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BRANCHING PROCESSES WITH IMMIGRATION AND INTEGER-VALUED TIME SERIES

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Abstract. In this paper, we indicate how integer-valued autoregressive time series Ginar(d) of ordre d, $d \geq 1$, are simple functionals of multitype branching processes with immigration. This allows the derivation of a simple criteria for the existence of a stationary distribution of the time series, thus proving and extending some results by Al-Osh and Alzaid [1], Du and Li [9] and Gauthier and Latour [11]. One can then transfer results on estimation in subcritical multitype branching processes to stationary Ginar(d) and get consistency and asymptotic normality for the corresponding estimators. The technique covers autoregressive moving average time series as well.

1. Introduction. Let us consider a multitype branching process $\{\boldsymbol{Z}_n\} = \{(Z_n(1), \dots, Z_n(d))\}$ having d types of particles and an independent immigration component $\{\boldsymbol{I}_n\}$ in each generation. In this Markovian model, which will be denoted BGWI(d), in honor of Bienaymé, Galton and Watson, all particles live one unit of time and then reproduce independently of each other. A particle of type j, $j = 1, 2, \dots, d$, gives rise to random numbers x_1, of particles of type $1, \dots, x_d$ particles of type d, according to an offspring distribution $p^j(\boldsymbol{x})$, where $\boldsymbol{x} = (x_1, \dots, x_d)$. Then $\boldsymbol{Z}_n$ is the vector of particles of each type in the n-th generation. Such models are studied extensively in a book by Mode [23] and its statistical counterpart is reviewed at length by

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Dion [8], for $d \geq 1$, covering the years 1970-1992. See also Badalbaev and Mukhitdinov [5] Athreya and Ney [4] and Sevastyanov [27].

Let then $\{\boldsymbol{I}_n\}$ be the immigration process of i.i.d. random vectors with values in $\mathbb{N}^d$, where $\mathbb{N}$ is the set of non-negative integers. Assume we are given, for each $j = 1, 2, \ldots, d$, independent sequences of i.i.d. random vectors $\left\{\boldsymbol{\xi}^j_{k,n}\right\}$, with values in $\mathbb{N}^d$, having a common offspring distribution $\{p^j(\boldsymbol{x}), \sum_{\boldsymbol{x}} p^j(\boldsymbol{x}) = 1\}$, and independent of $\{\boldsymbol{I}_n\}$. It is customary (though not essential) to let the process start with the first non-zero $\boldsymbol{I}_n$, conveniently labelled $\boldsymbol{I}_0 \equiv \boldsymbol{Z}_0$. Define recursively, for $n \geq 0$, the homogeneous Markov chain $\{\boldsymbol{Z}_n\}$ by

$$\boldsymbol{Z}_{n+1} = \begin{cases} \sum_{i=1}^{d} \sum_{k=1}^{Z_n(i)} \boldsymbol{\xi}^i_{k,n} + \boldsymbol{I}_{n+1}, & \text{if } \boldsymbol{Z}_n \neq \boldsymbol{0} \\ \boldsymbol{I}_{n+1} & \text{if } \boldsymbol{Z}_n = \boldsymbol{0} \end{cases} \tag{1}$$

then $\{\boldsymbol{Z}_n\}$ is a BGWI(d) process. The equality in (1) holds in distribution.

$$\begin{array}{l} \text{Let } \lambda = \mathbb{E}(\boldsymbol{I}_n) \text{ and } \sum(I) = Cov(\boldsymbol{I}_n); \\ \text{denote by } M \text{ the offspring mean matrix, i.e.} \\ M = ((m_{ij}))\,,\ m_{ij} = \sum_x x_j p^j(\boldsymbol{x}) = E\left(\xi^i(j)\right) \end{array} \tag{2}$$

where $\xi^i = (\xi^i(1), \ldots, \xi^i(d))$.

Finally let $\sum_0^i$ be the covariance matrix of the offspring distribution and assume throughout that $\sum_0^i$ and $\sum(I)$ are finite.

Reflecting properties of $\mathbb{E}(\boldsymbol{Z}_n | \boldsymbol{Z}_{n-1})$, several authors (Heyde and Seneta [12], Deistler and Feichtinger [6], Venkataraman [30], Suresh Chandra and Koteeswaran [29], Winnicki [33], Mills and Seneta [21, 22] ...) indicated the analogy with the classical **real-valued** autoregressive time series. A more intimate connection is actually attained by considering **integer-valued** autoregressive time series and it is the purpose of this paper to define and exploit such a connection.

