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ON A TWO-DIMENSIONAL SEARCH PROBLEM

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Abstract. In this article we explore the so-called two-dimensional tree^{-} search problem. We prove that for integers m of the form $m = (2^{st}-1)/(2^s-1)$ the rectangles $\mathcal{A}(m,n)$ are all tight, no matter what n is. On the other hand, we prove that there exist infinitely many integers m for which there is an infinite number of n's such that $\mathcal{A}(m,n)$ is loose. Furthermore, we determine the smallest loose rectangle as well as the smallest loose square ($\mathcal{A}(181,181)$). It is still undecided whether there exist infinitely many loose squares.

In this article we are concerned with a problem proposed by Katona [1]. It is formulated briefly below. Let us call the set $\mathcal{A}(m,n) = \{(i,j) \,|\, i,j \in \mathbf{Z}, 1 \le i \le m,\ 1 \le j \le n\}$ a *rectangle* of size m by n. A rectangle of size n by n will be referred to as a *square* of size n. Let $\mathbf{a} = (a_1, a_2) \in \mathcal{A}(m,n)$, $\mathbf{b} = (b_1, b_2) \in \mathcal{A}(m,n)$. We write $\mathbf{a} \le \mathbf{b}$ if and only if $a_1 \le b_1$, and $a_2 \le b_2$. Suppose an element $\mathbf{x} = (x_1, x_2)$ from $\mathcal{A}(m,n)$ is fixed but unknown to us. Now we want to know how many questions of the type "Is $\mathbf{x} \le \mathbf{a}$?" are necessary to determine $\mathbf{x}$. Let us mention at this point that every

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question will be considered as an element (point) from $\mathcal{A}(m,n)$. So, by saying "we ask the question $\mathbf{q} = (\alpha_1, \alpha_2)$" we mean the question "Is $\mathbf{x} \le \mathbf{q}$?". We restrict ourselves to the so- called tree-search, i.e. we suppose that the questions being asked depend on the previous answers.

Denote by $t(m,n)$ the minimum number of questions that are needed to identify $\mathbf{x}$. It is straightforward that

$$\lceil \log m + \log n \rceil \le t(m,n) \le \lceil \log m \rceil + \lceil \log n \rceil, \tag{1}$$

(see also [2], Proposition 2.1 and 2.2). All the logarithms here are taken to the base 2, and $\lceil a \rceil$ denotes, as usual, the smallest integer greater than or equal to a. The rectangle $\mathcal{A}(m,n)$ is said to be *tight* if there exists a searching algorithm identifying an element $\mathbf{x}$ (chosen from $\mathcal{A}(m,n)$) after $\lceil \log m + \log n \rceil$ questions. Otherwise $\mathcal{A}(m,n)$ is called *loose*. The problem of determining whether a rectangle is tight or loose becomes a nontrivial one if $\lceil \log m + \log n \rceil \ne \lceil \log m \rceil + \lceil \log n \rceil$.

Loose rectangles do exist. In fact, there exist infinitely such rectangles. It was pointed out in [2] that $\mathcal{A}(\frac{2^{10k}-1}{11}, 11)$ is loose for every $k \not\equiv 1 (\bmod 3)$. Below we state without proofs some straightforward results (see also [2]).

Proposition 1. *For every $m_1, n_1, m_2, n_2 \in \mathbf{N}$ with $m_1 \le n_1,\ m_2 \le n_2$ we have $t(m_1, n_1) \le t(m_2, n_2)$.*

Proposition 2. *For any $j, k, m, n \in \mathbf{N}$*

$$t(2^j m, 2^k n) \le j + k + t(m,n). \tag{2}$$

In particular, if $\mathcal{A}(m,n)$ is tight then $\mathcal{A}(2^j m, 2^k n)$ is also tight.

Now we are going to prove that for certain choices of m all rectangles $\mathcal{A}(m,n)$ are tight no matter what n is.

