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OPTIMIZATION OF DISCRETE-TIME, STOCHASTIC SYSTEMS*

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Abstract. In this paper we study discrete-time, finite horizon stochastic systems with multivalued dynamics and obtain a necessary and sufficient condition for optimality using the dynamic programming method. Then we examine a nonlinear stochastic discrete-time system with feedback control constraints and for it, we derive a necessary and sufficient condition for optimality which we then use to establish the existence of an optimal policy.

1. Introduction. In this paper we consider the problem of optimization of discrete-time, probabilistic dynamical-systems, monitored by a multivalued equation. Such systems stand in the interface of optimal control theory and mathematical economics. In fact in the last section of the paper, we consider a discrete-time, finite horizon, stochastic control system with nonlinear dynamics and feedback control constraints, and we derive for it necessary and sufficient conditions for optimality, using techniques from the theory of dynamic programming. More precisely, through a nonlinear stochastic functional equation (Bellman's equation), we establish these necessary and sufficient conditions, under generally mild hypotheses on the data. We also use them to obtain an optimal process (path).

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The study of such probabilistic discrete-time systems, was initiated by Morozan [9]. A comprehensive introduction ot the subject, with special emphasis on various stochastic economic models, can be found in the book of Arkin-Evstigneev [1].

2. Preliminaries. Let (Ω, Σ) be a measurable space and X a Polish space (i.e. a complete separable, metrizable space). By $P_f(X)$ (resp. $P_k(X)$) we will be denoting the family of all nonempty closed (resp. compact) subsets of X. A multifunction $F : \Omega \to P_f(X)$ is said to be measurable if it satisfies one of the following three equivalent statements:

(*i*) for every $U \subseteq X$ open, $F^-(U) = \{\omega \in \Omega : F(\omega) \cap U \neq \emptyset\} \in \Sigma$,

(*ii*) for every $x \in X$, $\omega \to d(x, F(\omega)) = \inf\{d(x, z) : z \in F(\omega)\}$ is measurable, where $d(\cdot, \cdot)$ is a complete metric generating the topology of X,

(*iii*) there exist $f_n : \Omega \to X$, $n \geq 1$ measurable functions, *s.t.* for all $\omega \in \Omega$, $F(\omega) = \overline{\{f_n(\omega)\}}_{n\geq 1}$.

A multifunction $G : \Omega \to 2^X$ is said to be "graph measurable", if $GrG = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\} \in \Sigma \times B(X)$, with $B(X)$ being the Borel σ-field of X. For a $P_f(X)$-valued multifunction, measurability implies graph measurability, while the converse is true if there exists a σ-finite measure $\mu(\cdot)$ defined on Σ, with respect to which Σ is complete. For further details on the measurability of multifunctions, we refer to the survey paper of Wagner [12].

Let $\mu(\cdot)$ be a finite measure on Σ and assume that (Ω, Σ, μ) is a complete measure space. Given a multifunction $G : \Omega \to 2^X \setminus \{\emptyset\}$, by S_G we will denote the set of Σ-measurable selectors of $G(\cdot)$; i.e. $S_G = \{g : \Omega \to X$ measurable $s.t. g(\omega) \in G(\omega), \omega \in \Omega\}$. The next theorem originally due to Aumann (see for example Wagner [12]), gives us conditions that guarantee the nonemptiness of S_G. Recall that a Souslin space, is a Hausdorff topological space X for which there exists a Polish space P and a continuous map from P onto X. So a separable Banach space Z equipped with the week topology (denoted by Z_ω), is clearly a Souslin space. Just let P be the space Z endowed with the strong (norm) topology and consider the identity map from P onto Z_ω. So a Souslin space is always separable, but need not be metrizable.