During the period 1978-1992, integer-valued autoregressive time series have been introduced and studied quite independently of BGWI(d) processes (see Jacob and Lewis [13], Steutel and Van Harn [28], McKenzie [18]–[20], Al-Osh and Alzaid [1]–[3], Du and Li [9], Gauthier and Latour [11]). The general integer-valued autoregressive process (GINAR) in its present formulation due to Gauthier and Latour is defined in the following way:

Let $\{\varepsilon_n\}$ be i.i.d. random variables with values in $\mathbb{N}$. With the help of independent sequences of i.i.d. random variables $\left\{\xi^j_{k,n}\right\}$, with values in $\mathbb{N}$, and all independent of $\{\varepsilon_n\}$, define recursively the GINAR(d) process $\{X_n\}$ by

(3) $$X_n = \sum_{i=1}^{d} \sum_{k=1}^{X_{n-i}} \xi_{k,n}^{i} + \varepsilon_n, \quad n \geq d,$$

where the equality holds in distribution.

Letting $\alpha_j = \mathbb{E}\left(\xi_{k,n}^{j}\right)$, $0 \leq \alpha_j < \infty$, $j = 1, \ldots, d$ and $\alpha_d \neq 0$, denote by $\alpha_j \circ X$ the expression $\sum_{k=1}^{X} \xi_{k,n}^{j}$. Then instead of (3) one may use the more suggestive autoregressive form:

(4) $$X_n = \sum_{i=1}^{d} \alpha_i \circ X_{n-i} + \varepsilon_n, \quad n \geq d.$$

Suppose that $(X_0, X_1, \ldots, X_{d-1})$ has been defined with some appropriate joint distribution on $\mathbb{N}^d$. Let $\lambda = \mathbb{E}(\varepsilon_n)$, $0 < \lambda < \infty$ and assume also $0 < Var\ \varepsilon_n < \infty$ as well as $0 < Var\left(\xi_{k,n}^{j}\right) < \infty$.

We will show that the GINAR(d) model (3) is a particular functional of the BGWI(d) model (1) and use this characterization to derive a general necessary and sufficient condition for the existence of a stationary distribution for $\{X_n\}$. Furthermore we will use statistical results on BGWI(d) processes to get consistent and asymptotically normal estimators for the means and covariance matrices of the GINAR(d) model. When $d = 1$, the two models are the same; hence all known results on BGWI($d = 1$) are valid for the GINAR(1) process and vice-versa, in particular those concerning autocorrelations, as well as those for non stationary (transient and null recurrent) processes. We conclude with an extension to autoregressive moving average models.

2. GINAR(d) viewed as BGWI(d). Let δ_{ij} be Kronecker's delta and put $e_j = (\delta_{j1}, \ldots, \delta_{jd})$ for each $j = 1, 2, \ldots, d$. By convention $e_{d+1} = \mathbf{0}$. Let $\{X_n\}$ be a GINAR(d) process, as defined by (3).

By taking $\boldsymbol{I}_n = (\varepsilon_n, 0, \ldots, 0)$,

(5) $$\boldsymbol{\xi}_{k,n}^{i} = \left(\xi_{k,n}^{i}, 0, \ldots, 0\right) + e_{i+1},$$

for $i = 1, 2, \ldots, d$ and defining $\boldsymbol{Z}_n = (Z_n(1), \ldots, Z_n(d))$ with $Z_n(i) = X_{n-i+1}$ for $1 \leq i \leq d$, it is easy to check that (1) is satisfied for $n \geq d - 1$. Hence $\{\boldsymbol{Z}_n\}_{n \geq d-1}$ is a BGWI(d) process and $X_n = Z_n(1)$; here $\boldsymbol{Z}_{d-1} = (X_{d-1}, X_{d-2}, \ldots, X_0)$ is to be seen as the initial state of the process $\{\boldsymbol{Z}_n\}$. Furtherinore, the mean matrix of the offspring distribution $M = ((m_{ij}))$ defined in (2) is

$$M = \begin{pmatrix} \alpha_1 & 1 & & 0 \\ \alpha_2 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 1 \\ \alpha_d & \vdots & \ddots & 0 \end{pmatrix} \quad \text{i.e} \quad m_{ij} = \begin{cases} \alpha, & j = 1 \\ 1, & j = i+1, \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

for $i = 1, 2, \ldots, d$, since $(m_{i1}, \ldots, m_{id}) = \mathbb{E}(\xi^i_{k,n}) = (\alpha_i, 0, \ldots, 0) + e_{i+1}$.

