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SOLUTIONS OF ANALYTICAL SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper are examined some classes of linear and non-linear analytical systems of partial differential equations. Compatibility conditions are found and if they are satisfied, the solutions are given as functional series in a neighborhood of a given point ($x = 0$).

1. Introduction. This paper is a continuation of the papers [2] – [5], and we will give a brief view of them.

In the paper [2] it was found a formula for the k-th covariant derivative. Further that formula was generalized for $k \in \mathbb{R}$. Especially, if $k = -1$ it yields to a general solution for a system of linear differential equations [3]. In the paper [4] is given an application of [3] for solving the Frenet equations. In [5] two main theorems are proved. The first theorem gives the solution of an analytical non-homogeneous linear system of differential equations of order k of n equations and n unknown functions. The second theorem gives the solution of a non-linear analytical system of differential equations (of the first order) of n equations and n unknown functions.

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In this paper we will prove two main theorems, considering linear and non-linear systems of partial differential equations. Without loss of generality, we will find the required solutions in a neighborhood of the point $(0,\dots,0)$. To the author's knowledge there are not similar results proved by other authors.

The results of this paper have applications in the differential geometry [6], in studying the non-linear connections [1]. For example the compatibility conditions in this paper are nothing but vanishing of the curvature tensor of the corresponding connections. If the systems of partial differential equations considered in this paper are tensor equations, then the obtained solutions also have tensor character.

2. Homogeneous system of linear partial differential equations. Let us consider the following system

$$\frac{\partial y_r}{\partial x_u}+\sum_{s=1}^{n} f_{rsu}y_s=0 \qquad (1\le r\le n, 1\le u\le k) \tag{2.1}$$

of unknown functions $y_1,\dots,y_n$ of k variables $x_1,\dots,x_k$ and f_{rsu} are given analytical functions of $x_1,\dots,x_k$, regular in a neighborhood of $(0,\dots,0)$. In order to consider the compatibility conditions, we introduce the following functions

$$R_{tsuv}=\frac{\partial f_{tsv}}{\partial x_u}-\frac{\partial f_{tsu}}{\partial x_v}+\sum_{p=1}^{n} f_{tpu}f_{psv}-\sum_{p=1}^{n} f_{tpv}f_{psu}. \tag{2.2}$$

$$(1\le u,v\le k, 1\le t,s\le n)$$

If (2.1) is an integrable system for arbitrary initial conditions, then using that

$$\frac{\partial}{\partial x_v}\frac{\partial y_r}{\partial x_u}=\frac{\partial}{\partial x_u}\frac{\partial y_r}{\partial x_v}$$

and the system (2.1), it is easy to obtain that

$$R_{tsuv}\equiv 0 \tag{2.3}$$

$1\le u,v\le k$ and $1\le t,s\le$n. Conversely, it is known that if (2.3) are satisfied, then the system (2.1) is integrable. Indeed this assertion also follows from the Theorem 2.1.

Theorem 2.1. *Let the system* (2.1) *with the initial conditions* $y_s(0,\dots,0)=C_s$ *(*$1\le s\le n$*) be given and the compatibility conditions* (2.3) *be satisfied. Then there exist functions* $P_{ts}^{<w_1,\dots,w_k>}(x_1,\dots,x_k)$, $w_1,\dots,w_k\in\mathbb{N}_0$ *and* $1\le t,s\le n$, *such that*

$$P_{ts}^{<0,\dots,0>}=\delta_{ts}; \tag{2.4a}$$

(2.4b) $$P_{ts}^{<w_1,\ldots,w_u+1,\ldots,w_k>} = \frac{\partial}{\partial x_u} P_{ts}^{<w_1,\ldots,w_k>} + \sum_{p=1}^{n} f_{tpu} P_{ps}^{<w_1,\ldots,w_k>}$$

and the solution of (2.1) *in a neighborhood of* $(0,\ldots,0)$ *is given by*

(2.5) $$y_r = \sum_{s=1}^{n}\sum_{w_1=0}^{\infty}\sum_{w_2=0}^{\infty}\cdots\sum_{w_k=0}^{\infty}\frac{(-x_1)^{w_1}}{w_1!}\cdot\frac{(-x_2)^{w_2}}{w_2!}\cdots\frac{(-x_k)^{w_k}}{w_k!}\cdot P_{rs}^{<w_1,\ldots,w_k>}C_s.$$

$$(1 \le r \le n)$$

This solution is unique with the given initial conditions in a neighborhood of $(0,\ldots,0)$.