Lemma 3. *Let $m = 2^t - 1$. Then $\mathcal{A}(m,n)$ is tight for every $n \in \mathbf{N}$.*

Proof. We use induction on n. For $n = 1$ the statement is trivial. Let $n = 2^l + A > 1,\ 0 < A < 2^l$, and let $\mathcal{A}(m, n')$ be tight for every $n' < n$. We can suppose that $mn \le 2^{t+l}$ for otherwise the lower and the upper bound in (1) coincide and there is nothing to prove.

We ask the following questions until we get an answer "yes".

$$\mathbf{q}_\alpha = (\sum_{i=1}^{\alpha} 2^{t-i}, 2^l),\ \alpha = 1, 2, \ldots, t. \tag{3}$$

If one of the questions has been answered by "yes" then we can apply a straightforward algorithm to obtain $\mathbf{x}$ in a rectangle whose "sides" are powers of 2. If all the questions $\mathbf{q}_1$, $\mathbf{q}_2$, ... , $\mathbf{q}_t$ have been asked and all the answers were "no" then $\mathbf{x}$ is contained in a rectangle of size m by A which is tight by the induction hypothesis. It remains to be noted that

$$mA = mn - m2^l \le 2^{t+l} - 2^l(2^t - 1) = 2^l,$$

whence it follows that $\mathbf{x}$ can be found after $t + l$ questions, as required. □

Theorem 4. *Let $m = \frac{2^{st}-1}{2^s-1}$ for some $s, t \in \mathbf{N}$. Then for every integer $n > 0$ the rectangle $\mathcal{A}(m, n)$ is tight.*

P r o o f. We shall prove the theorem by induction. Obviously, $\mathcal{A}(m, 1)$ is tight. Write $n = 2^l + A, 0 < A < 2^l$ and suppose that every rectangle $\mathcal{A}(m, n')$ with $n' < n$ is tight.

Without loss of generality we can assume that $mn \le 2^{s(t-1)+l+1}$. If $n \le 2^s - 1$ our theorem is settled by Lemma 3 and Proposition 1. Now let $n > 2^s - 1$ (i.e. $l > s > 1$). If $A \le 2^{l-1} + 2^{l-2} + \ldots + 2^{l-s+1}$ then $2^{s(t-1)+l} < mn < 2^{s(t-1)+l+1}$. The rectangle $\mathcal{A}(m, 2^{l-s-1}(2^s - 1))$ is tight by Proposition 2. Therefore, by Proposition 1 $\mathcal{A}(m, n)$ is also tight.

We have to consider only the case $A > 2^{l-1} + 2^{l-2} + \ldots + 2^{l-s+1}$. Now we ask the following st questions until an answer "yes" is obtained

$$\mathbf{q}_{\alpha s+\beta} = (\sum_{i=0}^{\alpha} 2^{s(t-i-1)}, \sum_{j=1}^{\beta} 2^{l-j+1}), \tag{4}$$

where $\alpha = 0, 1, \ldots, t - 1, \beta = 1, 2, \ldots, s$. Suppose the question $\mathbf{q}_{\alpha_0 s+\beta_0}$ has been answered by "yes". Then $\mathbf{x}$ is contained in a rectangle of size $2^{s(t-\alpha_0-1)}$ by $2^{l-\beta_0+1}$

and can be determined with $s(t-\alpha_0-1)+l-\beta_0+1$ questions, which gives a total of $s(t-1)+l+1$ questions, as required.

If all questions (4) have been answered by "no" then $\mathbf{x}$ is contained in a rectangle of size m by n', where $n' = n - 2^{l-s+1}(2^s-1)$, which is tight by the induction hypothesis. Further, we have

$$mn' \;=\; mn \;-\; m2^{l-s+1}(2^s-1) \;\leq 2^{s(t-1)+l+1} - 2^{l-s+1}(2^{st}-1) \;=\; 2^{l-s+1},$$

whence it follows that $\mathbf{x}$ can be determined after $st + (l-s+1) \;=\; s(t-1)+l+1$ questions, as required. $\square$

Now we are going to introduce some more notations. Let $\mathcal{A}(m,n)$ be a rectangle with $\lceil \log m + \log n \rceil \neq \lceil \log m \rceil + \lceil \log n \rceil$. The deficiency of $\mathcal{A}(m,n)$ is defined by