Let us summarize these remarks in the following proposition.

Proposition A. *Given a* GINAR(d) *process* $\{X_n\}$ *as defined by* (3), *the process* $\{\boldsymbol{Z}_n\}_{n \le d-1}$ *defined by* $Z_n(j) = X_{n+1-j}$, $1 \le j \le d$ *is a* BGWI(d) *process with immigration random vectors* $\{\boldsymbol{I}_n = (\varepsilon_n, 0, \ldots, 0)\}$ *and matrix of offspring means* M *given by* (6).

Note 1. Since $\alpha_d \neq 0$, M is irreducible, i.e. each type-offspring may be a descendant of any type-parent.

The matrix M in (6) is well known and arises in a variety of contexts of deterministic or probabilistic growth of population. Had Euler known the matrix notation, he may well have given it in his 1767 memoir no. 334. The first explicit reference to M appears to be the paper by Lewis [17] and the independent study by Leslie [15, 16] in a context of deterministic growth of population (without immigration). Pollard [25] studied the probabilistic analog of Leslie's model, a multitype branching process with M defined in (6); his analysis relies much on the direct matrix product and does not exploit the BGWI process.

This matrix (or particular cases of it) appears again in works by Steutel and Van Harn [28], McKenzie [20], Al-Osh and Alzaid [3] and specially Du and Li [9], who all seem unaware of the Lewis-Leslie-Pollard results. We now state a result due to Lewis and Leslie.

Proposition B. (i) *The characteristic polynomial of* M *is* $x^d - \sum_{k=1}^{d} \alpha_k x^{d-k} = 0$ *and*

(ii) *If* ρ *is the maximal eigenvalue of* M, *then* $\sum_1^d \alpha_k \lesseqqgtr 1$ *if and only if respectively* $\rho \lesseqqgtr 1$.

Note 2. The simple condition $\sum \alpha_k < 1$ implies that all roots of the characteristic polynomial of M lie inside the unit circle. This applies as well to **real-valued** autoregression time series with positive coefficients.

Let $\{\boldsymbol{Z}_n\}$ be a BGWI(d) process and M be its offspring mean matrix. M is said **irreducible** if $\forall i, j$, $\exists n = n_{ij}$ such that $m_{ij}(n) > 0$. where $m_{ij}(n)$ is the (i, j)-th entry of M^n. M is said **reducible** otherwise.

When $\exists n$ such that $m_{ij}(n) > 0$, $\forall i, j$ and n does not depend on (i, j), we say that M is **positively regular** and write $M^n \gg 0$ to mean that $m_{ij}(n) > 0\ \forall i, j$.

In order to avoid trivial cases, let us call the process $\{\boldsymbol{Z}_n\}$ **singular**, when M is irreducible, if each particle has exactly one offspring. When M is reducible, the set of its types partition itself into several (irreducible) classes $\{c_a\}$. In that situation, the process is called **singular** if at least one of the subprocesses $\{\boldsymbol{Z}_n(a) = \{Z_n(j), j \in c_a\}$ is singular.

The next result establishes the connection between the recurrence of any non-singular BGWI(d) $\{\boldsymbol{Z}_n\}$ and $\rho = \rho(M)$, the largest eigenvalue of M. It is a slight extension of theorem 7.1 (ii) in Mode ([23], p. 84) and its proof require the following trivial lemma:

Lemma 1. *Let* $\left\{\boldsymbol{Z}_n^{(\boldsymbol{i})}\right\}_0^\infty$ *be a Markov chain on* $\mathbb{N}^d$, *the superscript* $(\boldsymbol{i})$ *indicating that* $\boldsymbol{Z}_0 = \boldsymbol{i}$. *Suppose that* $\forall \boldsymbol{i} \in \mathbb{N}^d$, $\boldsymbol{Z}^{(\boldsymbol{i})} \xrightarrow{d} \boldsymbol{Z}$, *a proper random variable whose distribution does not depend on* $\boldsymbol{i}$. *Let* S *be the support of* $\boldsymbol{Z}$, $S \subset \mathbb{N}^d$. *Then* S *is irreducible, aperiodic and positive recurrent, while* $S' = \mathbb{N}^d - S$ *is transient.*

Recall that for aperiodic positive recurrent Markov chains, in view of Griffeath's maximal coupling theorem, (see also O'Brien, 1974 [24]) in as much as limiting results are concerned, there is no loss of generality in assuming that the chain is in stationary regime (which is the case if $P(\boldsymbol{Z}_0 = \boldsymbol{j}) = P(\boldsymbol{Z} = \boldsymbol{j})$, $\forall \boldsymbol{j} \in S$).