P r o o f. Let us suppose that the system (2.1) is given and the compatibility conditions (2.3) are satisfied. In order to prove that there exist functions

$$P_{ts}^{<w_1,\ldots,w_k>}(x_1,\ldots,x_k) \quad (w_1,\ldots,w_k \in \mathbb{N}_0, 1 \le t,s \le n)$$

such that (2.4a) and (2.4b) are satisfied, it is sufficient to prove that

(2.6) $$P_{ts}^{<w_1,\ldots,w_u^{(2)}+1,\ldots,w_v^{(1)}+1,\ldots,w_k>} = P_{ts}^{<w_1,\ldots,w_u^{(1)}+1,\ldots,w_v^{(2)}+1,\ldots,w_k>}$$

for each $t,s \in \{1,\ldots,n\}$ and $u,v \in \{1,\ldots,k\}, u \ne v$, where the notations (1) and (2) show the order of the two increased indices. In fact

$$P_{ts}^{<w_1,\ldots,w_u^{(2)}+1,\ldots,w_v^{(1)}+1,\ldots,w_k>} = \frac{\partial}{\partial x_u} P_{ts}^{<w_1,\ldots,w_v+1,\ldots,w_k>} +$$

$$+ \sum_{p=1}^{n} f_{tpr} P_{ps}^{<w_1,\ldots,w_v+1,\ldots,w_k>} =$$

$$= \frac{\partial}{\partial x_u}\left[\frac{\partial}{\partial x_v} P_{ts}^{<w_1,\ldots,w_k>} + \sum_{q=1}^{n} f_{tqv} P_{qs}^{<w_1,\ldots,w_k>}\right] +$$

$$+ \sum_{p=1}^{n} f_{tpu}\left[\frac{\partial}{\partial x_v} P_{ps}^{<w_1,\ldots,w_k>} + \sum_{a=1}^{n} f_{pav} P_{as}^{<w_1,\ldots,w_k>}\right]$$

and similarly

$$P_{ts}^{<w_1,\ldots,w_u^{(1)}+1,\ldots,w_v^{(2)}+1,\ldots,w_k>} =$$

$$= \frac{\partial}{\partial x_v}\left[\frac{\partial}{\partial x_u} P_{ts}^{<w_1,\ldots,w_k>} + \sum_{q=1}^{n} f_{tqu} P_{qs}^{<w_1,\ldots,w_k>}\right] +$$

$$+\sum_{p=1}^{n} f_{tpv}\left[\frac{\partial}{\partial x_u}P_{ps}^{<w_1,\dots,w_k>}+\sum_{a=1}^{n} f_{pau}P_{as}^{<w_1,\dots,w_k>}\right].$$

Hence we obtain

$$P_{ts}^{<w_1,\dots,w_u^{(2)}+1,\dots,w_v^{(1)}+1,\dots,w_k>}-P_{ts}^{<w_1,\dots,w_u^{(1)}+1,\dots,w_v^{(2)}+1,\dots,w_k>}=$$

$$=\sum_{q=1}^{n} R_{tquv}P_{qs}^{<w_1,\dots,w_k>},$$

and (2.6) is satisfied because $R_{tquv}\equiv 0$.

Now we should prove that the functions (y_r) of (2.5) satisfy the system (2.1). First we prove the uniform convergence of the right side of (2.5) in a neighborhood of $(0,\dots,0)$. We can consider analytical functions of complex variables. Suppose that $x=(x_1,\dots,x_k)$ is sufficiently close to $(0,\dots,0)$ such that all functions $\{f_{rsu}\}$ are regular in the disc $D_x=\{z=(z_1,\dots,z_k):|z-x|<\rho\}$ and $0\in D_x$. Hence $\left|\frac{x}{\rho}\right|<1$. Obviously, all functions $P_{ts}^{<w_1,\dots,w_k>}$ are regular in D_x. In order to find an estimation of $P_{ts}^{<w_1,\dots,w_k>}$ from (2.4a) and (2.4b), some additional results should be given. Let $D_{x_u}(1\le u\le k)$ be an operator defined by

$$D_{x_u}(y_r)=\frac{\partial y_r}{\partial x_u}+\sum_{s=1}^{n} f_{rsu}y_s \qquad (1\le r\le n). \tag{2.7}$$

If the compatibility conditions (2.3) are satisfied, then similarly to the result in [2], it holds

$$\begin{aligned}(D_{x_1}^{w_1}\circ D_{x_2}^{w_2}\circ\cdots\circ D_{x_k}^{w_k})(y_r)&=\sum_{m_1=0}^{w_1}\cdots\sum_{m_k=0}^{w_k}\sum_{s=1}^{n}P_{rs}^{<m_1,\dots,m_k>}\cdot\\ \cdot\frac{\partial^{w_1-m_1+\cdots+w_k-m_k}y_s}{\partial x_1^{w_1-m_1}\partial x_2^{w_2-m_2}\dots\partial x_k^{w_k-m_k}}&\cdot\frac{w_1!\dots w_k!}{m_1!\dots m_k!(w_1-m_1)!\dots(w_k-m_k)!}.\end{aligned}\tag{2.8}$$

Since $P_{rj}^{<w_1,\dots,w_k>}=(D_{x_1}^{w_1}\circ\cdots\circ D_{x_k}^{w_k})\delta_{rj}$ for fixed j, by putting $y_r=P_{rj}^{<1,\dots,1>}$, we obtain

$$P_{rj}^{<w_1+1,\dots,w_k+1>}=\sum_{m_1=0}^{w_1}\cdots\sum_{m_k=0}^{w_k}\sum_{s=1}^{n}P_{rs}^{<m_1,\dots,m_k>}\cdot$$

$$\cdot\left[\frac{\partial^{w_1-m_1+\cdots+w_k-m_k}}{\partial x_1^{w_1-m_1}\dots\partial x_k^{w_k-m_k}}P_{sj}^{<1,\dots,1>}\right]\frac{w_1!\dots w_k!}{m_1!\dots m_k!(w_1-m_1)!\dots(w_k-m_k)!}.$$

This equality is suitable for estimation of $\left|P_{rj}^{<w_1,\dots,w_k>}\right|$.