Theorem 2.1 (Aumann). *If (Ω, Σ, μ) is a finite, complete probability space, X a Souslin space and $G : \Omega \to 2^X \setminus \{\emptyset\}$ a graph measurable multifunction, then $S_G \neq \emptyset$.*

Let $\Sigma_0 \subseteq \Sigma$ be a complete sub-σ-field of Σ, X a Polish space and $f : \Omega \times X \to \mathbb{R}$ an integrand such that $f(\cdot, \cdot)$ is $\Sigma \times B(X)$-measurable, for every $\omega \in \Omega$, $f(\omega, \cdot)$ is upper semicontinuous ($u.s.c$) on X and there exists $\varphi(\cdot)$ a μ-integrable function such that $f(\omega, x) \leq \varphi(\omega)$ μ-$a.e.$ for all $x \in X$. The next result due to Klei [7], establishes the existence and uniqueness of the conditional expectation of $f(\cdot, \cdot)$ with respect to Σ_0.

Theorem 2.2 (Klei). *If (Ω, Σ, μ), Σ_0 and $f(\omega, x)$ are as above, then there exists a unique integrand $g : \Omega \times X \to \mathbb{R}$ which is $\Sigma_0 \times B(X)$-measurable, $g(\omega, \cdot)$ is u.s.c. and for all $u \in L^0(\Omega, \Sigma_0; X)$ and $B \in \Sigma_0$ we have*

$$\int_B f(\omega, u(\omega))d\mu(\omega) = \int_B g(\omega, u(\omega))d\mu(\omega).$$

Remark: We denote $g(\omega, x)$ by $E^{\Sigma_0} f(\omega, x)$. As always uniqueness should be understood up to sets of measure zero in the ω-variable. It should be mentioned that the above result of Klei extends earlier ones obtained by Dynkin-Evstigneev [5] and Thibault [11].

Finally let Y, Z be Hausdorff topological spaces. A multifunction $H : Y \to 2^Z \setminus \{\emptyset\}$ is said to be upper semicontinuous ($u.s.c$), if for all $C \subseteq Z$ closed, $H^-(C) = \{y \in Y : H(y) \cap C \neq \emptyset\}$. Recall (see for example, Berge [4]), that if $H(\cdot)$ has closed values, then upper semicontinuity in the above sense implies that $H(\cdot)$ has a closed graph; i.e. $GrH = \{(y, z) \in Y \times Z : z \in H(y)\}$ is closed in $Y \times Z$. The converse is true, if $\overline{H(Y)} = \overline{\bigcup_{y \in Y} H(y)}$ is compact in Z.

3. Multivalued stochastic systems. The mathematical setting is the following: Let (Ω, Σ, μ) be a probability space $T_N = \{0, 1, 2, \ldots, N\}$ is our finite, discrete-time horizon and $\{\Sigma_k\}_{k=0}^N$ is a finite sequence of increasing complete sub-σ-fields of Σ, with $\Sigma_N = \Sigma$. Each sub-σ-field Σ_k is interpreted as the information about states that are realized up to time $k \in T_N$ (i.e. the collection of all events prior to time instant k, including k). Let $\{X_k\}_{k=0}^N$ be Polish spaces, representing the state space for each time instant $k \in T_N$. The mutlivalued, discrete-time, finite horizon system under consideration, is the following:

$$(1) \qquad \left| \begin{array}{c} \displaystyle\int_\Omega \sum_{k=0}^N L_k(\omega, x_k(\omega))d\mu(\omega) \to \sup = M \\[2ex] s.t. \quad x_{k+1}(\omega) \in F_{k+1}(\omega, x_k(\omega)) \quad \mu\text{-}a.e., \quad k = 0, 1, 2, \ldots, N \\[2ex] x_0(\omega) = v_0(\omega). \end{array} \right|$$

This general formulation incorporates, in particular, the models of a developing economy, proposed by Gale (see for example Makarov-Rubinov [8] and Takayama [10]).

In Gale's model $X_k = \mathbb{R}_+^m \times \mathbb{R}_+^m$; i.e. the state space consists of pairs of nonnegative m-dimensional vectors. Subsets $G_1, \ldots, G_N$ of $\mathbb{R}_+^m \times \mathbb{R}_+^m$ representing the technological constraints at each time instant k are given and $F_k(x, y)$ is defined to be the collection of all pairs $(x', y') \in G_k$ such that $x' \leq y$. The pair (x, y) should be understood as a production process, with x the input vector and y the output vector. So G_k is the collection of all production processes that are technologically realizable at time k. Then the dynamical inclusion $z_k \in F_k(\omega, z_{k-1})$ describing an admissible program (path) of our system (1), can be rewritten in the form $(x_k, y_k) \in G_k$, $x_k \leq y_{k-1}$. As we already mentioned above, the first relation reflects technological limitations, while the second says that at each time instant k, the input can not exceed the output produced in $k-1$ (i.e. we can not live above our means). Also $L_k(\omega, x)$ represents the utility realized by the input x.