Proposition C. *Let* $\left\{\boldsymbol{Z}_n^{(\boldsymbol{i})}\right\}$ *be a* BGWI*(d) process as defined in* (1). *Suppose it is non-singular and* $\boldsymbol{0} \neq \mathbb{E}(\boldsymbol{I}_n) = \lambda$ *is finite. If* $\rho < 1$, *then* $\boldsymbol{Z}_n^{(\boldsymbol{i})} \xrightarrow{d} \boldsymbol{Z}$, $\forall \boldsymbol{i} \in \mathbb{N}^d$ *and* $\mathbb{E}\left(\boldsymbol{Z}_n^{(\boldsymbol{i})}\right) \to \mathbb{E}(\boldsymbol{Z}) = \lambda(I - M^{-1})$ *which is finite. Furthermore, the support of* Z *is irreducible, aperiodic and positive recurrent.*

P r o o f o f P r o p o s i t i o n C. Following Mode [23] p. 84, (1) can also be written as

$$\boldsymbol{Z}_n^{(\boldsymbol{i})} = \sum_{\nu=0}^{n-1} \boldsymbol{Y}(n, \nu) + \boldsymbol{X}_n^{(\boldsymbol{i})}, \tag{7}$$

where $\boldsymbol{X}_n^{(\boldsymbol{i})}$ is the n-th generation of a multitype branching process without immigration, started at $\boldsymbol{Z}_0 = \boldsymbol{X}_0 = \boldsymbol{i}$ and $\boldsymbol{Y}(n, \nu)$ is a ν-th generation multitype branching process without immigration with random initial vector $\boldsymbol{I}_{n-\nu}$, all vectors in the right-hand-side of (7) being independent. Let $Y(\nu) = Y(2\nu, \nu)$. Clearly $Y(\nu)$ is distributed as $Y(n, \nu)$, for $\nu = 0, 1, \ldots$

When M is positively regular, (the case treated explicitely by Mode [23] p. 84-86), $\boldsymbol{X}_n^{(\boldsymbol{i})} \xrightarrow{a.s.} \mathbf{0}$, $\forall \boldsymbol{i}$ (even when (1) holds in distribution only). When M is irreducible but not positive regular, (the periodic case), again $\boldsymbol{X}_n^{(\boldsymbol{i})} \xrightarrow{a.s.} \mathbf{0}$, $\forall \boldsymbol{i}$ in view of his theorem 2.1, (p. 54). Finally when M is reducible, theorem 3.1 (p. 65) imply $\boldsymbol{X}_n^{(\boldsymbol{i})} \xrightarrow{a.s.} \mathbf{0}$, $\forall \boldsymbol{i}$.

Since $E\left(\sum_{\nu=0}^{n-1} Y(\nu)\right) \to \lambda \sum M^n = \lambda(\boldsymbol{I} - M)^{-1}$ which is finite if $\rho < 1$ and λ exists, one has

$$\boldsymbol{Z} = \sum_{\nu=0}^{\infty} Y(\nu)$$

is finite $a.s.$ From (7),

$$\boldsymbol{Z}_n^{(\boldsymbol{i})} \xrightarrow{d} \boldsymbol{Z}$$

and also $E\left(\boldsymbol{Z}_n^{(\boldsymbol{i})}\right) \to \mathbb{E}(\boldsymbol{Z})$ since $E\boldsymbol{X}_n^{(\boldsymbol{i})} = \boldsymbol{i} \cdot M^n \to 0$ when $\rho < 1$. Lemma 1 then provides the last part of Proposition C.

3. A criteria for the stationarity of GINAR processes. We now prove that if $\sum \alpha_k < 1$, any GINAR(d) process admits a unique limiting stationary distribution. Of course any stationary GINAR(d) process must have $\sum \alpha_k < 1$ (take expectations in (3)).

