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PERTURBATIONS OF SYSTEMS DESCRIBING THE MOTION OF A PARTICLE IN CENTRAL FIELDS

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Abstract. The present paper deals with the KAM-theory conditions for systems describing the motion of a particle in central field.

1. Introduction. The question of integrability of Hamiltonian systems is one of the oldest problems of classical mechanics [1]. Classical results due to Poincare and Bruns show that most of the Hamiltonian systems are not integrable. This has lead Poincare to define the main problem of dynamics to be the study of Hamiltonian systems which are close to integrable ones. The most powerful approach to such systems is the KAM-theory. Before giving a brief account of KAM-theory we remind the structure of the integrable Hamiltonian systems.

The phase space of the generic integrable Hamiltonian systems with n-degrees of freedom is foliated into invariant manifolds the typical fibre being a n-dimensional torus, on which the motion is quasiperiodic. A natural question is whether small perturbations destroy these tori, KAM-theory gives conditions for the integrable systems which guarantee the survival of most of the invariant tori. The conditions are given in terms of the so-called action – angle variables $J_1, J_2, \dots, J_n$; $\varphi_1, \varphi_2, \dots, \varphi_n$. Without going into details, we remind that the action – angle variables can be introduced for any integrable system locally near a fixed torus and have a property that

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$\mathbf{J} = (J_1, J_2, \ldots, J_n)$ maps a neighbourhood of a fixed torus on an open subset of $\mathbb{R}^n$. The functions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are the co - ordinates on any of the nearby tori. Moreover, the first integrals become functions of the action variables $J_1, J_2, \ldots, J_n$. Finally to any fixed torus there corresponds an invariant torus on which the motion is quasiperiodic with frequencies $(\omega_1(\mathbf{J}), \ldots, \omega_n(\mathbf{J})) = (\partial H/\partial J_1, \ldots \partial H/\partial J_n)$ (see [2] for details).

One condition, stated by Kolmogorov (see [2], app. 8) on the Hamiltonian of the integrable system that ensures the survival of most of the invariant tori under small perturbations is that the frequency map

$$\mathbf{J} \to (\omega_1(\mathbf{J}), \omega_2(\mathbf{J}), \ldots, \omega_n(\mathbf{J}))$$

should be non-degenerate. Analytically this means that the Hesseian

$$\det\left(\frac{\partial^2 H}{\partial J_j \partial J_k}\right) \qquad j, k = 1, \ldots, n, \tag{1.1}$$

does not vanish. We should note that the measure of the surviving tori decreases with the increase of both perturbation and measure of the set where the above Hesseian is too close to zero.

Another condition of this type, stated by V. Arnold and J. Moser (see [2, app.8], [3]) is that of isoenergetical non-degeneracy, which can be explained as follows. Fix an energy level $H_0 = h_0$. If the Hamiltonian H_0 is written in action variables, then define the following map F_{h_0} from the $(n-1)$ dimensional variety $H_0^{-1}(h_0)$ into the projective space $\mathbb{P}^{n-1}$:

$$F_{h_0} : \mathbf{J} \to (\omega_1(\mathbf{J}) : \omega_2(\mathbf{J}) : \ldots : \omega_n(\mathbf{J})) .$$

The system is isoenergetically non-degenerate if the map $F_{h_0}$is a homeomorphism. Analytically the isoenergetical non-degeneracy is tantamount to the nonvanishing of the determinant

$$\det\begin{pmatrix} \dfrac{\partial^2 H_0}{\partial \mathbf{J}^2} & \dfrac{\partial H_0}{\partial \mathbf{J}} \\ \dfrac{\partial H_0}{\partial \mathbf{J}} & 0 \end{pmatrix} \tag{1.2}$$

The checking of the conditions (1.1) and (1.2) is a very difficult problem however, there exist several methods for solving such problems.

Knörrer [4] found a method for checking Kolmogorov's condition by reducing the number of degrees of freedom. Using this method he has proved that Kolmogorov's condition is fulfilled almost everywhere for several systems including the geodesic flow on the ellipsoid and K.Neumann's system.

In a recent paper Horozov [5] has proved that for the system describing the spherical pendulum, condition (1.1) is satisfied everywhere out of the bifurcation diagram of the energy – momentum map. The crucial role in [5] is played by certain algebraic curves and Abelian integrals on them. The condition (1.2) for the spherical pendulum is checked in [6].