The hypotheses on the data of (1) are the following:

$H(F)$: $F_k : \Omega \times X_{k-1} \to 2^{X_k} \setminus \{\emptyset\}$, $k = 1, 2, \ldots, N$ is a multifunction *s.t.*
- (a) $(\omega, x) \to F_k(\omega, x)$ is $\Sigma_k \times B(X_{k-1}) \times B(X_k)$-measurable and has compact values (i.e. for each $(\omega, x) \in \Omega \times X_{k-1}$, $F_k(\omega, x) \in P_k(X_k)$),
- (b) $x \to F_k(\omega, x)$ is *u.s.c.* on X_{k-1}.

$H(L)$: $L_k : \Omega \times X_k \to \mathbb{R}$, $k = 1, 2, \ldots, N$ is an integrand *s.t.*
- (a) $(\omega, x) \to L_k(\omega, x)$ is $\Sigma_k \times B(X_k)$-measurable,
- (b) $x \to L_k(\omega, x)$ is *u.s.c.* on X_k,
- (c) there exists an integrable function $\varphi_k(\cdot)$ *s.t.* $L_k(\omega, x) \leq \varphi_k(x)$ μ-*a.e.* for all $x \in X_k$.

For economy in the notation, set $J_k(x) = \int_\Omega L_k(\omega, x(\omega))d\mu(\omega)$, $x \in L^0(\Sigma_k, X_k) = \{y : \Omega \to X_k, \Sigma_k\text{-measurable}\}$ and $\Gamma_k(z) = S_{F_k(\cdot, z(\cdot))} = \{y \in L^0(\Sigma_k, X_k) : y(\omega) \in F_k(\omega, z(\omega)), \omega \in \Omega\}$ with $z \in L^0(\Sigma_{k-1}, X_{k-1})$.

Inductively, we define the following "global" Bellman functions (dynamic programming functions):

$$V_N(x) = J_N(x),\ x \in L^0(\Sigma_N, X_N)$$

$$\text{and}\ \ V_{k-1}(x) = \sup\left[J_{k-1}(x) + V_k(y)\ :\ y \in \Gamma_k(x)\right],\ k = 1, 2, \ldots, N.$$

By a "feasible program" (path) of (1), we mean a finite sequence $\{x_k\}_{k=0}^N$ *s.t.* $x_k \in L^0(\Sigma_k, X_k)$, $x_0 = v_0$ and $x_{k+1}(\omega) \in F_{k+1}(\omega, x_k(\omega))$. An "optimal program", is a feasible program that maximizes the intertemporal criterion $\displaystyle\int_\Omega \sum_{k=0}^N L_k(\omega, x_k(\omega))d\mu(\omega)$. Using the previously defined global Bellman functions, we can have our first optimality conditions for problem (1):

Proposition 3.1. *If hypotheses $H(F)$ and $H(L)$ hold, and $v_0 \in L^0(\Sigma_0, X_0)$,* *then program* $\{x_k^0\}_{k=0}^N$ *is optimal if and only if*

$$V_k(x_k^0) = J_k(x_k^0) + V_{k+1}(x_{k+1}^0),\ k = 0, 1, \ldots, N-1.$$

P r o o f. $\Downarrow$: First we will establish the equality for $k = N-1$. By definition we have:

$$\begin{aligned} V_{N-1}(x_{N-1}^0) &= \sup\left[J_{N-1}(x_{N-1}^0) + V_N(y) : y \in \Gamma_N(x_{N-1}^0)\right] \\ &= \sup\left[J_{N-1}(x_{N-1}^0) + J_N(y) : y \in \Gamma_N(x_{N-1}^0)\right]. \end{aligned}$$

Since by hypothesis program $\{x_k^0\}_{k=0}^N$ is optimal, it is clear that the above supremum is realized at $y = x_N^0$. So we have established the validity of the optimaly equation for $k = N-1$.

