $S(E)$ and $S(F')$, respectively, whose affine spans are dense in $B(E)$ and $B(F)$ (M and N may be infinite). Then for $U \in L(E,F)$ we have*

$$\gamma_{p,q}(U) = \sup_{\substack{\|\mu\|_{\ell_p^M} \le 1 \\ \|\nu\|_{\ell_{q'}^N} \le 1}} \inf_{U=\sum_{k=1}^{n} x'_k \otimes y_k} \sum_{k=1}^{n} \|(\mu_i x'_k(xi_i))_i\|_{\ell_p^M} \|(\nu_i \eta_i(y_k))_i\|_{\ell_{q'}^N}. \tag{2.11}$$

Corollary 2.3. ([16]) *For all $1 \le p, q \le \infty$*

$$[M_{p,q}, \mu_{p,q}] \subset [\Gamma_{p,q}, \gamma_{p,q}].$$

P r o o f. We always have ‘sup inf’ $\le$ ‘inf sup’ □

The following Proposition 2.4 can be obtained from the results of [8], its Corollary 2.5 was proved by Kwapien [5] in the case $p = q$. The case $p > q$ could be proved using the results of [8] combined with the fact that $[I_{p,q}, i_{p,q}]$ is a perfect ideal. This last fact was first proved by Lapreste [9], it is also a consequence of [16]. Here we give a proof based on the possibility of interchanging 'sup inf' with 'inf sup' in (2.11) in the case $p > q$ and thus identifying it with (1.9).

Proposition 2.4. *Let $1 \leq q \leq p \leq \infty$ then for all finite dimensional Banach spaces E and F and all $U \in L(E,F)$*

$$\gamma_{p,q}(U) = \eta_{p,q}(U).$$

Corollary 2.5. ([5], [9]) *For $1 \leq q \leq p \leq \infty$*

$$[\Gamma_{p,q}, \gamma_{p,q}] = [I_{p,q}, i_{p,q}].$$

In particular ([5]), *for $1 < p < \infty$ a Banach space E is isomorphic to a complemented subspace of L_p if and only if for every Banach space G, $V \in \Pi_{p'}(E,G)$ implies $V' \in I_{p'}(G',E')$.*

Proof of Proposition 2.4. We prove the case $1 < q \leq p < \infty$. By approximation we may suppose that the spaces E and F are polyhedral and the sets of extreme points of $B(E)$ and $B(F')$ are $(\xi_i)_{i=1}^M$ and $(\eta_i)_{i=1}^N$, respectively.

To show that the 'sup inf' in (2.11) may be replaced by the 'inf sup' of (1.9) we shall apply

Ky-Fan's minimax theorem [1]. *Let Γ and Λ be sets and f a real valued function on $\Gamma \times \Lambda$, which is convex-concave-like. Assume also that Λ is compact in some Hausdorff topology and $f(\gamma,\cdot)$ is upper semi-continuous on Λ for every $\gamma \in \Gamma$. Then*

$$\inf_\Gamma \max_\Lambda f(\gamma,\lambda) = \max_\Lambda \inf_\Gamma f(\gamma,\lambda).$$

We recall that f is *convex-concave-like* on $\Gamma \times \Lambda$ if for all $0 < \alpha < 1$ we have

- For all $\gamma_1, \gamma_2 \in \Gamma$ there exists $\gamma_3 \in \Gamma$ so that for all $\lambda \in \Lambda$

$$f(\gamma_3,\lambda) \leq \alpha f(\gamma_1,\lambda) + (1-\alpha) f(\gamma_2,\lambda) \tag{2.12}$$

- For all $\lambda_1, \lambda_2 \in \Lambda$ there exists $\lambda_3 \in \Lambda$ so that for all $\gamma \in \Gamma$

$$f(\gamma,\lambda_3) \geq \alpha f(\gamma,\lambda_1) + (1-\alpha) f(\gamma,\lambda_2). \tag{2.13}$$

We let $\Lambda = B(\ell_p^M)_+ \times B(\ell_{q'}^N)_+$, equipped with the product of the norm topologies and we define Γ as the set of sequences $((x'_k, y_k)) \in (E' \times F)^{\mathbb{N}}$ with finite number of non-zero elements, for which $U = \sum_{k \in \mathbb{N}} x'_k \otimes y_k$.