Theorem 1. *Let $\{X_n\}_{n\geq 0}$ be a GINAR(d) process as defined in* (3) *and let $\boldsymbol{Z}_n = (X_n, X_{n-1}, \ldots, X_{n-d+1})$ for $n \geq d-1$. If $\sum \alpha_k < 1$ and $0 < \mathbb{E}(\varepsilon_n) = \lambda < \infty$, then $\boldsymbol{Z}_n \xrightarrow{d} \boldsymbol{Z}$ for any initial distribution on $\boldsymbol{Z}_{d-1} = (X_{d-1}, \ldots, X_0)$ and*

$$\mathbb{E}(\boldsymbol{Z}_n) \to \mathbb{E}(\boldsymbol{Z}) = (\mu_1, \ldots, \mu_d)$$

where

$$\mu_i = \lambda\left(1 - \sum \alpha_k\right)^{-1}, \quad i = 1, 2, \ldots, d.$$

Furthermore if S is the support of $\boldsymbol{Z}$, then $\{\boldsymbol{Z}_n\}_{n\geq d-1+n_0}$ is an aperiodic irreducible positive recurrent Markov chain on S, for some finite constant n_0.

Proof. If $\sum \alpha_k < 1$, then $\rho < 1$ by Proposition B. The conclusion follows from Proposition A and C. In this case

$$(I - M)^{-1} = ((c_{ij})) \cdot \left(1 - \sum_1^d \alpha_k\right)^{-1}$$

where

$$c_{1j} = 1, \ \forall j = 1, 2, \ldots, d$$

$$c_{ij} = 1 - \sum_{1}^{i-1} \alpha_k, \quad \text{for} \quad j \geq i > 1,$$

and

$$c_{ij} = \sum_{1}^{d} \alpha_k \quad \text{for} \quad j < i, \ i = 2, \ldots, d.$$

It is easy to check that $E(\boldsymbol{Z}) = (\mu_1, \ldots, \mu_d)$ with $\mu_i = \lambda \left(1 - \sum \alpha_k\right)^{-1}$.

Let $n_0 = \inf\{n : P(\boldsymbol{V}_n^i = 0) > 0, \ \forall i = 1, \ldots, d\}$ where $\boldsymbol{V}_n^i$ is a n-th generation of a multitype branching process without immigration, starting with $\boldsymbol{V}_0 = e_i$. To prove that from $(n_0 + d - 1)$ onwards, $\boldsymbol{Z}_n^i$ has its support on S, $\forall \boldsymbol{i}$, proceed now as in the proof of lemma 1 of Quine and Durham [26] even though M is not necessarily positive regular when $d > 1$. It is sufficient to prove that $\forall \boldsymbol{i} \in S$, $\exists \boldsymbol{j}$, with $\boldsymbol{j} \geq \boldsymbol{i} + \boldsymbol{1}$ which is in S (or equivalently, such that $\boldsymbol{i} \rightsquigarrow \boldsymbol{j}$).

For that, let $K = \inf\{j \geq 1 : P(\varepsilon_n = j) > 0\}$ and $\gamma = \inf\{j \geq 1 : P(\boldsymbol{V}_n^d(1) > 0\}$ where $\boldsymbol{V}_n^d(1)$ is the first coordinate of $\boldsymbol{V}_n^d$. K and γ are well defined since $P(\varepsilon_n = 0) < 1$ and $\alpha_d > 0$. Since $\boldsymbol{Z}_n = (X_n, \ldots, X_{n-d+1})$ it is possible to move in one step from any $\boldsymbol{i} \in S$, $\boldsymbol{i} = (i_1, \ldots, i_d)$ to $(j_1, i_1, \ldots, i_{d-1})$ for some $j_1 \geq \gamma i_d + K$, in one more step to $(j_2, j_1, i_1, \ldots, i_{d-2})$, with $j_2 \geq \gamma i_{d-1} + K$, etc... After d steps, $(j_d, \ldots, j_1)$ can be reached, with $j_k \geq K \geq 1$. Obviously for some n,

$$\boldsymbol{i} \rightsquigarrow \boldsymbol{j} = (j_n, \ldots, j_{n-d+1}) \quad \text{with} \quad \boldsymbol{j} \geq \boldsymbol{i} + \boldsymbol{1}.$$

In terms of the GINAR(d) process itself, one has:

Corollary 1. *Let $\{X_n\}$ be a* GINAR(d) *process with $\sum_1^d \alpha_k < 1$ and $0 < \lambda < \infty$. Then $\{X_n\}$ admits a unique limiting stationary distribution and there exists a distribution on $(X_{d-1}, \ldots, X_0)$ (the distribution of $\boldsymbol{Z}$) such that $\{X_n\}_{n \geq d-1}$ is strictly stationary. Moreover, $\mathbb{E}(X) = \lambda(1 - \sum \alpha_k)^{-1} = \lim \mathbb{E}(X_n)$, where X has the limiting distribution.*

Corollary 1 contains theorem 2.1 of Du and Li [9] who considered only the case where the random variables $\left(\xi_{k,n}^i\right)$ in (3) are Bernoulli. At the same time we provide a simpler criteria which can easily be checked even if the α_k are unknown (see section 4).