Now suppose the equation holds for $k = N-1, \ldots, m+1$. Then by definition we have:

$$V_m(x_m^0) = \sup\left[J_m(x_m^0) + V_{m+1}(y) \ : \ y \in \Gamma_{m+1}(x_m^0)\right].$$

Let $\varepsilon > 0$ and choose $\{y_k\}_{k=m}^N$ such that $y_m = x_m^0$, $y_{k+1} \in \Gamma_{k+1}(y_k)$ and

$$V_k(y_k) - \frac{\varepsilon}{N-m} \leq J_k(y_k) + V_{k+1}(y_{k+1}),\ k = m, m+1, \ldots, N-1.$$

Then we get

$$V_m(x_m^0) - \left(\frac{\varepsilon}{N-m}\right)(N-m) \leq \sum_{k=m}^N J_k(y_k).$$

But since $\{x_k^0\}_{k=0}^N$ is an optimal program, we have that

$$\sum_{k=m}^N J_k(y_k) \leq \sum_{k=m}^N J_k(x_k^0)$$

$$\Rightarrow V_m(x_m^0) - \varepsilon \leq \sum_{k=m}^N J_k(x_k^0).$$

Because of the hypothesis, we have

$$\sum_{k=m}^N J_k(x_k^0) = J_m(x_m^0) + V_{m+1}(x_{m+1}^0)$$

$$\Rightarrow V_m(x_m^0) - \varepsilon \leq J_m(x_m^0) + V_{m+1}(x_{m+1}^0).$$

Since $\varepsilon > 0$ was arbitrary, let $\varepsilon \downarrow 0$, to get

$$V_m(x_m^0) \leq J_m(x_m^0) + V_{m+1}(x_{m+1}^0).$$

The opposite inequality is always true from the definitions of the global Bellmann functions. Therefore

$$V_m(x_m^0) = J_m(x_m^0) + V_{m+1}(x_{m+1}^0)$$

and so by induction, we have established the validity of the optimality equation for all $k = 0, 1, \ldots, N$.

$\Uparrow$: Let $\{w_k\}_{k=0}^N$ be a feasible program. From the optimality equation, we have:

$$V_0(x_0^0) = \sum_{k=0}^N J_k(x_k^0).$$

But by the definition,
$$\begin{aligned} V_0(x_0^0) &= \sup\left[J_0(x_0^0) + V_1(y) : y \in \Gamma_1(x_0^0)\right] \\ &\geq J_0(x_0^0) + V_1(w_1) = J_0(v_0) + V_1(w_1) \\ &\geq J_0(v_0) + J_1(w_1) + V_2(w_2) \\ &\geq \ldots \geq \sum_{k=0}^N J_k(w_k). \end{aligned}$$

Since $\{w_k\}_{k=0}^N$ was an arbitrary feasible program, we conclude that $\{x_k^0\}_{k=0}^N$ is indeed optimal.

Q.E.D.

Now we define the pointwise Bellman functions as follows:

$$v_N(\omega, x) = L_N(\omega, x)$$

and $v_k(\omega, x) = E^{\Sigma_k} \sup[L_k(\omega, x) + v_{k+1}(\omega, y) \; : \; y \in F_{k+1}(\omega, x), \, k = 0, 1, 2, \ldots, N-1]$.

First we will show that these pointwise Bellman functions are well defined. To this end, let $h_n(\omega, x) = v_N(\omega, x)$ and $h_k(\omega, x) = \sup[L_k(\omega, x) + v_{k+1}(\omega, y) : y \in F_{k+1}(\omega, x)]$.