For $\gamma = ((x'_k, y_k))_{k\in\mathbb{N}} \in \Gamma$ and $\lambda = (\mu, \nu) \in \Lambda$ we define

$$f(\gamma,\lambda) = \sum_k \|(\mu_i x'_k(\xi_i))_i\|_{\ell_p^M} \|(\nu_i \eta_i(y_k))_i\|_{\ell_{q'}^N}.$$

Now (1.9) and (2.11) are:

$$\eta_{p,q}(U) = \inf_\Gamma \sup_\Lambda f(\gamma,\lambda)\,; \quad \gamma_{p,q}(U) = \sup_\Lambda \inf_\Gamma f(\gamma,\lambda).$$

It is clear that $f(\gamma,\cdot)$ is continuous on the compact Λ for each $\gamma \in \Gamma$, so it remains to show that f is convex-concave-like on $\Gamma \times \Lambda$. We first show (2.12). For $\gamma_j = ((x'_{j,k}, y_{j,k}))_{k\in\mathbb{N}} \in \Gamma\,, \quad j = 1,2$ and $0 < \alpha < 1$, let

$$\gamma_3 = ((z'_{j,k}, w_{j,k}))_{\substack{j=1,2\\ k\in\mathbb{N}}}$$

be defined by

$$z'_{1,k} = \alpha x'_{1,k}\,, \quad z'_{2,k} = x'_{2,k}$$

$$w_{1,k} = y_{1,k}\,, \quad w_{2,k} = (1-\alpha) y_{2,k}.$$

Then, of course $U = \sum_{j,k} z'_{j,k} \otimes w_{j,k}$ hence $\gamma_3 \in \Gamma$ and for every $\lambda \in \Lambda$ we have:

$$f(\gamma_3,\lambda) = \sum_{j,k} \|(\mu_i z'_{j,k}(\xi_i))_i\|_{\ell_p^M} \|(\nu_i \eta_i(w_{j,k}))_i\|_{\ell_{q'}^N} = \alpha f(\gamma_1,\lambda) + (1-\alpha) f(\gamma_2,\lambda).$$

We turn now to (2.13). Let $\lambda_1 = (\mu,\nu)$ and $\lambda_2 = (\tilde\mu,\tilde\nu)$ be in Λ and $0 < \alpha < 1$. We may assume that $\mu_i, \nu_i, \tilde\mu_i, \tilde\nu_i$ are positive for all i. Define $\lambda_3 = (\hat\mu,\hat\nu)$ by

$$\hat\mu_i = (\alpha\mu_i^p + (1-\alpha)\tilde\mu_i^p)^{1/p}$$

$$\hat\nu_i = (\alpha\nu_i^{q'} + (1-\alpha)\tilde\nu_i^{q'})^{1/q'}.$$

Then

$$\|\hat\mu\|_{\ell_p^M} = \left(\alpha\sum_{i=1}^M \mu_i^p + (1-\alpha)\sum_{i=1}^M \tilde\mu_i^p\right)^{1/p} \le \max(\|\mu\|_{\ell_p^M}, \|\tilde\mu\|_{\ell_p^M}) \le 1$$

a similar inequality holds for $\hat\nu$, hence $\lambda_3 \in \Gamma$.

Let $\gamma = ((x'_k, y_k)) \in \Gamma$ and denote

$$a_k = \left(\sum_{i=1}^{M} \mu_i^p |x'_k(\xi_i)|^p\right)^{1/p}, \quad \tilde{a}_k = \left(\sum_{i=1}^{M} \tilde{\mu}_i^p |x'_k(\xi_i)|^p\right)^{1/p}$$

$$b_k = \left(\sum_{i=1}^{N} \nu_i^{q'} |\eta_i(y_k)|^{q'}\right)^{1/q'}, \quad \tilde{b}_k = \left(\sum_{i=1}^{N} \tilde{\nu}_i^{q'} |\eta_i(y_k)|^{q'}\right)^{1/q'}$$

$$A_k = \alpha a_k^p + (1-\alpha)\tilde{a}_k^p\,, \qquad B_k = \alpha b_k^{q'} + (1-\alpha)\tilde{b}_k^{q'}\,.$$

Then

$$f(\gamma, \lambda_3) = \sum_k A_k^{1/p} B_k^{1/q'}\,. \tag{2.14}$$

As we assumed $\mu, \tilde{\mu}, \nu, \tilde{\nu} > 0$ there is no loss of generality in assuming a_k, $\tilde{a}_k$, b_k, $\tilde{b}_k > 0$ for all k. We use the following simple formula:

If $0 < p, q', r < \infty$ *and* $1/p + 1/q' = 1/r$ *then*

$$\min_{0<t}\left(\frac{1}{p}t^p A + \frac{1}{q'}t^{-q'} B\right)^{1/r} = \frac{1}{r}A^{1/p}B^{1/q'} \tag{2.15}$$

(in our case $r \geq 1$). Using (2.15) for $A = A_k$, $B = B_k$ and rearranging the terms, we get, with certain positive numbers t_k:

$$f(\gamma, \lambda_3) = r\sum_k \left[\alpha\left(\frac{1}{p}t_k^p a_k^p + \frac{1}{q'}t_k^{-q'} b_k^{q'}\right) + (1-\alpha)\left(\frac{1}{p}t_k^p \tilde{a}_k^p + \frac{1}{q'}t_k^{-q'}\tilde{b}_k^{q'}\right)\right]^{1/r}$$

another use of (2.15) yields

$$f(\gamma, \lambda_3) \geq \sum_k [\alpha(a_k b_k)^r + (1-\alpha)(\tilde{a}_k\tilde{b}_k)^r]^{1/r} \geq \sum_k [\alpha a_k b_k + (1-\alpha)\tilde{a}_k\tilde{b}_k] \tag{2.16}$$