The purpose of this paper is to check the KAM-theory conditions (1.1) and (1.2) for the system describing the motion of a particle in central fields with potentials $V = ar^{n+1}$. The problem of the motion of the particle in a central field with potential $V(r)$, where r is the distance between the attracting centre and the particle, is not new and is considered in [6, 7], for instance. It is known that the motion takes place in a plane through the origin. Therefore, the problem is reduced to a system with two degrees of freedom. It is also well known that the problems with such potentials are completely integrable [7, 8]. The natural co - ordinates in which the system takes the simplest form are the polar co - ordinates $x = r\cos\varphi$, $y = r\sin\varphi$, where $r \in (0, \infty)$ and $\varphi \in [0, 2\pi]$. Without loss of generality we put the mass of the particle equal to 1 $(m = 1)$. Then, the Lagrangian of the system is

$$L = \dot{r}^2/2 + r^2\dot{\varphi}^2/2 - ar^{n+1}. \quad (\text{ fig.1})$$

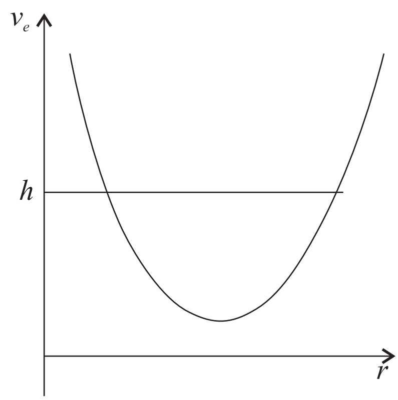

Fig. 1. The effective potential for $n = 0$

Using the Legendre transformation, we obtain the corresponding Hamiltonian system via

$$H = \frac{p_r^2}{2} + \frac{p_\varphi^2}{2r^2} + ar^{n+1},$$

where $p_r = \partial L/\partial \dot{r} = \dot{r}$, $p_\varphi = \partial L/\partial \dot{\varphi} = \dot{\varphi}r^2$. The symplectic structure is $\omega = d\sigma$ where

$$\sigma = p_r dr + p_\varphi d\varphi. \tag{1.3}$$

As φ is a cyclic co-ordinate, the system is integrable with obvious integrals

$$F = p_\varphi = f = \text{const}\,,$$

$$H = \frac{p_r^2}{2} + \frac{f^2}{2r^2} + ar^{n+1} = h = \text{const}\,. \tag{1.4}$$

In this paper we will study the cases in which there exist tori and the problem is solved via elliptic functions – $n = 0, 3$ $(a > 0)$ and $n = -5/3$, $-7/3$ $(a < 0)$, which we call cubic, because algebraic curves are polynomials of 3-rd degree and cases $n = 5$, $(a > 0)$, $5/2$ $(a < 0)$ which are reduced to polynomials of 4-th degree (quartic).

The paper consists of two parts. The Part A is devoted to the cubic cases and only the case $n = 0$ is considered in details. In Section 2 the action variables are introduced and the result is stated. The proof is given in Section 3. For the proof of the result we use the method from [5]. Section 4 gives the keypoints for the other cubic cases – $n = 3$, $-5/3$, $-7/3$. The quartic cases are studied in Part B and again only one case: $n = -5/2$ is considered in details. We introduce the action variables and state the corresponding result for the quartic cases in Section 5. The proof is given in Section 6, by using Picard-Fuchs and Riccati equations for evaluating the number of zeros of an elliptic integral.

Part A. CUBIC CASES

2. Statement of the result for the cubic cases. For the case $n = 0$ the first integrals are

$$F = p_\varphi = f = \text{const}\,,$$

$$H = \frac{p_r^2}{2} + \frac{f^2}{2r^2} + ar = h = \text{const}\,. \tag{2.1}$$

The values of H and F for which the real movement takes place define the set

$$U = \left\{(h, f) : h \geq 0, 0 \leq f^2 \leq \frac{8h^3}{27a^2}\right\}.$$

In order to introduce the action – angle variables we need to exclude from U the critical values of the energy – momentum map (H, F). It is easy to calculate that these points are the boundaries of U i.e. the points satisfying the equations