(5). $$d(m,n) \;=\; 2^{\lceil \log mn \rceil} - mn.$$

Let $\mathbf{x} \in \mathcal{A}(m,n)$ be fixed but unknown and let $\mathbf{q}_1$, $\mathbf{q}_2$, $\ldots$, $\mathbf{q}_s$ be a sequence of questions for this $\mathbf{x}$. Suppose further that they have been answered by i_1, i_2, $\ldots$, i_s; $i_j \in \{0,1\}$ (0 is viewed as "no", and 1 as "yes"). By $\mathcal{S}^{i_1,i_2,\ldots,i_s}_{\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_s}$ we denote the subset of $\mathcal{A}(m,n)$ containing those of its elements for which the answers on $\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_s$ are precisely $i_1,i_2,\ldots,i_s$. Obviously, if $\mathcal{A}(m,n)$ is tight and if $\mathbf{q}_1 = (\alpha_1,\alpha_2)$ we have

(6) $$\alpha_1\alpha_2 \;=\; |\mathcal{S}^1_{\mathbf{q}_1}| \;\leq\; 2^{\lceil \log mn \rceil - 1},$$

(7) $$mn - \alpha_1\alpha_2 \;=\; |\mathcal{S}^0_{\mathbf{q}_1}| \;\leq\; 2^{\lceil \log mn \rceil - 1},$$

whence $2^{\lceil \log mn \rceil - 1} - d(m,n) \;\leq\; \alpha_1\alpha_2 \;\leq\; 2^{\lceil \log mn \rceil - 1}$. Taking into account the obvious restrictions $1 \leq \alpha_1 \leq m$, $1 \leq \alpha_2 \leq n$, we can easily find all possible starting questions by investigating the decompositions into primes of the numbers

(8) $$2^{\lceil \log mn \rceil - 1} - d(m,n),\; 2^{\lceil \log mn \rceil - 1} - d(m,n) + 1,\; \ldots\; , 2^{\lceil \log mn \rceil - 1}.$$

In the same way we can define the deficiency of the set $\mathcal{S}^{i_1,i_2,\ldots,i_s}_{\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_s}$ as

(9) $$d^{i_1,i_2,\ldots,i_s}_{\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_s} \;=\; 2^{\lceil \log mn \rceil - s} - |\mathcal{S}^{i_1,i_2,\ldots,i_s}_{\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_s}|.$$

From $\mathcal{S}^{i_1,i_2,\ldots,i_{s-1}}_{\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_{s-1}} = \mathcal{S}^{i_1,\ldots,i_{s-1},0}_{\mathbf{q}_1,\ldots,\mathbf{q}_{s-1},\mathbf{q}_s} \cup \mathcal{S}^{i_1,\ldots,i_{s-1},1}_{\mathbf{q}_1,\ldots,\mathbf{q}_{s-1},\mathbf{q}_s}$ we can easily obtain

$$d^{i_1,i_2,\ldots,i_{s-1}}_{\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_{s-1}} = d^{i_1,\ldots,i_{s-1},0}_{\mathbf{q}_1,\ldots,\mathbf{q}_{s-1},\mathbf{q}_s} + d^{i_1,\ldots,i_{s-1},1}_{\mathbf{q}_1,\ldots,\mathbf{q}_{s-1},\mathbf{q}_s}. \tag{10}$$

In particular, $d^0_{\mathbf{q}_1} + d^1_{\mathbf{q}_1} = d(m,n)$. Thus every algorithm finding out an $\mathbf{x} \in \mathcal{A}(m,n)$ in $\lceil \log mn \rceil$ steps must have a nonnegative deficiency for each set $\mathcal{S}^{i_1,i_2,\ldots,i_s}_{\mathbf{q}_1,\mathbf{q}_2,\ldots,\mathbf{q}_s}$, for each integer $s \le \lceil \log mn \rceil$ and each s-tuple of zeros and ones $i_1, i_2, \ldots, i_s$.

The following lemma will be needed later.