the last inequality holds since $r \geq 1$. Hence

$$f(\gamma, \lambda_3) \geq \alpha f(\gamma, \lambda_1) + (1-\alpha) f(\gamma, \lambda_2)$$

which completes the proof. $\square$

There exist examples of Banach spaces E which admit unconditional decomposition

$$E = \sum_{n=1}^{\infty} \oplus E_n \tag{2.17}$$

such that the G.L. constants of E_n are bounded (even this: all E_n are isometric to ℓ_2) but E fails to have the G.L. property (such are, for example, spaces of 'triangular' compact operators on a separable Hilbert space – cf. [10] p.51).

The following application of Theorem 2.1 shows that if the / decomposition (2.17) is unconditional in a stronger sense, then such examples do not exist.

Theorem 2.6. *Let X be a Banach space with an unconditional basis $(e_n)_{n=1}^{\infty}$, which is p-convex and q-concave as a Banach lattice. Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of Banach spaces which satisfy: $\sup_n \gamma_{p,q}(I_{E_n}) = K < \infty$ and*

$$E = \sum_{n=1}^{\infty} \oplus_X E_n$$

(that is, for $x = \sum_{n=1}^{\infty} x_n \in E$, $x_n \in E_n$, $\|x\| = \|\sum_{n=1}^{\infty} \|x_n\|_{E_n} e_n \|_X$).

Then E is a p,q-G.L.-space and

$$\gamma_{p,q}(I_E) \leq K K^{(p)}(X) K_{(q)}(X).$$

In particular, a direct sum in the sense of an unconditional basis of Banach spaces with bounded G.L.-constants, is a G.L.-space.

Proof. Using perfectness of $[\Gamma_{p,q}, \gamma_{p,q}]$ we may assume that X and all E_n are finite dimensional. In view of Theorem 2.1 we have to show that given $(x_i) \in B(\ell_p(E))$ and $(y_i') \in B(\ell_{q'}(E'))$ we have, for every $\varepsilon > 0$, a representation

$$I_E = \sum_j x_j' \otimes y_j \tag{2.18}$$

such that

$$\sum_j \|(x_j'(x_i))_i\|_{\ell_p} \|(y_i'(y_j))_i\|_{\ell_{q'}} \leq (1+\varepsilon) K K^{(p)}(X) K_{(q)}(X).$$

For all i, x_i and y_i' have decompositions

$$x_i = \sum_n x_{i,n}\,, \;\; x_{i,n} \in E_n\,; \quad y_i' = \sum_n y_{i,n}'\,, \;\; y_{i,n}' \in E_n'.$$

By the assumption $\gamma_{p,q}(I_{E_n}) \leq K$ and by Theorem 2.1 we have representations

$$I_{E_n} = \sum_{k=1}^{m(n)} x_{k,n}' \otimes y_{k,n}\,, \quad x_{k,n}' \in E_n'\,,\; y_{k,n} \in E_n$$

satisfying

$$\sum_{k=1}^{m(n)} \|(x'_{k,n}(x_{i,n})_i\|_{\ell_p}\|(y'_{i,n}(y_{k,n}))_i\|_{\ell_{q'}} \leq (1+\varepsilon)K\|(x_{i,n})_i\|_{\ell_p(E_n)}\|(y'_{i,n})_i\|_{\ell_{q'}(E'_n)}.$$

We claim that the representation

$$I_E = \sum_{k,n} x'_{k,n} \otimes y_{k,n}$$

is a good representation to substitute in (2.18). In fact

$$\sum_{k,n} \|(x'_{k,n}(x_i))_i\|_{\ell_p}\|(y'_i(y_{k,n}))_i\|_{\ell_{q'}} = \sum_{k,n} \|(x'_{k,n}(x_{i,n})_i\|_{\ell_p}\|(y'_{i,n}(y_{k,n}))_i\|_{\ell_{q'}} \leq$$

$$(1+\varepsilon)K\sum_n \|(x_{i,n})_i\|_{\ell_p(E_n)}\|(y'_{i,n})_i\|_{\ell_{q'}(E'_n)} \leq$$

$$(1+\varepsilon)K\left\|\sum_n \|(x_{i,n})_i\|_{\ell_p(E_n)}e_n\right\|_X \left\|\sum_n \|(y'_{i,n})_i\|_{\ell_{q'}(E'_n)}e'_n\right\|_{X'} \leq$$

$$(1+\varepsilon)KK^{(p)}(X)K^{(q')}(X')\left(\sum_i \left\|\sum_n \|x_{i,n}\|_{E_n}e_n\right\|_X^p\right)^{1/p}\left(\sum_i \left\|\sum_n \|y'_{i,n}\|_{E'_n}e'_n\right\|_{X'}^{q'}\right)^{1/q'} \leq$$

$$(1+\varepsilon)KK^{(p)}(X)K_{(q)}(X).$$

$\square$

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