$$f = 0, \quad f^2 = \frac{8h^3}{27a^2}.$$

Denote by U_r the set of regular values of the energy - momentum map

$$U_r = \{(h,f) : h > 0,\ 0 < f^2 < \frac{8h^3}{27a^2}\}.\ \text{(fig.2)}$$

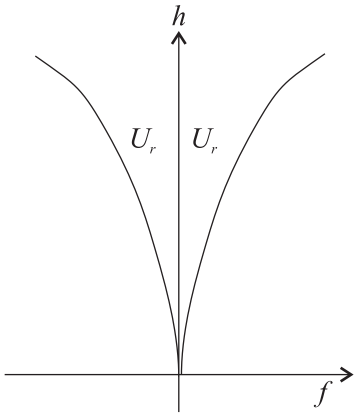

Fig. 2. The set of the regular values of the energy-momentum

For the points $(h,f) \in U_r$ the level surface determined by the equations $H = h$, $F = f$ is a torus $T_{h,f}$. Choose a basis γ_1, γ_2 of the homology group $H_1(T_{h,f}, \mathbb{Z})$ with the following representatives. For γ_1 take the curve on $T_{h,f}$ defined by fixing r, p_r, p_φ and letting φ run through $[0, 2\pi]$. For γ_2 fix φ and let r, p_r make one circle on the curve given by the equation

$$\frac{p_r^2}{2} + \frac{f^2}{2r^2} + ar = h.$$

Now we can define the action co - ordinates J_1, J_2 by the formula

$$J_j = \int_{\gamma_j} \sigma, \quad j = 1, 2,$$

where σ is canonical one - form (1.3). Trivial computations give

$$J_1 = 2\pi f, \tag{2.2}$$

$$J_2 = \int_{\gamma_2} p_r dr = 2 \int_{r_-}^{r_+} \sqrt{(2h - 2ar - f^2/r^2)} dr, \tag{2.3}$$

where $r_+ > r_-$ are the two roots of the equation

$$h - ar - f^2/2r^2 = 0.$$

Put $y^2 = 2hr^2 - 2ar^3 - f^2$. Denote the oval of the curve $\Gamma_{h,f} = \{(y,r) : y^2 = 2hr^2 - 2ar^3 - f^2\}$ by γ. Then we have

$$\psi(h,f) \overset{def}{=} J_2 = \int_\gamma \frac{ydr}{r}. \tag{2.4}$$

Denote by $\widetilde{H}(J_1, J_2)$ the Hamiltonian of the considered system in action - angle co - ordinates. We state the theorem which is one of the aims of this paper.

Theorem 1. *For $(h,f) \in U_r$ the following determinants do not vanish*

$$\det \begin{pmatrix} \dfrac{\partial^2 \widetilde{H}}{\partial J_1^2} & \dfrac{\partial^2 \widetilde{H}}{\partial J_1 \partial J_2} \\ \dfrac{\partial^2 \widetilde{H}}{\partial J_1 \partial J_2} & \dfrac{\partial^2 \widetilde{H}}{\partial J_2^2} \end{pmatrix} \neq 0. \tag{2.5}$$

$$\det \begin{pmatrix} \dfrac{\partial^2 \widetilde{H}}{\partial J_1^2} & \dfrac{\partial^2 \widetilde{H}}{\partial J_1 \partial J_2} & \dfrac{\partial \widetilde{H}}{\partial J_1} \\ \dfrac{\partial^2 \widetilde{H}}{\partial J_1 \partial J_2} & \dfrac{\partial^2 \widetilde{H}}{\partial J_2^2} & \dfrac{\partial \widetilde{H}}{\partial J_2} \\ \dfrac{\partial \widetilde{H}}{\partial J_1} & \dfrac{\partial \widetilde{H}}{\partial J_2} & 0 \end{pmatrix} \neq 0, \tag{2.6}$$

i.e. the system with Hamiltonian (2.1) *is nondegenerate and isoenergetically nondegenerate.*

Remark. The condition of isoenergetical non-degeneracy guarantees stability of action variables in sense that they remain always near their initial values when the perturbations are small (see [2, app. 8]).

We shall give the conditions (2.5) and (2.6) explicit form in terms of Abelian integrals of second kind. Using the expressions (2.2) and (2.4) for J_1 and J_2 we can determine $\widetilde{F}$ and $\widetilde{H}$ implicitly from the equations

$$J_1 = 2\pi \widetilde{F},$$

$$J_2 = \psi(\widetilde{F}, \widetilde{H}).$$

Lemma 1 (Horozov [5]).

$$(2\pi)^2(\partial\psi/\partial h)^4 \det\begin{pmatrix} \dfrac{\partial^2\widetilde{H}}{\partial J_1^2} & \dfrac{\partial^2\widetilde{H}}{\partial J_1\partial J_2} \\ \\ \dfrac{\partial^2\widetilde{H}}{\partial J_1\partial J_2} & \dfrac{\partial^2\widetilde{H}}{\partial J_2^2} \end{pmatrix} = \det\begin{pmatrix} \dfrac{\partial^2\psi}{\partial h^2} & \dfrac{\partial^2\psi}{\partial h\partial f} \\ \\ \dfrac{\partial^2\psi}{\partial h\partial f} & \dfrac{\partial^2\psi}{\partial f^2} \end{pmatrix}.$$