Note 3. If the process $\{\boldsymbol{Z}_n\}$ is viewed as started from the infinite past, it is obvious that equation (7) would become

$$\boldsymbol{Z}_n^{(\boldsymbol{i})} = \sum_{\nu=0}^{\infty} \boldsymbol{Y}(n,\nu) \tag{8}$$

which is distributed as $\boldsymbol{Z}$, confirming that $\boldsymbol{Z}_n^{(\boldsymbol{i})}$ is in stationary regime. Equation (8) can be written more suggestively as:

$$\boldsymbol{Z}_n^{(\boldsymbol{i})} \stackrel{d}{=} \sum_{\nu=0}^{\infty} M^{\nu} \circ \boldsymbol{I}_{n-\nu} \stackrel{d}{=} \sum_{\nu=0}^{\infty} M^{\nu} \circ \boldsymbol{I}_{\nu} \tag{9}$$

where

$$M^{\nu} \circ \boldsymbol{Y} = \sum_{i=1}^{d} \sum_{k=1}^{Y(i)} \zeta_{k,\nu}^{i},$$

$\zeta_{k,\nu}^{i}$ being a ν-th generation multitype branching process with initial vector $\boldsymbol{e}_i$, and $E\left(\boldsymbol{\xi}_{k,\nu}^{i}\right) = (m_{i1}(\nu), \ldots, m_{id}(\nu))$.

This provides a representation for $\{\boldsymbol{Z}_n\}$ in terms of the immigrations $(\boldsymbol{I}_n)$ and the operator $\circ$, when $\{\boldsymbol{Z}_n\}$ is initiated from the infinite past. Since $X_n = Z_n(1) = \boldsymbol{Z}_n \circ \boldsymbol{e}_1'$, where $\boldsymbol{e}'$ means *transpose of* $\boldsymbol{e}$, the representation for the GINAR(d) process is

$$X_n = \sum_{\nu=0}^{\infty} \sum_{i=1}^{d} [m_{i1}(\nu) \circ \varepsilon_{n-\nu}]. \tag{10}$$

Expression (10) is similar and yet different from that of Al-Osh and Alzaid [3] (p. 318) (except when $d = 1$).

Example. Bernoulli-Poisson GINAR(d).

Let $f^i(s_1) = \alpha_i s_1 + \overline{\alpha}_i$, $\overline{\alpha}_i = 1 - \alpha_i$, for $0 \leq \alpha_i \leq 1$, $i = 1, \ldots, d$. Suppose the GINAR(d) $\{X_n\}$ is in stationary regime ($\sum_{i=1}^{d} \alpha_i < 1$). Suppose ε_n is Poisson(λ). Since $\alpha \circ \varepsilon_n$ is Poisson($\alpha\lambda$) when the offspring distribution is Bernoulli(α), one obtains directly from Equation (10) that

$$X_n \text{ is Poisson} \left(\lambda \sum_{\nu=0}^{\infty} m_{11}(\nu) \right).$$

From the general explicit expression for $(\boldsymbol{I}-\boldsymbol{M})^{-1}$, $\sum_{\nu=0}^{\infty} m_{11}(\nu) = \left(1 - \sum_{i=1}^{d} \alpha_i\right)^{-1}$.
Thus, X_n is Poisson$\left(\lambda \left(1 - \sum_{i=1}^{d} \alpha_i\right)^{-1}\right)$. Looking back at Equation (3), this provides an example where the sum of several **dependent** Poisson random variables is itself Poisson.

4. Estimation in GINARD(d). In the last two decades, numerous results have been obtained in estimation and hypothesis testing for BGWI(d) processes in all instances $\rho \lesseqgtr 1$. (For a general review, see Dion [8]). In view of our Proposition A, they apply in particular to GINAR(d) processes. A translation of all these results might prove tedious, we shall be satisfied with the fofiowing comments.

When $\rho > 1$, $(\sum \alpha_k > 1)$, under the hypotheses of Proposition C, $\{\boldsymbol{Z}_n\}$ is transient and behaves as a BGWI(d) process does on the set of non-extinction. In particular for the GINAR(d) process, $\frac{X_n}{X_{n-1}} \xrightarrow{a.s.} \rho$. Furthermore, there cannot exist a consistent estimator for $\lambda = E(\varepsilon_n)$.