If $Q_{rj}^{<w_1,\dots,w_k>} = \dfrac{P_{rj}^{<w_1,\dots,w_k>}}{w_1!\cdots w_k!}$, then

$$Q_{rj}^{<w_1+1,\dots,w_k+1>} = \frac{1}{(w_1+1)\cdots(w_k+1)} \sum_{m_1=0}^{w_1}\cdots\sum_{m_k=0}^{w_k}\sum_{s=1}^{n} Q_{rs}^{<m_1,\dots,m_k>}\cdot \tag{2.9}$$
$$\cdot\frac{\partial^{w_1-m_1+\cdots+w_k-m_k}}{\partial x_1^{w_1-m_1}\dots\partial x_k^{w_k-m_k}} P_{sj}^{<1,\dots,1>} \frac{1}{(w_1-m_1)!\dots(w_k-m_k)!}.$$

According to the Cauchy integral formula, it holds

$$\max_{s,j}\left|\frac{\partial^{w_1-m_1+\cdots+w_k-m_k}}{\partial x_1^{w_1-m_1}\dots\partial x_k^{w_k-m_k}} P_{sj}^{<1,\dots,1>}\right| \le \frac{M\cdot(w_1-m_1)!\dots(w_k-m_k)!}{\rho^{w_1-m_1+\dots+w_k-m_k}}, \tag{2.10}$$

where M depends (continuously) only on $x_1,\dots,x_k$.

Let $A_r^{<w_1,\dots,w_k>} = \max_j\left|Q_{rj}^{<w_1,\dots,w_k>}\right|$. Then (2.9) and (2.10) imply

$$A_r^{<w_1+1,\dots,w_k+1>} \le \frac{1}{(w_1+1)\cdots(w_k+1)}\sum_{m_1=0}^{w_1}\cdots\sum_{m_k=0}^{w_k} A_r^{<m_1,\dots,m_k>}\cdot$$
$$\cdot\frac{nM}{\rho^{w_1-m_1+\cdots+w_k-m_k}}.$$

Now if ρ is sufficiently small such that $nM\rho^k \le 1$, then

$$A_r^{<w_1+1,\dots,w_k+1>}\rho^{(w_1+1)+\dots+(w_k+1)} \le$$
$$\le \frac{1}{(w_1+1)\cdots(w_k+1)}\sum_{m_1=0}^{w_1}\cdots\sum_{m_k=0}^{w_k} A_r^{<m_1,\dots,m_k>}\rho^{m_1+\dots+m_k}.$$

Moreover, we can suppose that instead of (2.10) it holds

$$\max_{s,j}\left|\frac{\partial^{w_1-m_1+\cdots+w_k-m_k}}{\partial x_1^{w_1-m_1}\dots\partial x_k^{w_k-m_k}} P_{sj}^{<a_1,\dots,a_k>}\right| \le \frac{M(w_1-m_1)!\dots(w_k-m_k)!}{\rho^{w_1-m_1+\cdots+w_k-m_k}}$$

for each $a_1,\dots,a_k \in \{0,1\}$, and ρ is such that $nM\rho^u \le 1$ for $1 \le u \le$k. Now by induction of k it is easy to verify that

$$A_r^{<m_1,\dots,m_k>}\rho^{m_1+\cdots+m_k} \le 1.$$

Thus

$$\left|\frac{1}{w_1!\cdots w_k!}P_{rj}^{<w_1,\dots,w_k>}(x_1,\dots,x_k)\right| \leq \frac{1}{\rho^{m_1+\dots+m_k}},$$

and we have uniform convergence in (2.5) for $|x_1|,\dots,|x_k| \leq (1-\epsilon)\rho$. According to the Weierstrass theorem, the functions y_r are regular in a neighborhood of $(0,\dots,0)$ and we can differentiate them by parts.

If $C_1,\dots,C_n$ are arbitrary constants, using (2.5) and (2.4b) we obtain

$$\frac{\partial y_r}{\partial x_u} = \sum_{s=1}^{n}\sum_{w_1,\dots,w_k\in\mathbb{N}_0}\frac{(-x_1)^{w_1}}{w_1!}\cdots(-1)\frac{(-x_u)^{w_u-1}}{(w_u-1)!}\cdots\frac{(-x_k)^{w_k}}{w_k!}\cdot$$

$$\cdot P_{rs}^{<w_1,\dots,w_k>}C_s +$$

$$+\sum_{s=1}^{n}\sum_{w_1,\dots,w_k\in\mathbb{N}_0}\frac{(-x_1)^{w_1}}{w_1!}\cdots\frac{(-x_k)^{w_k}}{w_k!}\cdot\frac{\partial}{\partial x_u}P_{rs}^{<w_1,\dots,w_k>}C_s$$