P r o o f. Differentiating equality $J_2 = \psi(J_1/2\pi, \widetilde{H}(J_1, J_2))$ by J_1 and J_2, we obtain

$$0 = \psi_f\frac{1}{2\pi} + \psi_h\frac{\partial\widetilde{H}}{\partial J_1},$$

$$1 = \psi_h\frac{\partial\widetilde{H}}{\partial J_2}.$$

From here we find

$$\frac{\partial\widetilde{H}}{\partial J_1} = -\frac{1}{2\pi}\frac{\psi_f}{\psi_h},$$

$$\frac{\partial\widetilde{H}}{\partial J_2} = \frac{1}{\psi_h}.$$

Differentiating again these equalities we have

$$\begin{aligned} \frac{\partial^2\widetilde{H}}{\partial J_1^2} &= -\frac{1}{(2\pi)^2}\frac{\psi_{ff}\psi_h^2 - 2\psi_h\psi_f\psi_{hf} + \psi_f^2\psi_{hh}}{\psi_h^3}, \\ \frac{\partial^2\widetilde{H}}{\partial J_1\partial J_2} &= -\frac{1}{2\pi}\frac{\psi_{fh} - \psi_{hh}(\psi_f/\psi_h)}{\psi_h^2}, \\ \frac{\partial^2\widetilde{H}}{\partial J_2^2} &= -\frac{\psi_{hh}}{\psi_h^3}. \end{aligned}$$

Consequently

$$\det\begin{pmatrix} \dfrac{\partial^2\widetilde{H}}{\partial J_1^2} & \dfrac{\partial^2\widetilde{H}}{\partial J_1\partial J_2} \\ \\ \dfrac{\partial^2\widetilde{H}}{\partial J_1\partial J_2} & \dfrac{\partial^2\widetilde{H}}{\partial J_2^2} \end{pmatrix} = \frac{1}{(2\pi)^2}\,\frac{1}{\psi_h^4}(\psi_{hh}\psi_{ff} - \psi_{hf}^2).$$

This completes the proof of Lemma 1. $\square$

Similarly, we have

Lemma 2.

$$(2\pi)\psi_h^3 \det \begin{pmatrix} \dfrac{\partial^2 \widetilde{H}}{\partial J_1^2} & \dfrac{\partial^2 \widetilde{H}}{\partial J_1 \partial J_2} & \dfrac{\partial \widetilde{H}}{\partial J_1} \\ \dfrac{\partial^2 \widetilde{H}}{\partial J_1 \partial J_2} & \dfrac{\partial^2 \widetilde{H}}{\partial J_2^2} & \dfrac{\partial \widetilde{H}}{\partial J_2} \\ \dfrac{\partial \widetilde{H}}{\partial J_1} & \dfrac{\partial \widetilde{H}}{\partial J_2} & 0 \end{pmatrix} = \psi_{ff}.$$

The proof of Lemma 2 is a straightforward computation using Lemma 1 and therefore is omitted. It is easy to see that

$$\psi_h = \int_\gamma \frac{rdr}{y} \neq 0$$

in U_r, because $r_+ > r_- > 0$ and $\displaystyle\int_\gamma \frac{dr}{y} \neq 0$ since it is the period. So, instead of Theorem 1 we shall prove the equivalent to it.

Theorem 2. (*i*) *For the* $(h, f) \in U_r$ *the determinant*

$$D = \det \begin{pmatrix} \dfrac{\partial^2 \psi}{\partial h^2} & \dfrac{\partial^2 \psi}{\partial h \partial f} \\ \dfrac{\partial^2 \psi}{\partial h \partial f} & \dfrac{\partial^2 \psi}{\partial f^2} \end{pmatrix}.$$

does not vanish.

(*ii*) *For the* $(h, f) \in U_r$ *the expression* $D_1 = \psi_{ff}$ *does not vanish.*

The proof will be given in Section 4.

Next we would like to show that the entries of D (and D_1) can be represented as elliptic integrals. If we differentiate (2.4) twice formally, we get the following expressions

$$\begin{aligned} \frac{\partial^2 \psi}{\partial h^2} &= -\int_\gamma \frac{r^3 dr}{y^3}, \\ \frac{\partial^2 \psi}{\partial h \partial f} &= f \int_\gamma \frac{rdr}{y^3}, \qquad (2.7) \\ \frac{\partial^2 \psi}{\partial f^2} &= -\left(\int_\gamma \frac{dr}{ry} + f^2 \int_\gamma \frac{dr}{ry^3} \right). \end{aligned}$$

(We have used that $y\partial y/\partial f = -f$, $y\partial y/\partial h = r^2$ which follows by differentiating the curve $\Gamma_{h,f}$.)