The case $\rho = 1$ is the most difficult, since $\{\boldsymbol{Z}_n\}$ could be transient or null recurrent, when second moments are finite. When $d = 1$, $\rho = \alpha = 1$, Wei and Winnicki [31, 32] provided conditional least squares estimators which are consistent for ρ and λ. When the process is null recurrent, consistent estimators for $Var(\xi)$ and $Var(\varepsilon)$ are possible while when $\{X_n\}$ is transient, no parameter of the immigration distribution, except λ, can have a consistent estimator.

When $\rho < 1$, which is the case of stationary GINAR(d) processes $(\sum \alpha_k < 1)$, the results of Quine and Durham [26] and Badalbaev and Mukhitdinov [5] apply to provide estimators for the α_k, λ and the autocorrelations. Extending Quine, Durham's results to matrices M not necessarily positive regular, one has:

Proposition D. *Let* $\{\boldsymbol{Z}_k\}$ *be the* BGWI(d) *process associated with the* GINAR(d) *process. Let*

$$\hat{\boldsymbol{\mu}}_n = \frac{\boldsymbol{S}_n}{n} = \frac{\sum_1^n \boldsymbol{Z}_k}{n},$$

$$\hat{M}_n = \left[\sum (\boldsymbol{Z}_k - n^{-1}\boldsymbol{S}_n)(\boldsymbol{Z}_k - n^{-1}\boldsymbol{S}_n)'\right]^{-1} \left[\sum \boldsymbol{Z}_k(\boldsymbol{Z}_{k+1} - n^{-1}\boldsymbol{S}_n)'\right],$$

and

$$\hat{D}_n = n^{-1}\left[\sum (\boldsymbol{Z}_k - n^{-1}S_n)(\boldsymbol{Z}_k - n^{-1}S_n')\right].$$

Assume M is finite $\rho < 1$ and $0 < |\lambda| < \infty$. Let μ be the mean of the stationary distribution and D be its covariance matrix. Then

(i) $\hat{\boldsymbol{\mu}}_n \to \boldsymbol{\mu}$ *a. s.*

(ii) *If the variances are finite,*

$$\hat{M}_n \to M a.s., \ \hat{D}_n \xrightarrow{a.s.} D$$

$$\sqrt{n}(\mu_n - \mu) \xrightarrow{\mathcal{L}} N\left(0, \sum \mu\right)$$

and

(iii) *If third moments exits,*

$$\sqrt{n}\left(\hat{\mathcal{M}}_n - \mathcal{M}\right) \xrightarrow{\mathcal{L}} N\left(0, \sum \mathcal{M}\right)$$

where $\hat{\mathcal{M}}_n$ is the row vector of the rows of $\mathcal{M}_n$ and $\mathcal{M}$ is the row vector of the rows of $\mathcal{M}$, given in dictionary order. $\sum_\mu$ and $\sum \mathcal{M}$ are covariances matrices given by the authors.

These results apply to GINAR processes provided we know that $\rho < 1$ i.e. $\sum \alpha_k < 1$. It is a difficult problem to provide a test that $\rho > 1$, $\rho = 1$ or $\rho < 1$, or equivalently to provide a confidence interval for ρ that would be valid whether $\rho \lesseqqgtr 1$. When $d = 1$, this has been solved, under the assumption that a BGWI(d) (or a GINAR(d)) is an appropriate model for the data. To test that such a model is appropriate (without the prior knowledge about ρ, $\rho \lesseqqgtr 1$) is even more problematic.

5. Autoregressive and moving average processes. These results can be extended to autoregressive moving average processes GINARMA$(d, q)\{X_n\}$, defined by

$$X_n = \sum_{i=1}^{d} \sum_{k=1}^{X_{n-i}} \xi^i_{k,n} + \sum_{i=1}^{q} \sum_{k=1}^{\varepsilon_{n-i}} \zeta^i_{k,n} + \varepsilon_n, \tag{11}$$

where $\left\{\xi^i_{k,n}\right\}$ are independent and, for each i, identically distributed on $\mathbb{N}$, with $\alpha_i = \mathbb{E}\left(\xi^i_{k,n}\right)$, $\left(\xi^i_{k,n}\right)_k$ being independent of X_{n-i}; the $\left\{\zeta^i_{k,n}\right\}$ are independent of $\left\{\xi^i_{k,n}\right\}$, have the same properties, with $\beta_i = \mathbb{E}\left(\zeta^i_{k,n}\right)$. The (ε_k) are i.i.d. with values in $\mathbb{N}$, and independent of all the other variables. Let $\lambda = E(\varepsilon_k)$ be finite and positive,

$$\alpha_j \geq 0, \forall j, \alpha_d \neq 0, \beta_j \geq 0, \forall j, \beta_q \neq 0.$$