$$= -\sum_{s=1}^{n}\sum_{w_1,\dots,w_k\in\mathbb{N}_0}\frac{(-x_1)^{w_1}}{w_1!}\cdots\frac{(-x_u)^{w_u}}{w_u!}\cdots\frac{(-x_k)^{w_k}}{w_k!}\cdot$$

$$\cdot P_{rs}^{<w_1,\dots,w_{u+1},\dots,w_k>}C_s +$$

$$+\sum_{s=1}^{n}\sum_{w_1,\dots,w_k\in\mathbb{N}_0}\frac{(-x_1)^{w_1}}{w_1!}\cdots\frac{(-x_k)^{w_k}}{w_k!}\cdot\frac{\partial}{\partial x_u}P_{rs}^{<w_1,\dots,w_k>}C_s =$$

$$= -\sum_{s=1}^{n}\sum_{w_1,\dots,w_k\in\mathbb{N}_0}\frac{(-x_1)^{w_1}}{w_1!}\cdots\frac{(-x_k)^{w_k}}{w_k!}\cdot$$

$$\left[P_{rs}^{<w_1,\dots,w_u+1,\dots,w_k>} - \frac{\partial}{\partial x_u}P_{rs}^{<w_1,\dots,w_k>}\right]C_s =$$

$$= -\sum_{s=1}^{n}\sum_{w_1,\dots,w_k\in\mathbb{N}_0}\frac{(-x_1)^{w_1}}{w_1!}\cdots\frac{(-x_k)^{w_k}}{w_k!}\sum_{p=1}^{n}f_{rpu}P_{ps}^{<w_1,\dots,w_k>}C_s =$$

$$= -\sum_{p=1}^{n}f_{rpu}\left[\sum_{s=1}^{n}\sum_{w_1,\dots,w_k\in\mathbb{N}_0}\frac{(-x_1)^{w_1}}{w_1!}\cdots\frac{(-x_k)^{w_k}}{w_k!}P_{ps}^{<w_1,\dots,w_k>}C_s\right]$$

$$= -\sum_{p=1}^{n}f_{rpu}y_p,$$

i.e. (2.1) is satisfied. Moreover, using (2.4a) we obtain

$$y_r(0,\dots,0) = P_{rs}^{<0,\dots,0>}C_s = \delta_{rs}C_s = C_r.$$

To the end of the proof we have only to prove the uniqueness of the solution of (2.1). Since the system (2.1) is linear, it is sufficient to prove that $y_s(0,\dots,0) = C_s = 0$ $(1 \le s \le n)$ implies $y_s = 0$ $(1 \le s \le n)$. Since the functions f_{rsu} are analytical, each solution of (2.1) is analytical. Using that $y_s(0,\dots,0) = 0$, it follows from (2.1) that the first partial derivatives of y_s vanish at $(0,\dots,0)$. By successive partial differentiations of (2.1), all partial derivatives of y_s vanish at $(0,\dots,0)$. Hence $y_s(x_1,\dots,x_k) = 0$ in a neighborhood of $(0,\dots,0)$. □

3. Non-linear system of partial differential equations. Let us consider the following non-linear system of partial differential equations

$$\frac{\partial y_r}{\partial x_u} + F(x_1,\dots,x_k,y_1,\dots,y_n) = 0 \qquad (1 \le r \le n, 1 \le u \le k).$$

We suppose that $F(x_1,\dots,x_k,y_1,\dots,y_n)$ can be written in a Laurent's series, i.e.

$$\frac{\partial y_r}{\partial x_u} + \sum_{i_1,\dots,i_n \in \mathbb{Z}} f_{ri_1\dots i_n u}(x_1,\dots,x_k) y_1^{i_1} y_2^{i_2} \dots y_n^{i_n} = 0 \tag{3.1}$$

$$1 \le r \le n, 1 \le u \le k$$

where $f_{ri_1\dots i_n u}$ are analytical functions. Moreover, suppose that there exist a neighborhood U of $(0,\dots,0)$ such that all functions $f_{ri_1\dots i_n u}$ are regular in U. Let W be such that the Laurent's series in (3.1) converge for $(y_1,\dots,y_n) \in W$ and $(x_1,\dots,x_k) \in U$. Before we consider the compatibility conditions and the solution of the system (3.1), we will introduce some notations.

If $f_{ri_1\dots i_n u}$ $(1 \le r \le n,\ 1 \le u \le k,\ i_1,\dots,i_n \in \mathbb{Z})$ are given functions of $x_1,\dots,x_k$, then we define new functions $h_{i_1\dots i_n j_1\dots j_n u}$ and $R_{i_1\dots i_n j_1\dots j_n uv}$ $(i_1,\dots,i_n,$ $j_1,\dots,j_n \in \mathbb{Z},\ 1 \le u,v \le k)$. First define

$$h_{i_1\dots i_n j_1\dots j_n u} = \sum_{s=1}^{n} i_s f_{s(j_1-i_1)\dots(j_s-i_s+1)\dots(j_n-i_n)u}. \tag{3.2}$$

Now we will prove the convergence of the series

$$\sum_{t_1,\dots,t_n \in \mathbb{Z}} h_{t_1\dots t_n j_1\dots j_n v} h_{i_1\dots i_n t_1\dots t_n u}.$$