Lemma 5. *Let $m = 2^{l_1} + A$, $n = 2^{l_2} + B$, where $A < 2^{l_1}$, $B < 2^{l_2}$, and let $\mathbf{q}_1 = (2^{l_1}, 2^{l_2})$, $\mathbf{q}_2 = (\alpha_1, \alpha_2)$, with $\alpha_1 > 2^{l_1}$, $\alpha_2 > 2^{l_2}$. Then*

$$|\mathcal{S}^{0,1}_{\mathbf{q}_1,\mathbf{q}_2}| \neq 2^{l_1+l_2-1}.$$

Proof. Let us assume that $|\mathcal{S}^{0,1}_{\mathbf{q}_1,\mathbf{q}_2}| = 2^{l_1+l_2-1}$. Note that neither of α_1 and α_2 is a power of 2. Therefore both of them have an odd prime divisor. On the other hand we have

$$\alpha_1\alpha_2 = |\mathcal{S}^1_{\mathbf{q}_1} \cup \mathcal{S}^{0,1}_{\mathbf{q}_1,\mathbf{q}_2}| = 3.2^{l_1+l_2-1}.$$

This is not possible because there is only one odd prime divisor on the right. $\square$

The next theorem provides a large infinite class of loose rectangles. It claims that for appropriate m's there exist infinitely many choices for n so that $\mathcal{A}(m,n)$ is loose.

Theorem 6. *Let $m = 2^{l_1} + A$, $2^{l_1-2} + 2^{l_1-3} \le A < 2^{l_1-1}$ be an odd integer and let*

$$k = lcm\left\{ r_u \mid \frac{m}{2} < u \le m, u - \text{odd} \right\}, \tag{11}$$

where r_u and r are the orders of 2 in $\mathbf{Z}^*_u$ and $\mathbf{Z}^*_m$, respectively. Then the rectangle $\mathcal{A}(m,n)$ with $n = (2^{\alpha k r} - 1)/m$ is loose for every $\alpha \in \mathbf{N}$.

Remark. The theorem still holds if k is a number with the property $(k, r_u) \neq 1$ for every odd u with $m/2 < u \le m$.

P r o o f. Our first question $\mathbf{q}_1 = (\alpha_1, \alpha_2)$ must satisfy $\alpha_1\alpha_2 = 2^{\alpha kr-1}$, or $\alpha_1\alpha_2 = 2^{\alpha kr-1} - 1$. Suppose $\alpha_1\alpha_2 = 2^{\alpha kr-1} - 1$. Obviously α_1 is odd and $m/2 < \alpha_1 \leq m$. From $2^{\alpha kr-1} \equiv 1 (\mathrm{mod}\, \alpha_1)$ it follows that $\alpha kr \equiv 1 \,(\mathrm{mod}\, r_{\alpha_1})$ which is impossible by (11).

Thus $\mathbf{q}_1 = (\alpha_1, \alpha_2)$ with $\alpha_1\alpha_2 = 2^{\alpha kr-1}$. If we write $n = 2^{l_2} + B$, then it follows from the condition on A that $2^{l_2-2} \leq B < 2^{l_2-1}$, $\alpha kr - 1 = l_1 + l_2$, and $\alpha_1 = 2^{l_1}, \alpha_2 = 2^{l_2}$.

Let the second question be $\mathbf{q}_2 = (\beta_1, \beta_2)$. Then $|\mathcal{S}^{0,1}_{\mathbf{q}_1,\mathbf{q}_2}| = 2^{\alpha kr-2} - 1$ or $2^{\alpha kr-2}$ which implies that $\beta_1 > 2^{l_1}$, $\beta_2 > 2^{l_2}$. According to Lemma 5 we have $|\mathcal{S}^{0,1}_{\mathbf{q}_1,\mathbf{q}_2}| \neq 2^{\alpha kr-2}$, whence we get

$$|\mathcal{S}^{1}_{\mathbf{q}_1} \cup \mathcal{S}^{0,1}_{\mathbf{q}_1,\mathbf{q}_2}| = 2^{\alpha kr-1} + 2^{\alpha kr-2} - 1 = \beta_1\beta_2. \tag{12}$$

This yields

$$3.2^{\alpha kr-2} \equiv 1 (\mathrm{mod}\, \beta_1), \tag{13}$$

where $(\beta_1, 2) = (\beta_1, 3) = 1$, $m/2 < \beta_1 \leq m$.