Equation (11) can be written more suggestively as

(12) $$X_n = \sum_1^d \alpha_i \circ X_{n-i} + \sum_1^q \beta_q \circ \varepsilon_{n-i} + \varepsilon_n.$$

Again this corresponds to a BGWI$(d+q)$ process $\{\boldsymbol{Z}_n\}$, where

(13) $$\boldsymbol{Z}_n = \sum_{i=1}^{d+q} \sum_{k=1}^{Z_{n-1}(i)} \boldsymbol{\xi}_{k,n}^i + \boldsymbol{I}_n$$

where

(14) $$\begin{aligned} \boldsymbol{Z}_n &= (Z_n(1), \dots, Z_n(d+q)), \quad Z_{n-1}(i) = \begin{cases} X_{n-i}, & i \le d \\ \varepsilon_{n-i}, & i > d \end{cases} \\ \boldsymbol{\xi}_{k,n}^i &= (\xi_{k,n}^i, 0, \dots, 0) + e_{i+1},\ i < d \\ &= (\xi_{k,n}^i, 0, \dots, 0),\ i = d \\ &= (\zeta_{k,n}^i, 0, \dots, 0) + e_{i+1},\ i \ge d+1,\ (\boldsymbol{e}_{d+q+1} \equiv \boldsymbol{0}) \end{aligned}$$

and finally,

(15) $$\boldsymbol{I}_n = (\varepsilon_n, 0, \dots, \varepsilon_n, \dots, 0)$$

the ε_n being the value of the first and $(d+1)$-th coordinates.

The mean matrix $M = ((m_{ij}))$ corresponding to (13) is

(16) $$\begin{aligned} m_{i1} &= \alpha_i && \text{if } i \le d \\ m_{i+d,1} &= \beta_i && \text{if } i = 1, 2, \dots, q \\ m_{i,i+1} &= 1 && \text{if } i \ne d, i = 1, 2, dots, d+q-1 \\ m_{ij} &= 0 && \text{otherwise.} \end{aligned}$$

This is a reducible matrix, containg in its left upper corner the submatrix $M(1)$ studied previously in (6). The largest eigenvalue of M, $\rho(M)$ is equal to $\rho(M(1)) = \rho$. Consequently $\rho < 1 \iff \sum \alpha_k < 1$, whatever the values of (β_k).

In complete analogy to Corollary 1, one has

Corollary 2. *Let* $\{X_n\}$ *be a* GINARMA(d,q) *process as defined in* (11) (*or* (12)), *with* $\sum_1^d \alpha_k < 1$, $\alpha_d \neq 0$, $\beta_q \neq 0$, *and* $0 < \lambda < \infty$.

Then $\{X_n\}$ *admits a unique limiting stationary distribution and there exists a distribution on* $(X_{d-1}, \ldots, X_0)$ *such that* $\{X_n\}_{n \geq d-1}$ *is strictly stationary. Moreover,* $\mathbb{E}(X) = \lambda(1+\sum \beta_k)(1-\sum \alpha_k)^{-1} = \lim \mathbb{E}(X_n)$, *where* X *has the stationary distribution.*

Consistency of Quine, Durham's estimators follow readily from the ergodic theorem, and asymptotic normality of $\hat{\boldsymbol{\mu}}_n$ and $\hat{\mathcal{M}}_n$ is a consequence of Proposition D, whose conclusions remain valid for GINARMA(d,q) processes with $\sum_{k=1}^{d} \alpha_k < 1$.

5. Conclusion. Using the relationship between GINAR(d) (or GINARMA(d,q) and BGWI(d) processes, we gave necessary and sufficient conditions for the time series to admit a unique limiting stationary distribution. By showing that Quine and Durham's technique [26] apply even though our matrix $\boldsymbol{M}$ in (6) or (16) is not necessarily positive regular we provided consistent and asymptotically normal estimators for the parameters involved. This unifies and improves on several previous results by Al-Osh and Alzaid, McKenzie, Du and Li, Gauthier and Latour.

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