According to the definition (3.2), it is sufficient to prove the convergence of the series

$$\sum_{t_1,\dots,t_n \in \mathbb{Z}} f_{p(j_1-t_1)\dots(j_p-t_p+1)\dots(j_n-t_n)v} \cdot f_{s(t_1-i_1)\dots(t_s-i_s+1)\dots(t_n-i_n)u}.$$

Indeed, it converges because that is the coefficient before

$$z_1^{j_1-i_1}\cdots z_s^{j_s-i_s+1}\cdots z_p^{j_p-i_p+1}\cdots z_n^{j_n-i_n}$$

of the product of the Laurent's series

$$\sum_{t_1,\dots,t_n\in\mathbb{Z}} f_{p(j_1-t_1)\dots(j_p-t_p+1)\dots(j_n-t_n)v}\cdot z_1^{j_1-t_1}\cdots z_p^{j_p-t_p+1}\cdots z_n^{j_n-t_n}$$

and

$$\sum_{t_1,\dots,t_n\in\mathbb{Z}} f_{s(t_1-i_1)\dots(t_s-i_s+1)\dots(t_n-i_n)u}\cdot z_1^{t_1-i_1}\cdots z_s^{t_s-i_s+1}\cdots z_n^{t_n-i_n},$$

which are convergent for $(z_1,\dots,z_n)\in W$. Now we can define

$$\begin{aligned}R_{i_1\dots i_n j_1\dots j_n uv} &= \frac{\partial}{\partial x_u}h_{i_1\dots i_n j_1\dots j_n v}-\frac{\partial}{\partial x_v}h_{i_1\dots i_n j_1\dots j_n u}+\\ &+\sum_{t_1,\dots,t_n\in\mathbb{Z}} h_{t_1\dots t_n j_1\dots j_n v}h_{i_1\dots i_n t_1\dots t_n u}-\sum_{t_1,\dots,t_n\in\mathbb{Z}} h_{t_1\dots t_n j_1\dots j_n u}h_{i_1\dots i_n t_1\dots t_n v}.\end{aligned}\tag{3.3}$$

Note that the series

$$\sum_{t_1,\dots,t_n\in\mathbb{Z}} h_{i_1\dots i_n j_1\dots j_n u}y_1^{j_1}\cdots y_n^{j_n}\qquad\text{and}\qquad R_{i_1\dots i_n j_1\dots j_n uv}y_1^{j_1}\cdots y_n^{j_n}$$

converge for $(y_1,\dots,y_n)\in W$ and $(x_1,\dots,x_k)\in U$. In order to simplify the notations, sometimes we will denote by the Greek indices $\alpha,\beta,\gamma,\dots$ a set of n integer indices $i_1\dots i_n$; $j_1\dots j_n$; ... We will denote by $\{r\}$ the set of n indices $0\dots010\dots0$ where 1 appears at the r-th place. Now $\alpha+\beta$ and $\alpha-\beta$ are defined by

$$i_1\dots i_n\pm j_1\dots j_n=(i_1\pm j_1)(i_2\pm j_2)\dots(i_n\pm j_n).$$

Theorem 3.1. *The quantities $h_{\alpha\beta u}$ and $R_{\alpha\beta uv}$ satisfy the following properties:*

$$h_{(\alpha+\beta)\gamma u}=h_{\alpha(\gamma-\beta)u}+h_{\beta(\gamma-\alpha)u},\tag{3.4}$$

$$R_{(\alpha+\beta)\gamma uv}=R_{\alpha(\gamma-\beta)uv}+R_{\beta(\gamma-\alpha)uv},\tag{3.5}$$

$$h_{\alpha\beta u}=\sum_{s=1}^{n} i_s h_{\{s\}(\beta-\alpha+\{s\})u},\tag{3.6}$$

$$R_{\alpha\beta uv} = \sum_{s=1}^{n} i_s R_{\{s\}(\beta-\alpha+\{s\})uv}, \tag{3.7}$$

where $\alpha = i_1 \dots i_n$.

Proof. Using the definition (3.2) we obtain

$$h_{\alpha(\gamma-\beta)u} + h_{\beta(\gamma-\alpha)u} =$$

$$= h_{i_1\dots i_n(t_1-j_1)\dots(t_n-j_n)u} + h_{j_1\dots j_n(t_1-i_1)\dots(t_n-i_n)u} =$$

$$= \sum_{s=1}^{n} i_s f_{s(t_1-j_1-i_1)\dots(t_n-j_n-i_n)u} + \sum_{s=1}^{n} j_s f_{s(t_1-i_1-j_1)\dots(t_n-i_n-j_n)u} =$$

$$= \sum_{s=1}^{n}(i_s + j_s) f_{s(t_1-(i_1+j_1))\dots(t_n-(i_n+j_n))u} =$$

$$= h_{(i_1+j_1)\dots(i_n+j_n)t_1\dots t_n u} = h_{(\alpha+\beta)\gamma u},$$

and the identity (3.4) is proved.