Proposition 3.2. *If hypoteses $H(F)$ and $H(L)$ hold, then $(\omega, x) \to h_k(\omega, x)$ is $\Sigma_{k+1} \times B(X_k)$-measurable, $k = 0, 1, \ldots, N$, with*

$$\Sigma_{N+1} = \Sigma_N = \Sigma \text{ and } x \to h_k(\omega, x) \text{ is u.s.c.}$$

P r o o f. First note that by a simple induction argument and by using Theorem 2.1 of this paper, as well as Theorem 2, p. 122 of Berge [4], we can have that for all $k = 0, 1, 2, \dots, N$ and $\omega \in \Omega$, $x \to h_k(\omega, x)$ is *u.s.c.*

Next note that because of hypothesis $H(L)$, for $k = N$, $h_N = v_N = L_N$ is $\Sigma_N \times B(X_N)$-measurable. Then assume that we have established the measurability of $h_k(\omega, x)$ for $k = N, N-1, \dots, m+1$. Set $\eta_k(\omega, x, y) = L_k(\omega, x) + v_{k+1}(\omega, y)$. Then since by the induction hypotesis, $h_{k+1}(\omega, y)$ is $\Sigma_{k+2} \times B(X_{k+1})$-measurable and we know that $h_{k+1}(\omega, \cdot)$ is *u.s.c.*, we deduce from Theorem 2.2, that $v_{k+1}(\omega, y)$ is $\Sigma_{k+1} \times B(X_{k+1})$-measurable and $v_{k+1}(\omega, \cdot)$ is *u.s.c.* Thus, $\eta_k(\omega, x, y)$ is $\Sigma_{k+1} \times B(X_k) \times B(X_{k+1})$-measurable and $\eta_k(\omega, \cdot, \cdot)$ is *u.s.c.* Invoking Lemma 2 of Balder [3], we can find $\eta_k^n : \Omega \times X_k \times X_{k+1} \to \mathbb{R}$ integrands *s.t.* $\eta_k^n(\cdot, x, y)$ is Σ_{k+1}-measurable, $\eta_k^n(\omega, \cdot, \cdot)$ is continuous and $\eta_k^n(\omega, x, y) \downarrow \eta_k(\omega, x, y)$ as $n \to \infty$.

Set $h_k^n(\omega, x) = \sup[\eta_k^n(\omega, x, y) : y \in F_{k+1}(\omega, x)]$. Because of hypothesis $H(F)$ (a), we can find $y_m : \Omega \times X_k \to X_{k+1}$, $m \geq 1$ functions which are $(\Sigma_{k+1} \times B(X_k), B(X_{k+1}))$-measurable *s.t.* $\overline{\{y_m(\omega, x)\}}_{m\geq 1} = F_{k+1}(\omega, x)$ (see Section 2). So

$$h_k^n(\omega, x) = \sup_{m\geq 1} \eta_k^n(\omega, x, y_m(\omega, x)).$$

Since $\eta_k^n(\omega, x, y)$ is jointly measurable (being measurable in ω, continuous in (x, y)), we have that $(\omega, x) \to \eta_k^n(\omega, x, y_m(\omega.x))$ is $\Sigma_{k+1} \times B(X_k)$–measurable for all $m \geq 1 \Rightarrow (\omega, x) \to h_k^n(\omega, x)$ is $\Sigma_{k+1} \times B(X_k)$–measurable. Furthermore from Theorem 1.44, p. 101 of Attouch [2], we have

$$h_k^n(\omega, z) \downarrow h_k(\omega, x) \quad \text{for all} \quad (\omega, x) \in \Omega \times X_k \quad \text{as} \quad n \to \infty,$$

$$\Rightarrow h_k(\omega, x) \quad \text{is} \quad \Sigma_{k+1} \times B(X_k)\text{-measurable.}$$

So by induction we have established the claim of this proposition.

Q.E.D.

Therefore invoking Theorem 2.2, we deduce that $v_k(\omega, x)$ is a well-defined $\Sigma_k \times B(X_k)$-measurable function and for all $\omega \in \Omega$, $x \to v_k(\omega, x)$ is *u.s.c.*

The next result will be needed in the proof of the main theorem in this section.

Proposition 3.3. *If hypotheses $H(E)$ and $H(L)$ hold and $x(\cdot) \in L^0(\Sigma_k, X_k)$, then $V_k(x) = \int_\Omega h_k(\omega, x(\omega))d\mu(\omega)$.*

P r o o f. We will establish the equality by induction.

For $k = N$, we have $\int_\Omega h_N(\omega, x(\omega))d\mu(\omega) = \int_\Omega L_N(\omega, x(\omega))d\mu(\omega) = J_N(x) = V_N(x)$.