The differential forms containing y^{-3} have poles along γ. A standard way to get rid of the poles on the integration path is to consider $\Gamma_{h,f}$ as a complex curve. Topologically it is a torus from which one point is removed [10]. The differential form $yr^{-1}dr$ is holomorphic on γ. If we deform cycle γ into cycle γ' homological to γ on which functions y and r have neither poles, nor zeroes. Then by Cauchy's theorem [9] the function $\psi(h,f)$ can be defined by the integral (2.4) on γ' instead of γ. After these notes it is clear that the derivatives (2.7) are well defined. We again denote γ' by γ.

3. Proof of the result for the cubic cases. The proof will be made with the help of several lemmas. First, we shall need the functions

$$w_j = \int_\gamma \frac{r^j dr}{y^3}, \qquad j = 0, 1. \tag{3.1}$$

The next lemma gives a representation of D as a quadratic form in w_0, w_1.

Lemma 3. *The determinant D has the following representation*

$$D = -f^2 w_1 (w_1 - 2hw_0/3a). \tag{3.2}$$

P r o o f. We have to express derivatives (2.7) by the integrals w_0, w_1. Obviously $\psi_{hf} = fw_1$. For ψ_{ff} we make the following transformations:

$$\begin{aligned}
\frac{\partial^2\psi}{\partial f^2} &= -\left(\int_\gamma \frac{dr}{ry} + f^2 \int_\gamma \frac{dr}{ry^3}\right) = -\int_\gamma \frac{y^2+f^2}{ry^3}dr \\
&= -\int_\gamma \frac{2hr - 2ar^2}{y^3}dr.
\end{aligned}$$

So,

$$\psi_{ff} = -2hw_1 + 2a\int_\gamma \frac{r^2 dr}{y^3}.$$

Now, we express integrals $\int_\gamma \frac{r^2dr}{y^3}$ and $\int_\gamma \frac{r^3dr}{y^3}$ via w_0 and w_1 in the following way:

$$\int_\gamma \frac{r^2dr}{y^3} = \frac{1}{3}\int_\gamma \frac{dr^3}{y^3} = -\frac{1}{6a}\int_\gamma \frac{d(-2ar^3)}{y^3} =$$

$$= -\frac{1}{6a}\int_\gamma \frac{d(y^2-2hr^2+f^2)}{y^3} = -\frac{1}{3a}\left(\int_\gamma \frac{dy}{y^2} - 2h\int_\gamma \frac{rdr}{y^3}\right) =$$

$$(2h/3a)w_1.$$

Similarly,

$$\int_\gamma \frac{r^3dr}{y^3} = -\frac{1}{6a}\int_\gamma \frac{rd(-2ar^3)}{y^3} = -\frac{1}{6a}\int_\gamma \frac{rd(y^2-2hr^2+f^2)}{y^3} =$$

$$= -\frac{1}{3a}\left(\int_\gamma \frac{rdy}{y^2} - 2h\int_\gamma \frac{r^2dr}{y^3}\right).$$

Integrating first summand by parts, we obtain

$$\int_\gamma \frac{r^3dr}{y^3} = (f^2/a)w_0.$$

Consequently

$$\psi_{hh} = -(f^2/a)w_0, \quad \psi_{ff} = -(2h/3)w_1$$

from where we obtain the representation (3.2). This completes the proof of Lemma 3.

Earlier in [9] was proved that $w_0 < 0$ in U_r. We introduce the function $\delta = w_1/w_0$. Then D and D_1 have the following form

$$D = -f^2w_0^2\delta(\delta - (2h/3a)), \tag{3.3}$$

$$D_1 = -(2/3)hw_0\delta. \tag{3.4}$$

Next, we need some other functions for the study of δ. In order to introduce them, we put the family of elliptic curves $\Gamma_{h,f}$ into the normal form:

$$\Gamma_p = \{(u,v) \in \mathbb{C}^2, \quad v^2 = 2(u^3 - 3u + p)\} \tag{3.5}$$

by the translation $r = x + (h/3a)$ and the rescaling $y = \alpha v$, $x = \beta u$, where

$$\beta = -h/(3a), \quad \alpha^2 = -a\beta^3. \tag{3.6}$$

If we put

$$p(h,f)=\frac{4h^3-27a^2f^2}{2h^3}, \tag{3.7}$$

we get (4.5). In these variables the integrals w_0 and w_1 become

$$w_0=\frac{\beta}{\alpha^3}\int_{\gamma(p)}\frac{du}{v^3},\quad w_1=\frac{\beta}{\alpha^3}\int_{\gamma(p)}\frac{(\beta u+h/3a)du}{v^3}.$$