From the definition of $R_{\lambda\mu uv}$, i.e.

$$R_{\lambda\mu uv} = \frac{\partial}{\partial x_u} h_{\lambda\mu v} - \frac{\partial}{\partial x_v} h_{\lambda\mu u} + \sum_{\delta} h_{\delta\mu v} h_{\lambda\delta u} - \sum_{\delta} h_{\delta\mu u} h_{\lambda\delta v}$$

and the identity (3.4) we obtain

$$R_{(\alpha+\beta)\gamma uv} =$$

$$= \frac{\partial}{\partial x_u} h_{\alpha(\gamma-\beta)v} + \frac{\partial}{\partial x_u} h_{\beta(\gamma-\alpha)v} - \frac{\partial}{\partial x_v} h_{\alpha(\gamma-\beta)u} - \frac{\partial}{\partial x_v} h_{\beta(\gamma-\alpha)u} +$$

$$+ \sum_{\delta} h_{\delta\gamma v}(h_{\alpha(\delta-\beta)u} + h_{\beta(\delta-\alpha)u}) - \sum_{\delta} h_{\delta\gamma u}(h_{\alpha(\delta-\beta)v} + h_{\beta(\delta-\alpha)v}) =$$

$$= \frac{\partial}{\partial x_u} h_{\alpha(\gamma-\beta)v} - \frac{\partial}{\partial x_v} h_{\alpha(\gamma-\beta)u} + \frac{\partial}{\partial x_u} h_{\beta(\gamma-\alpha)v} - \frac{\partial}{\partial x_v} h_{\beta(\gamma-\alpha)u} +$$

$$+ \sum_{\delta}(h_{(\delta-\beta)(\gamma-\beta)v} + h_{\beta(\gamma-\delta+\beta)v}) h_{\alpha(\delta-\beta)u} +$$

$$+ \sum_{\delta}(h_{(\delta-\alpha)(\gamma-\alpha)v} + h_{\alpha(\gamma-\delta+\alpha)v}) h_{\beta(\delta-\alpha)u} -$$

$$- \sum_{\delta}(h_{(\delta-\beta)(\gamma-\beta)u} + h_{\beta(\gamma-\delta+\beta)u}) h_{\alpha(\delta-\beta)v} -$$

$$-\sum_{\delta}(h_{(\delta-\alpha)(\gamma-\alpha)u}+h_{\alpha(\gamma-\delta+\alpha)u})h_{\beta(\delta-\alpha)v}=$$

$$=R_{\alpha(\gamma-\beta)uv}+R_{\beta(\gamma-\alpha)uv}+\sum_{\delta}h_{\beta(\gamma-\delta+\beta)v}h_{\alpha(\delta-\beta)u}+$$

$$+\sum_{\delta}h_{\alpha(\gamma-\delta+\alpha)v}h_{\beta(\delta-\alpha)u}-\sum_{\delta}h_{\beta(\gamma-\delta+\beta)u}h_{\alpha(\delta-\beta)v}-$$

$$-\sum_{\delta}h_{\alpha(\gamma-\delta+\alpha)u}h_{\beta(\delta-\alpha)v}=R_{\alpha(\gamma-\beta)uv}+R_{\beta(\gamma-\alpha)uv}$$

because

$$\sum_{\delta}h_{\beta(\gamma-\delta+\beta)v}h_{\alpha(\delta-\beta)u}=\sum_{\delta}h_{\alpha(\gamma-\delta+\alpha)u}h_{\beta(\delta-\alpha)v}$$

and

$$\sum_{\delta}h_{\alpha(\gamma-\delta+\alpha)v}h_{\beta(\delta-\alpha)u}=\sum_{\delta}h_{\beta(\gamma-\delta+\beta)u}h_{\alpha(\delta-\beta)v}.$$

Hence the identity (3.5) is proved.

Finally, (3.6) and (3.7) are direct consequences of (3.4) and (3.5). Indeed, using (3.4) and (3.5) one can verify that if (3.6) and (3.7) hold for the set of indices $i_1\dots i_n$, then they also hold for the set of indices $i_1\dots(i_s\pm 1)\dots i_n$ for each $s\in\{1,\dots,n\}$.

We notice that (3.6) can be proved simpler as follows. From (3.2) it follows

$$h_{\{r\}j_1\dots j_n u}=f_{rj_1\dots j_n u}$$

and now (3.6) is a consequence of (3.2).

Finally, we notice that (3.4) and (3.5) are also consequences of (3.6) and (3.7), i.e.

$$(3.4)\Leftrightarrow(3.6)\text{ and }(3.5)\Leftrightarrow(3.7).\quad\square$$

Now we are ready to give the main theorem.

Theorem 3.2. (i) *The compatibility conditions for the system* (3.1) *for arbitrary initial conditions* $y_i(0,\dots,0)=C_i$, $1\le i\le n$, *are*

$$R_{\alpha\beta uv}\equiv 0\qquad i.e.\qquad R_{\{r\}\beta uv}\equiv 0. \tag{3.8}$$