Suppose we have established the claim of the proposition for $k = N, N - 1, \dots, m+1$. Then we have

$$\int_\Omega h_m(\omega, x(\omega))d\mu(\omega) = \int_\Omega \sup[L_m(\omega, x) + h_{m+1}(\omega, y) : y \in F_{m+1}(\omega, x)]d\mu(\omega).$$

Using Theorem 2.2 of Hiai-Umegaki [6], we get

$$\int_\Omega \sup[L_m(\omega, x) + h_{m+1}(\omega, y) : y \in F_{m+1}(\omega, x)]d\mu(\omega)$$

$$= \sup\left[\int_\Omega (L_m(\omega, x(\omega)) + h_{m+1}(\omega, y(\omega)))d\mu(\omega) : y \in S_{F_{m+1}(\cdot, x(\cdot))}\right]$$

$$\Rightarrow \int_\Omega h_m(\omega, x(\omega))d\mu(\omega) = \sup\left[J_m(x) + V_{m+1}(y) : y \in S_{F_{m+1}(\cdot, x(\cdot))}\right] = V_m(x).$$

So by induction, we have proved the claim of the proposition.

Q.E.D.

Now we are ready to state and prove our main result of this section, which gives us a pointwise necessary and sufficient condition for optimality of a feasible program. This condition is a stochastic version of Bellman's dynamic programming functional equation.

Theorem 3.1. *If hypotheses $H(F)$ and $H(L)$ hold, $v_0 \in L^0(\Sigma_0, X_0)$ and $\{x_k\}_{k=0}^N$ is a feasible program, then program $\{x_k\}_{k=0}^N$ is optimal for* (1) *if and only if*

$$v_k(\omega, x_k^0(\omega)) = L_k(\omega, x_k^0(\omega)) + E^{\Sigma_k} v_{k+1}(\omega, x_{k+1}^0(\omega)) \ \mu\text{-a.e.}, \ k = 0, 1, \dots, N-1.$$

P r o o f. $\Downarrow$: From Proposition 3.1, we have that

$$V_k(x_k^0) = J_k(x_k^0) + V_{k+1}(x_{k+1}^0), \ k = 0, 1, \dots, N-1.$$

Using Proposition 3.3, we can write

$$V_{k+1}(x_{k+1}^0) = \int_\Omega h_{k+1}(\omega, x_{k+1}^0(\omega))d\mu(\omega) = \int_\Omega E^{\Sigma_{k+1}} h_{k+1}(\omega, x_{k+1}^0(\omega))d\mu(\omega)$$

since by definition $x_{k+1}^0 \in L^0(\Sigma_{k+1}, X_{k+1})$ (see Theorem 2.2). But then from the definition of the pointwise Bellman functions, we have

$$\int_\Omega E^{\Sigma_{k+1}} h_{k+1}(\omega, x_{k+1}^0(\omega))d\mu(\omega) = \int_\Omega v_{k+1}(\omega, x_{k+1}^0(\omega))d\mu(\omega)$$

$$\Rightarrow V_{k+1}(x^0_{k+1}) = \int_\Omega v_{k+1}(\omega, x^0_{k+1}(\omega))d\mu(\omega).$$

Therefore using Theorem 2.2, we have

$$0 = \int_\Omega \left[h_k(\omega, x^0_k(\omega)) - L_k(\omega, x^0_k(\omega)) - v_{k+1}(\omega, x^0_{k+1}(\omega))\right] d\mu(\omega)$$

$$= \int_\Omega (E^{\Sigma_k} h_k(\omega, x^0_k(\omega)) - E^{\Sigma_k} L_k(\omega, x^0_k(\omega)) - E^{\Sigma_k} v_{k+1}(\omega, x^0_{k+1}(\omega)))d\mu(\omega)$$

$$= \int_\Omega E^{\Sigma_k}(h_k(\omega, x^0_k(\omega)) - L_k(\omega, x^0_k(\omega)) - v_{k+1}(\omega, x^0_{k+1}(\omega)))d\mu(\omega).$$