Introduce the new functions

$$\theta_0(p)=\int_{\gamma(p)}\frac{du}{v^3},\quad \theta_1(p)=\int_{\gamma(p)}\frac{udu}{v^3}$$

and their ratio

$$\rho(p)=\theta_1(p)/\theta_0(p).$$

In these notations we have

$$\delta=\beta\rho+h/3a=(1-\rho)h/3a.$$

Then, in these variables the expressions D and D_1 become

$$D=-f^2\frac{\beta^2}{\alpha^6}\ \frac{h^2}{9a}\theta_0^2(\rho+1)(\rho-1), \tag{3.8}$$

$$D_1=-2\frac{\beta}{\alpha^3}\ \frac{h^2}{9a}\theta_0(1-\rho). \tag{3.9}$$

Note that $p_{|f=0}=2$ and $p_{|f^2=8h^3/27a^2}=-2$. The following result from [5] is crucial for the proof of the theorem.

Lemma 4 (Horozov, [5]). (i) *The function $\rho(p)$ is strictly monotonous decreasing in the interval* $[-2,2]$,
(i) $\rho(-2)=7/5,\ \ \rho(2)=1.$

P r o o f o f T h e o r e m 2. For D and D_1 we have expressions (4.8) and (4.9)

$$D=-f^2\frac{\beta^2}{\alpha^6}\ \frac{h^2}{9a}\theta_0^2(\rho+1)(\rho-1),$$

$$D_1=-2\frac{\beta}{\alpha^3}\ \frac{h^2}{9a}\theta_0(1-\rho).$$

Lemma 4 shows that D and D_1 vanish on the boundary of the set U_r. This finishes the proof of the theorems.

4. Remarks on the perturbations of another cubic cases. The results of the Sections 2 and 3 can easily be expanded on the cases $n = 3,\ -5/3,\ -7/3$. Here we list the keypoints of these cases.

4.1. $n = 3$. $V(r) = ar^4,\ \ a > 0,\ \ U_r = \left\{(h, f) : h > 0,\ 0 < f^2 < \dfrac{4h^{3/2}}{3(3a)^{1/2}}\right\}$. After a change of the variable $x = r^2$, the algebraic curve is

$$y^2 = 2hx - 2ax^3 - f^2.$$

Then

$$D = w_0^2((h/3)\delta - (f^2/4)), D_1 = (2/3)hw_0 \neq 0.$$

After putting the algebraic curve into the normal form we obtain

$$D = -\frac{(3a)^2}{h^2}\theta_0^2\left(\rho - \frac{p}{2}\right).$$

4.2. $n = -5/3$. $V(r) = ar^{-2/3},\ \ a < 0$,
$U_r = \left\{(h, f) : h < 0,\ 0 < f^2 < -(2a/3)(2a/3h)^2\right\}$. After a change of the variable $x = r^{2/3}$, the algebraic curve is

$$y^2 = 2hx^3 - 2ax^2 - f^2.$$

Then, if we put $\beta = -(a/3h)$, $\alpha^2 = h\beta^3$

$$D = -54f^2w_0^2\delta(\delta + 2\beta)),\quad D_1 = aw_0\delta.$$

After putting the algebraic curve into the normal form we obtain

$$D = -54f^2\frac{\beta^6}{\alpha^6}\theta_0^2(\rho + 1)(\rho - 1),\quad D_1 = a^{\frac{\beta^2}{\alpha^3}}\theta_0(\rho - 1).$$

4.3. $n = -7/3$. $V(r) = ar^{-4/3},\ \ a < 0$,
$U_r = \left\{(h, f) : h < 0,\ 0 \leq f^2 < -(4a/3)(a/3h)^{1/2}\right\}$. After a change of the variable $x = r^{2/3}$, the algebraic curve is

$$y^2 = 2hx^3 - 2ax - f^2.$$

Then, if we put $\beta = -(a/3h)^{1/2},\ \ \alpha^2 = h\beta^3$,

$$D = 45w_0^2\beta^4(a\delta + f^2 3/4)),\quad D_1 = 2aw_0 \neq 0.$$

After putting the algebraic curve into the normal form we obtain

$$D = 45a\frac{\beta^7}{\alpha^6}\theta_0^2(\rho - p/2).$$

The proof of Theorem 1 (Theorem 2) proceeds the lines explained above.