(ii) *If the compatibility conditions* (3.8) *are satisfied, then there exist functions*

$$P^{<w_1,\dots,w_k>}_{i_1\dots i_n j_1\dots j_n}(x_1,\dots,x_k), w_1,\dots,w_n\in\mathbb{N}_0, i_1,\dots,i_n,j_1,\dots,j_n\in\mathbb{Z}$$

in a neighborhood of $(0,\dots,0)$ *such that*

$$P^{<0,\dots,0>}_{i_1\dots i_n j_1\dots j_n} = \delta_{i_1 j_1}\delta_{i_2 j_2}\dots\delta_{i_n j_n}, \tag{3.9a}$$

$$\begin{aligned} P^{<w_1,\dots,w_u+1,\dots,w_k>}_{i_1\dots i_n j_1\dots j_n} &= \frac{\partial}{\partial x_u} P^{<w_1,\dots,w_k>}_{i_1\dots i_n j_1\dots j_n} + \\ &+ \sum_{t_1,\dots,t_n\in\mathbb{Z}} \Big(\sum_{s=1}^{n} i_s f_{s(t_1-i_1)\dots(t_s-i_s+1)\dots(t_n-i_n)u}\Big) P^{<w_1,\dots,w_k>}_{t_1\dots t_n j_1\dots j_n}. \end{aligned} \tag{3.9b}$$

If $(C_1,\dots,C_n)\in W$*, then the solution of* (3.1) *in a neighborhood of* $(0,\dots,0)$ *is given by*

$$\begin{aligned} y_1 &= \sum_{w_1,\dots,w_k\in\mathbb{N}_0} \Big[\frac{(-x_1)^{w_1}}{w_1!}\cdots\frac{(-x_k)^{w_k}}{w_k!} \sum_{j_1,\dots,j_n\in\mathbb{Z}} P^{<w_1,\dots,w_k>}_{10\dots0 j_1\dots j_n} C_1^{j_1}\cdot C_2^{j_2}\cdots C_n^{j_n}\Big] \\ y_2 &= \sum_{w_1,\dots,w_k\in\mathbb{N}_0} \Big[\frac{(-x_1)^{w_1}}{w_1!}\cdots\frac{(-x_k)^{w_k}}{w_k!} \sum_{j_1,\dots,j_n\in\mathbb{Z}} P^{<w_1,\dots,w_k>}_{01\dots0 j_1\dots j_n} C_1^{j_1}\cdot C_2^{j_2}\cdots C_n^{j_n}\Big] \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_n &= \sum_{w_1,\dots,w_k\in\mathbb{N}_0} \Big[\frac{(-x_1)^{w_1}}{w_1!}\cdots\frac{(-x_k)^{w_k}}{w_k!} \sum_{j_1,\dots,j_n\in\mathbb{Z}} P^{<w_1,\dots,w_k>}_{0\dots01 j_1\dots j_n} C_1^{j_1}\cdot C_2^{j_2}\cdots C_n^{j_n}\Big]. \end{aligned} \tag{3.10}$$

This solution is unique with the given initial conditions in a neighborhood of $(0,\dots,0)$.

P r o o f. Let us introduce the following functions

$$y_\alpha = y_{i_1 i_2\dots i_n} = y_1^{i_1}\cdot y_2^{i_2}\cdots y_n^{i_n}, \qquad (i_1,\dots,i_n\in\mathbb{Z})$$

such that $y_1 = y_{\{1\}},\dots,y_n = y_{\{n\}}$. These functions satisfy

$$\frac{\partial y_{\{r\}}}{\partial x_u} + \sum_\alpha f_{r\alpha u} y_\alpha = 0 \qquad (1\le r\le n, 1\le u\le k)$$

and hence

$$\begin{aligned} \frac{\partial y_\alpha}{\partial x_u} &= \frac{\partial}{\partial x_u}(y_1^{i_1}\cdot y_2^{i_2}\cdots y_n^{i_n}) \\ &= i_1 y_{\alpha-\{1\}}\frac{\partial y_1}{\partial x_u} + \dots + i_n y_{\alpha-\{n\}}\frac{\partial y_n}{\partial x_u} \end{aligned}$$

$$= i_1 y_{\alpha-\{1\}}\Big(-\sum_\beta f_{1\beta u} y_\beta\Big) + \cdots + i_n y_{\alpha-\{n\}}\Big(-\sum_\beta f_{n\beta u} y_\beta\Big)$$

$$= -i_1 \sum_\beta f_{1\beta u} y_{\alpha+\beta-\{1\}} - \cdots - i_n \sum_\beta f_{n\beta u} y_{\alpha+\beta-\{n\}}$$

$$= -\sum_{s=1}^n i_s \sum_\beta f_{s\beta u} y_{\alpha+\beta-\{s\}}$$

$$= -\sum_{s=1}^n i_s \sum_\gamma f_{s(\gamma-\alpha+\{s\})u} y_\gamma, \qquad \text{i.e.}$$

$$\frac{\partial}{\partial x_u} y_\alpha + \sum_\gamma h_{\alpha\gamma u} y_\gamma = 0 \tag{3.11}$$

for $\alpha \in \mathbb{Z}^n$, $u \in \{1, \dots, k\}$.

Thus we obtain that the system (3.1) induces the system (3.11). The converse also holds, i.e. one can prove that if the functions $f_{r\alpha u}$ are given, and

(i) the system (3.11) is satisfied, where $h_{\alpha\gamma u}$ are defined by (3.2),

(ii) $y_{i_1\dots i_n}(0, \dots, 0) = C_1^{i_1} \cdot C_2^{i_2} \cdots C_n^{i_n}$ (C_i are constants),

then the system (3.1) is satisfied, where $y_r = y_{\{r\}}$ for $1 \le r \le n$.