If $3 \notin \langle 2 \rangle$ in $\mathbf{Z}^*_{\beta_1}$ then (13) is impossible. Now suppose that

$$2^s \equiv 3 \,(\mathrm{mod}\, \beta_1), \quad 0 < s < r_{\beta_1}. \tag{14}$$

Then we have $2^{\alpha kr+s-2} \equiv 1 \,(\mathrm{mod}\, \beta_1)$. Therefore r_{β_1} divides $s-2$, which is possible only for $s = 2$. We get a contradiction to (14). $\square$

It was pointed out in [2] that the smallest loose rectangle (i.e. rectangle with minimal number of elements) the author was able to find was $\mathcal{A}(23, 89)$. It turns out that there exists a smaller loose rectangle.

Proposition 7. *The smallest loose rectangle is* $\mathcal{A}(11, 93)$. *It is the only loose rectangle of cardinality less than 1024.*

P r o o f. We shall consider rectangles $\mathcal{A}(m, n)$ with $m \leq n$, and $2^{l-1} < m.n \leq 2^l$. It can be easily checked that if neither m nor n is of the type $2^k(2^{st} - 1)/(2^s - 1)$, where s is a positive and k, t nonnegative integers, there exist the following possibilities:

$$\begin{array}{llll}
(A) & m = 11, & n = 11, & l = 7; \\
(B) & m = 11, & n \leq 23, & l = 8; \\
(C) & m = 13, & n \leq 19, & l = 8; \\
(D) & m = 11, & n \leq 46, & l = 9; \\
(E) & m = 13, & n \leq 39, & l = 9; \\
(F) & m = 19, & n \leq 26, & l = 9; \\
(G) & m = 22, & n \leq 23, & l = 9; \\
(H) & m = 11, & n \leq 93, & l = 10; \\
(I) & m = 19, & n \leq 53, & l = 10; \\
(J) & m = 27, & n \leq 37, & l = 10; \\
(K) & m = 29, & n \leq 35, & l = 10.
\end{array}$$

Case (A) is settled in [2]. It can be proved that the rectangles $\mathcal{A}(11, 23)$, $\mathcal{A}(13, 19)$, $\mathcal{A}(13, 39)$, $\mathcal{A}(19, 53)$, $\mathcal{A}(27, 37)$, $\mathcal{A}(29, 35)$ are tight which settles cases (B)–(G), (I)–(K). (See the Appendix for the corresponding algorithms.)

Now we are going to prove that $\mathcal{A}(11, 93)$ is loose. Let our first question be $\mathbf{q}_1 = (\alpha_1, \alpha_2)$, where $\alpha_1\alpha_2 = 511$, or 512. So we have two possibilities

(i) $\alpha_1 = 8,\ \alpha_2 = 64$;

(ii) $\alpha_1 = 7,\ \alpha_2 = 73$.

Let $\mathbf{q}_2 = (\beta_1, \beta_2)$.

(i) $|\mathcal{S}^{0,1}_{\mathbf{q}_1,\mathbf{q}_2}| = 255$, or 256. Hence $\beta_1 > \alpha_1$, $\beta_2 > \alpha_2$. It follows from Lemma 5 that $|\mathcal{S}^{0,1}_{\mathbf{q}_1,\mathbf{q}_2}| = 255$, and $|\mathcal{S}^{0,0}_{\mathbf{q}_1,\mathbf{q}_2}| = 256$. Therefore, we have $1023 - \beta_1\beta_2 = 256$, with $8 < \beta_1 \leq 11$, $64 < \beta_2 \leq 93$. We cannot find such integers, which rejects (i).