But by definition we have

$$h_k(\omega, x^0_k(\omega)) - L_k(\omega, x^0_k(\omega)) - v_{k+1}(\omega, x^0_{k+1}(\omega)) \geq 0 \quad \mu\text{-}a.e.$$

$$\Rightarrow E^{\Sigma_k}(h_k(\omega, x^0_k(\omega)) - L_k(\omega, x^0_k(\omega)) - v_{k+1}(\omega, x^0_{k+1}(\omega))) = 0 \quad \mu\text{-}a.e.$$

$$\Rightarrow E^{\Sigma_k} h_k(\omega, x^0_k(\omega)) = v_k(\omega, x^0_k(\omega)) = L_k(\omega, x^0_k(\omega)) + E^{\Sigma_k} v_{k+1}(\omega, x^0_{k+1}(\omega)) \quad \mu\text{-}a.e.$$

for $k = 0, 1, \ldots, N-1$.

$\Uparrow$: From the Bellman equation we have

$$\int_\Omega v_k(\omega, x^0_k(\omega))d\mu(\omega) = \int_\Omega L_k(\omega, x^0_k(\omega))d\mu(\omega)$$

$$= \int_\Omega v_{k+1}(\omega, x^0_{k+1}(\omega))d\mu(\omega) \quad k = 0, 1, \ldots, N-1.$$

But note that from the definition of $v_k(\omega, x)$ and using Proposition 3.2, we have

$$\int_\Omega v_k(\omega, x^0_k(\omega))d\mu(\omega) = \int_\Omega E^{\Sigma_k} h_k(\omega, x^0_k(\omega))d\mu(\omega) = \int_\Omega h_k(\omega, x^0_k(\omega))d\mu(\omega) = V_k(x^0_k)$$

and

$$\int_\Omega v_{k+1}(\omega, x^0_{k+1}(\omega))d\mu(\omega) = \int_\Omega E^{\Sigma_{k+1}} h_{k+1}(\omega, x^0_{k+1}(\omega))d\mu(\omega)$$

$$= \int_\Omega h_{k+1}(\omega, x^0_{k+1}(\omega))d\mu(\omega) = V_{k+1}(x^0_{k+1}).$$

So $V_k(x^0_k) = J_k(x^0_k) + V_{k+1}(x^0_{k+1})$ for $k = 0, 1, 2, \ldots, N-1$, which by Proposition 3.1 implies that the feasible program $\{x^0_k\}_{k=0}^N$ is indeed optimal.

Q.E.D.

4. Discrete-time, stochastic optimal control. In this section we turn our attention to a discrete-time stochastic optimal control system, with feedback control constraints. Using Bellman functions, we derive a necessary and sufficient condition for optimality of a feasible program, which is then used to establish the existence of an optimal program.

Let (Ω, Σ, μ), $\{\Sigma_k\}_{k=0}^N$ and $\{X_k\}_{k=0}^N$ be as in Section 3. Also, we are given a collection of Polish spaces $\{Y_k\}_{k=0}^N$ modelling the control space at each time instant k. The problem under consideration is the following:

$$(2) \quad \left| \begin{array}{l} \displaystyle\int_\Omega \sum_{k=0}^N L_k(\omega, x_k(\omega), u_k(\omega)) d\mu(\omega) \to \sup = M' \\ s.t.\ x_{k+1}(\omega) = f_k(\omega, x_k(\omega), u_k(\omega))\ \mu\text{-}a.e. \\ x_0(\omega) = \gamma_0(\omega),\ u_k(\omega) \in U_k(\omega, x_k(\omega))\ \mu\text{-}a.e. \end{array} \right.$$

By a feasible "state-control" policy we mean two sequences $\{x_k\}_{k=0}^N$ and $\{u_k\}_{k=0}^N$ such that $x_k \in L^0(\Sigma_k, X_k)$ and $u_k \in L^0(\Sigma_k, X_k)$ and they satisfy the constraints of problem (2) above. An optimal policy, is a feasible policy that maximizes the intertemporal integral criterion.

We will need the following hypotheses on the data of (2):

$H(f)$: $f_k : \Omega \times X_k \times Y_k \to X_{k+1}$, $k = 0, 1, 2, \ldots, N-1$ are maps $s.t.$

(1) $\omega \to f_k(\omega, x, y)$ is Σ_k-measurable,

(2) $(x, y) \to f_k(\omega, x, y)$ is continuous.