Similarly to the compatibility conditions for the system (2.1), the compatibility conditions for the homogeneous linear system (3.11) are given by $R_{\alpha\beta uv} \equiv 0$, i.e. $R_{\{s\}\beta uv} \equiv 0$, because (3.7) is satisfied. Hence, the compatibility conditions of (3.1) are given by (3.8), and (i) is proved.

Similarly to the proof of Theorem 2.1, if the compatibility conditions (3.8) are satisfied, then there exist functions

$$P_{\alpha\beta}^{<w_1,\dots,w_k>}(x_1, \dots, x_k), \qquad w_1, \dots, w_k \in \mathbb{N}_0, \alpha, \beta \in \mathbb{Z}^n$$

such that (3.9a,b) are satisfied. In order to prove that they are well defined, the convergence in (3.9b) should be verified. It is easy to prove from (3.9a) and (3.9b) that for each $w_1, \dots, w_k \in \mathbb{N}_0$ the series

$$\sum_{i_1,\dots,i_n \in \mathbb{Z}} P_{i_1\dots i_n j_1 \dots j_n}^{<w_1,\dots,w_k>} z_1^{j_1-i_1} \cdot z_2^{j_2-i_2} \cdots z_n^{j_n-i_n}$$

uniformly converge for $(z_1, \dots, z_n)$ in a closed subset of W. The proof is by induction of $w_1, \dots, w_k$ and it is analogous to the proof of the convergence of $\sum_\gamma h_{\gamma\alpha v} h_{\beta\gamma u}$.

The convergency in (3.9b) follows simultaneously from here. Further by induction of $w_1, \ldots, w_k$ it is also verified the uniform convergence of

$$\sum_{j_1,\ldots,j_n \in \mathbb{Z}} P_{i_1 \ldots i_n j_1 \ldots j_n}^{<w_1,\ldots,w_k>} z_1^{j_1} \cdot z_2^{j_2} \cdots z_n^{j_n}$$

for $(z_1, \ldots, z_n)$ in a closed subset of W. Moreover, for fixed $i_1, \ldots, i_n$ there exist constants $M_{i_1 \ldots i_n}$ in a neighborhood of the considered point, such that

$$\left| \sum_{j_1,\ldots,j_n \in \mathbb{Z}} P_{i_1 \ldots i_n j_1 \ldots j_n}^{<w_1,\ldots,w_k>} C_1^{j_1} \cdots C_n^{j_n} \cdot \frac{1}{w_1! \cdots w_k!} \right| \leq M_{i_1 \ldots i_n}$$

for arbitrary $w_1, \ldots, w_n \in \mathbb{N}_0$ and $(C_1, \ldots, C_n) \in W$. The proof follows from a formula analogous to (2.9). Indeed, using the same notations as in the proof of the Theorem 2.1, by induction of $w_1, \ldots, w_k \in \mathbb{N}_0$, it is verified that

$$\left| \sum_{\beta} Q_{\alpha\beta}^{<w_1,\ldots,w_k>} \sum_{\gamma} M_{\beta\gamma} C_1^{j_1} \cdots C_n^{j_n} \right| \leq N_\alpha \qquad (\gamma = j_1 \ldots j_n)$$

where N_α do not depend on $w_1, \ldots, w_k$, and where

$$M_{\beta\gamma} = \max_{a_1,\ldots,a_k \in \{0,1\}} \left| \frac{\partial^{w_1 - m_1 + \ldots + w_k - m_k}}{\partial x_1^{w_1 - m_1} \ldots \partial x_k^{w_k - m_k}} P_{\beta\gamma}^{<a_1,\ldots,a_k>} \right| \cdot$$

$$\cdot \frac{\rho^{w_1 - m_1 + \cdots + w_k - m_k}}{(w_1 - m_1)! \ldots (w_k - m_k)!},$$

according to the Cauchy integral formula.

Similarly to the proof of Theorem 2.1, one can verify that the solution of (3.1) with $y_{i_1 \ldots i_n}(0, \ldots, 0) = C_1^{i_1} \cdot C_2^{i_2} \ldots C_n^{i_n}$ is given by

$$y_\alpha = \sum_{w_1,\ldots,w_k \in \mathbb{N}_0} \left[\frac{(-x_1)^{w_1}}{w_1!} \cdots \frac{(-x_k)^{w_k}}{w_k!} \sum_{\beta} P_{\alpha\beta}^{<w_1,\ldots,w_k>} C_1^{j_1} C_2^{j_2} \ldots C_n^{j_n} \right]$$

where $\beta = j_1 \ldots j_n$. Its convergence follows from the previous discussion. Especially, if $\alpha \in \{1\}, \ldots, \alpha \in \{n\}$ we obtain the required solution (3.10).

Each solution of (3.1) is analytical function. On the other hand, by successive differentiation of (3.1), we notice that all partial derivatives of y_s can be calculated uniquely at $(0,\dots,0)$. Hence (3.1) does not have more than one solution in a neighborhood of $(0,\dots,0)$, and the obtained solution is unique. □

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