PART B. QUARTIC CASES

5. Statement of the results for the quartic cases. As it is mentioned above only the case $n = -5/2$ will be considered in details. For this case the integrals of the motion are:

$$F = p_\varphi = f = \text{const}\,,$$

$$H = \frac{p_r^2}{2} + \frac{f^2}{2r^2} + ar^{-3/2} = h = \text{const}\,. \tag{5.1}$$

Similarly, as in Part A, Section 2 we introduce action variables

$$J_1 = 2\pi f,$$

$$J_2 = \int_{\gamma_2} p_r dr = 2\int_{r_-}^{r_+} \sqrt{(2h - 2ar^{-3/2} - f^2/r^2)}dr, \tag{5.2}$$

where $r_+ > r_-$ are the two roots of the equation

$$h - ar^{-\frac{3}{2}} - \frac{f^2}{2r^2} = 0.$$

After changing variables $z = r^{1/2}$ we put

$$y^2 = 2hz^4 - 2az - f^2. \tag{5.3}$$

Denote the oval of the curve $\Gamma_{h,f} = \{(y,z) : y^2 = 2hz^4 - 2az - f^2\}$ by γ. Then,

$$\psi(h,f) \stackrel{\text{def}}{=} J_2 = \int_\gamma \frac{ydz}{z}.$$

Here $U_r = \left\{f, h < 0,\ 0 \le f^2 < -3a\left(2\sqrt[3]{a/(4h)}\right)\right\}.$

Theorem 3. *For $(h,f) \in U_r$ the determinants (2.5) and (2.6) do not vanish.*

Now, using Lemma 1 and Lemma 2 from Part A, Section 2 and that $\psi_h \not\equiv 0$ in U_r, we shall prove instead of Theorem 3 equivalent to it

Theorem 4. *For $(h, f) \in U_r$ the expressions*

$$D = \psi_{hh}\psi_{ff} - \psi_{hf}^2,$$

$$D_1 = \psi_{ff}$$

do not vanish.

At last, we express the entries of D (D_1) as elliptic integrals. Using the same arguments as in Part A we obtain

$$\frac{\partial^2\psi}{\partial h^2} = -2\int_\gamma \frac{z^7 dz}{y^3},$$

$$\frac{\partial^2\psi}{\partial h \partial f} = 2f\int_\gamma \frac{z^3 dz}{y^3},$$

$$\frac{\partial^2\psi}{\partial f^2} = -\left(\int_\gamma \frac{dz}{zy} + f^2\int_\gamma \frac{dz}{zy^3}\right).$$

6. Proof of the result for the quartic cases. First, we shall need the functions

$$w_j = \int_\gamma \frac{z^j dz}{y^3}, \qquad j = 0, 1. \tag{6.1}$$

Lemma 5. *The determinant D has the following representation*

$$D = \frac{7a^2}{4h^2} w_0(2f^2 w_0 + 3aw_1).$$

Proof. Obviously for the proof we need to express the integrals ψ_{hh}, ψ_{hf}, ψ_{ff} via w_0 and w_1. This may be done by a way similar to that in Lemma 3. So,

$$\psi_{hh} = (a/4h)(5f^2 w_0 + 7aw_1),$$

$$\psi_{hf} = (fa/2h)w_0,$$

$$\psi_{ff} = 3aw_0,$$

from where the representation for D is obtained.

Next, we bring the algebraic curve (5.3) into the normal form $v^2 = (a/2)u^4 + bu^2 + 2ku + 2p$, where $a = \pm 1$, $b = \pm 2$. Using that in our case $b = 0$, we bring our curve into the following normal form

$$v^2 = -(1/2)u^4 + 2u + 2p \tag{6.2}$$

putting $y = \alpha v$, $z = \beta u$, where $\beta = \sqrt[3]{\dfrac{a}{4h}}$, $\alpha^2 = -a\sqrt[3]{\dfrac{a}{4h}}$ and $p = \dfrac{f^2}{\left(2a\sqrt[3]{\frac{a}{4h}}\right)}$.