(ii) Now we must have $|\mathcal{S}^{0,1}_{\mathbf{q}_1,\mathbf{q}_2}| = 256$, and the only possibility is $\beta_1 = 11,\ \beta_2 = 64$. Write $\mathbf{q}_3 = (\gamma_1, \gamma_2)$ for our third question. We have $|\mathcal{S}^{0,0,0}_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}| = 128$. This cannot be achieved unless $\gamma_1 > 7, \gamma_2 > 73$. Thus we have

$$128 \ = \ 29.11 \ - \ \gamma_1(\gamma_2 - 64)$$

$$\gamma_1(\gamma_2 - 64) \ = \ 191,$$

which is impossible.

To finish the proof it remains to be noted that the rectangle $\mathcal{A}(11, 92)$ is tight since 92=4.23 and $\mathcal{A}(11, 23)$ was proved to be tight. $\square$

It is known that the squares $\mathcal{A}(5, 5)$, $\mathcal{A}(11, 11)$, $\mathcal{A}(45, 45)$ are tight [2]. This fact combined with Proposition 1 and Proposition 2 yields that every square $\mathcal{A}(n, n)$ with $n \leq 180$ is loose. Hence the first undecided case is $\mathcal{A}(181, 181)$.

Theorem 8. *The square* $\mathcal{A}(181, 181)$ *is loose.*

Proof. (sketch) We have $d(181, 181) = 7$, and the first question can be computed using the decomposition of the numbers (8). Similarly, we can obtain all possible second questions. It turns out that there exist 13 possibilities which are listed below

$$
\begin{array}{lll}
1A) & \mathbf{q}_1 = (140, 117), & \mathbf{q}_2 = (128, 181); \\
1B) & \mathbf{q}_1 = (140, 117), & \mathbf{q}_2 = (130, 180); \\
2A) & \mathbf{q}_1 = (156, 105), & \mathbf{q}_2 = (117, 175); \\
2B) & \mathbf{q}_1 = (156, 105), & \mathbf{q}_2 = (126, 170); \\
2C) & \mathbf{q}_1 = (156, 105), & \mathbf{q}_2 = (128, 169); \\
2D) & \mathbf{q}_1 = (156, 105), & \mathbf{q}_2 = (130, 168); \\
3A) & \mathbf{q}_1 = (159, 103), & \mathbf{q}_2 = (128, 167); \\
4A) & \mathbf{q}_1 = (180, 91), & \mathbf{q}_2 = (91, 181); \\
4B) & \mathbf{q}_1 = (180, 91), & \mathbf{q}_2 = (105, 169); \\
4C) & \mathbf{q}_1 = (181, 91), & \mathbf{q}_2 = (117, 161); \\
4D) & \mathbf{q}_1 = (181, 91), & \mathbf{q}_2 = (126, 156); \\
4E) & \mathbf{q}_1 = (181, 91), & \mathbf{q}_2 = (128, 155); \\
4F) & \mathbf{q}_1 = (181, 91), & \mathbf{q}_2 = (130, 154);
\end{array}
$$

In order to give an idea of the proof let us consider some cases in more detail.

3A) Here we have $e^{0,0}_{\mathbf{q}_1, \mathbf{q}_2} = 0$, which means that $|\mathcal{S}^{0,0,i}_{\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3}| = 2^{12}$, $i = 0, 1$.

Further we write

$$\mathcal{S}^{0,0}_{\mathbf{q}_1,\mathbf{q}_2} = \bigcup_{i=1}^{6} \mathcal{T}_i,$$

where the $\mathcal{T}_i$'s are defined as follows

$$\begin{aligned}
\mathcal{T}_1 &= \{(a,b) \,|\, 128 < a \le 181;\ 0 < b \le 103\};\\
\mathcal{T}_2 &= \{(a,b) \,|\, 128 < a \le 159;\ 103 < b \le 167\};\\
\mathcal{T}_3 &= \{(a,b) \,|\, 159 < a \le 181;\ 103 < b \le 167\};\\
\mathcal{T}_4 &= \{(a,b) \,|\, 0 < a \le 128;\ 167 < b \le 181\};\\
\mathcal{T}_5 &= \{(a,b) \,|\, 128 < a \le 159;\ 167 < b \le 181\};\\
\mathcal{T}_6 &= \{(a,b) \,|\, 159 < a \le 181;\ 167 < b \le 181\}.
\end{aligned}$$

The question $\mathbf{q}_3$ cannot be in $\mathcal{T}_1$, $\mathcal{T}_2$, or $\mathcal{T}_4$ because of $|\mathcal{S}^{0,0,1}_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}| \le |\mathcal{T}_i| < 2^{12}$, $i = 1,2,4$. We cannot have $\mathbf{q}_3 \in \mathcal{T}_5$ because of Lemma 5. If $\mathbf{q}_3 \in \mathcal{T}_6$ then obviously

$$181^2 - \alpha_1\alpha_2 = 2^{12},$$

which does not have any solutions with $159 < \alpha_1 \le 181$, and $167 < \alpha_2 \le 181$. The case when $\mathbf{q}_3 \in \mathcal{T}_3$ is treated similarly – we must have

$$|\mathcal{S}^{0,0,1}_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}| = (\alpha_1 - 159)\alpha_2 + (\alpha_2 - 103)31 = 2^{12},$$

which has no integer solutions α_1, α_2 with $159 < \alpha_1 \le 181$, $103 < \alpha_2 \le 167$.

2B) Now $e^{0,0}_{\mathbf{q}_1,\mathbf{q}_2} = 1$ and we must have $|\mathcal{S}^{0,0,i}_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3}| = 2^{12} - 1$, or 2^{12}. The set $\mathcal{S}^{0,0}_{\mathbf{q}_1,\mathbf{q}_2}$ is divided as above in six subsets. Investigating each one of them we find the only possiblity $\mathbf{q}_3 = (171, 161)$. It is easily checked that $e^{0,0,0}_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3} = 0$ and for each fourth question $\mathbf{q}_4 = (\delta_1, \delta_2)$ we must have $|\mathcal{S}^{0,0,0,i}_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4}| = 2^{11}$, $i = 0, 1$. This turns out to be impossible. To show this we must first note that necessarily $126 < \delta_1 \le 181$, and $170, \delta_2 \le 181$. There is a straightforward check that it cannot be found a $\mathbf{q}_4$ which satisfies $|\mathcal{S}^{0,0,0,1}_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{q}_3,\mathbf{q}_4}| = 2^{11}$.

The rest of the cases are treated in a similar way. $\square$

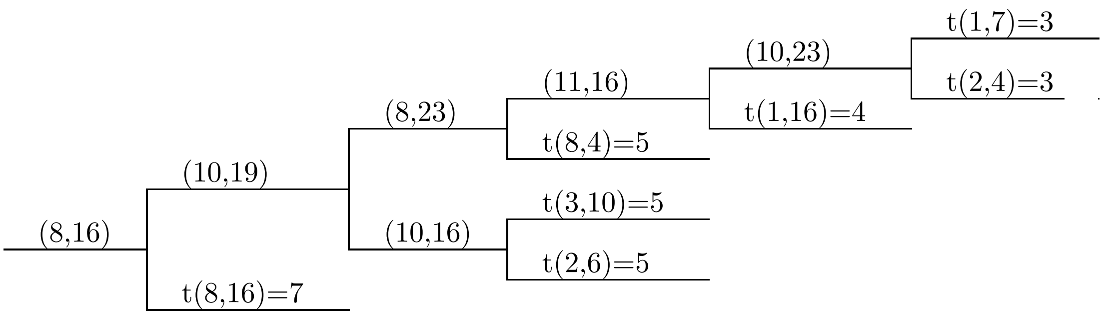
Let us finish with an open problem. Although we could show that $\mathcal{A}(181, 181)$ is loose we were not able to prove that there exist infinitely many loose squares. So the problem is:

Problem. Are there infinitely many loose squares?

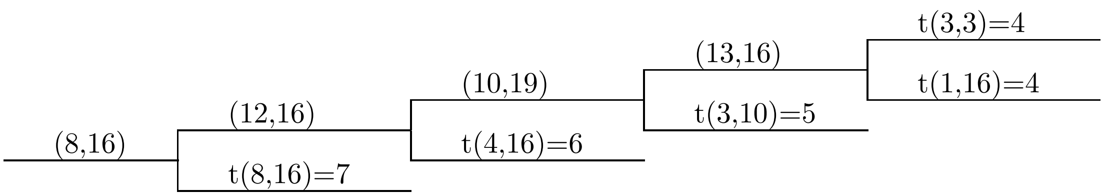
Appendix.