Note, that when $(h, f) \in U_r$, $p \in (-3/4, 0]$. Denote by

$$\theta_j(p) = \int_{\gamma(p)} \frac{u^j\,du}{v^3}, j = 0, 1.$$

Then, the expressions D and D_1 take the following forms

$$\begin{aligned} D &= \frac{7f^2a^2\beta^2}{2h^2\alpha^6}\theta_0\left(\theta_0 + \frac{3}{4p}\theta_1\right), \\ D_1 &= 3a\beta\theta_0/\alpha^3. \end{aligned}$$

Next, we shall express the integrals θ_0 and θ_1 (and consequently D and D_1) via two Abelian integrals which do not vanish when $p \in (-3/4, 0]$. In order to do this we shall use the Picard-Fuchs equations for the algebraic curve (6.2), which are derived in [9]. Following the notations in [9], we put

$$I_j = \int_{\gamma(p)} vu^j\,du, \quad j = 0, 1, 2\dots$$

It is obvious that $\theta_j = -I_j''$, $j = 0, 1$. ($' = d/dp$)

The Picard-Fuchs equations for the algebraic curve (6.2) are:

$$\begin{bmatrix} I_0'' \\ \\ I_2'' \end{bmatrix} = \frac{1}{\sigma}\begin{bmatrix} -16p^2 & 9 \\ \\ 12p & 16p^2 \end{bmatrix}\begin{bmatrix} I_0' \\ \\ I_2' \end{bmatrix},$$

where $\sigma = 64p^3 + 27$. Actually we need I_1'', so we add $4pI_1'' + 3I_2'' = 0$ (see [8, appendix]) to these equations. Then

$$I_0'' = (-16p^2 I_0' + 9I_2')/\sigma, \tag{6.3}$$

$$I_1'' = -3(3I_0' + 4pI_2')/\sigma. \tag{6.4}$$

Using these equations the expressions D and D_1 become

$$D = \frac{7a^3\beta^3}{4h^2\alpha^6} I_0' I_0'',$$

$$D_1 = -3a\beta I_0''/\alpha^3.$$

Since I_0' and I_2' do not vanish when $p \in (-3/4, 0]$ and $I_0' = T$ ($I_0'' = T'$) where T is the period, the proof of the theorem is reduced to proving that the period T has no critical points when $p \in (-3/4, 0]$.

Lemma 6. *$T' \neq 0$, when $p \in (-3/4, 0]$.*

P r o o f. The period function $T(p)$ satisfies a second order Picard-Fuchs equation [9]

$$\sigma T'' + \sigma' T' + 28p\, T = 0$$

and $x(p) = T'(p)/T(p)$ satisfies a Riccati equation

$$\sigma(x' + x^2) + \sigma' x + 28p = 0. \tag{6.5}$$

We shall prove that $x(p) \neq 0$ when $p \in (-3/4, 0]$. It is easily obtained from [8] that $\lim\limits_{p\to -3/4} x(p) = 7/36 > 0$. Suppose that there exist $p_0 \in (-3/4, 0)$ and $x(p_0) = 0$. It is clear from (6.5) that p_0 is not a critical point with multiplicity and

$$x'(p_0) = -28p_0/(64p_0^3 + 27) > 0.$$

But this is a contradiction because for $p \leq p_0$ $x'(p)$ must be ≤ 0 (fig. 3). Now , let $p_0 = 0$. From the equation (6.2) we obtain

$$T' = I_0'' = I_2'/3_{|p_0=0} \neq 0.$$

This completes the proof of Lemma 6 and Theorem 4.

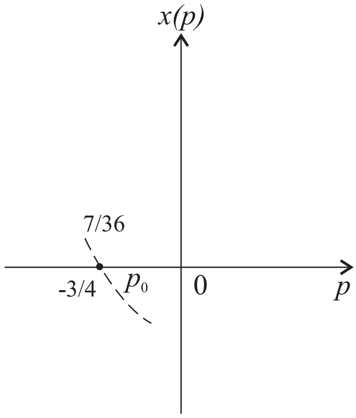

Fig. 3. Possible behaviour of $x(p)$

7. Remark on the perturbation of another quartic case. The results from the Sections 5 and 6 are valid also for the case $n = 5$ ($V(r) = ar^6$, $a > 0$). Here are the keypoints.

$$U_r = \{f, h > 0,\ 0 \le f^2 < 3h\sqrt[3]{h/(4a)}/2\}$$

$$D = \tfrac{1}{4}w_0(-f^2 w_0 + 3hw_1/2).$$

$$D_1 = -3hw_0/4.$$

Then after putting the corresponding algebraic curve into the normal form, we obtain

$$D = -\tfrac{f^2\beta^2}{4\alpha^6}\theta_0\left(\theta_0 + \tfrac{3}{4p}\theta_1\right),$$

$$D_1 = -3h\beta\theta_0/4\alpha^3.$$

Here, the Picard-Fuchs equations and Lemma 6 are applied.

Unfortunately, the condition (1.1) for the quartic cases $n = -1/3$ ($a > 0$) and $n = 3/2$ ($a < 0$) cannot be shown by the present analysis.

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