We present below the missing algorithms from the proof of Proposition 7. The pictures have to be understood in the following way. We move from left to right asking the questions (a, b), where the numbers a and b are specified on the pictures. If we get an answer ”no” we move upwards, in case of answer “yes” we move downwards. When we arrive at a rectangle of size u by v which is tight and for which the algorithm is known (Proposition 2, Theorem 4), we indicate this by giving the value $t(u, v)$.

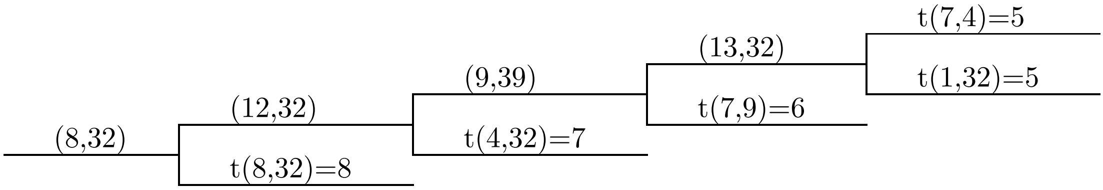
$\mathcal{A}(11, 23)$



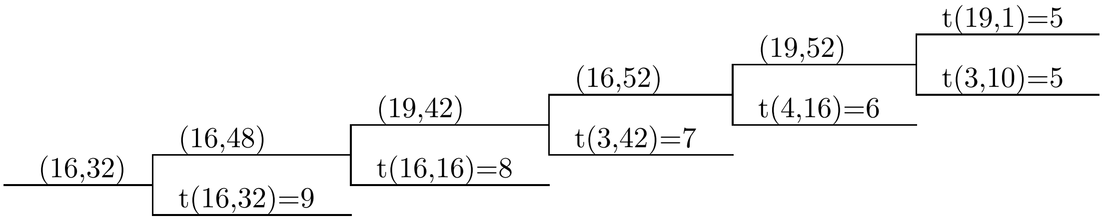
$\mathcal{A}(13, 19)$



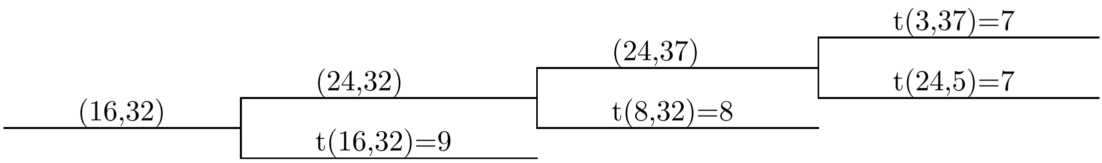
$\mathcal{A}(13, 39)$



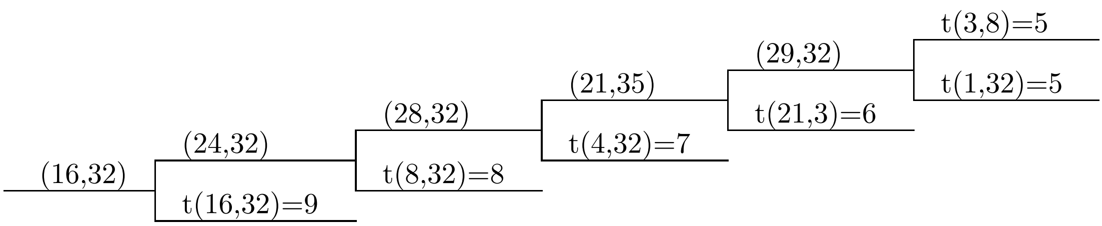
$\mathcal{A}(19, 53)$



$\mathcal{A}(27, 37)$



$\mathcal{A}(29, 35)$



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