$H(U)$: $U_k : \Omega \times X_k \to P_k(Y_k)$, $k = 0, 1, \ldots, N$, are multifunctions $s.t.$

(1) $(\omega, x) \to U_k(\omega, x)$ is $(\Sigma_k \times B(X_k), B(Y_k))$-measurable,

(2) $x \to U_k(\omega, x)$ is $u.s.c.$

$H(L)'$: $L_k : \Omega \times X_k \times Y_k \to \mathbb{R}$, $k = 0, 1, \ldots, N$ are integrands $s.t.$

(1) $(\omega, x, y) \to L_k(\omega, x, y)$ is $\Sigma_k \times B(X_k) \times B(Y_k)$-measurable,

(2) $(x, y) \to L_k(\omega, x, y)$ is $u.s.c.$,

(3) $L_k(\omega, x, y) \le \varphi(\omega)$ μ-$a.e.$ for all $x \in X_k$, $u \in U_k(\omega, x)$ and with $\varphi_k(\cdot)$ an integrable Σ_k-measurable function.

Define inductively the following pointwise Bellman functions:

$$v_N(\omega, x) = \sup\left[L_N(\omega, x, u) : u \in U_N(\omega, y)\right]$$

and $v_k(\omega,x)=\max\left[L_k(\omega,x,u)+E^{\Sigma_k}v_{k+1}(\omega,f_k(\omega,x,u)):u\in U_k(\omega,x)\right].$

First we will check that these are well-defined functions.

Proposition 4.1. *If hypotheses $H(f)$, $H(U)$ and $H(L)'$ hold, then for every $k=0,1,\dots,N$, $(\omega,x)\to v_k(\omega,x)$ is $\Sigma_k\times B(X_k)$-measurable and $x\to v_k(\omega,x)$ is u.s.c.*

P r o o f. For $k=N$, we have by definition:

$$v_N(\omega,x)=\sup\left[L_N(\omega,x,u):u\in U_N(\omega,x)\right].$$

Using Lemma 2.2 of Balder [3], we can find $L_N^n:\Omega\times X_N\times Y_N\to\mathbb{R}$ Caratheodory integrands (i.e. $\omega\to L_N^n(\omega,x,y)$ is Σ_N-measurable, $(x,y)\to L_N^n(\omega,x,y)$ is continuous; hence $(\omega,x,y)\to L_N^n(\omega,x,y)$ is jointly measurable, see for example Arkin-Evstigneev [1] *s.t.* $L_N^n(\omega,x,y)\downarrow L_N(\omega,x,y)$. Also let $u_l:\Omega\times X_N\to Y_N$, $l\geq 1$, be measurable functions *s.t.* $U_N(\omega,x)=\overline{\{u_l(\omega,x)\}}_{l\geq 1}$. The existence of this sequence is guaranteed by hypothesis $H(U)$ (1) (see Section 2). Then $v_N^n(\omega,x)=\sup\left[L_N^n(\omega,x,u):u\in U_N(\omega,x)\right]$ $=\sup_{l\geq 1}L_N^n(\omega,x,u_l(\omega,x))\Rightarrow(\omega,x)\to v_N^n(\omega,x)$ is $\Sigma_N\times B(X_N)$-measurable. Because $U_N(\omega,x)\in P_k(Y_N)$ from Theorem 1.44, p. 101 of Attouch [2], we have that $v_N^n(\omega,x)\downarrow v_N(\omega,x)$ as $n\to\infty\Rightarrow(\omega,x)\to v_N(\omega,x)$ is $\Sigma_N\times B(X_N)$-measurable. Also from Theorem 2, p. 122 of Berge [4], we have that $x\to v_N(\omega,x)$ is *u.s.c.*

Next, assume that we have established the claim of the proposition for $k=N,N-1,\dots,m+1$. We have:

$$v_m(\omega,x)=\sup\left[L_m(\omega,x,u)+E^{\Sigma_m}v_{m+1}(\omega,f_m(\omega,x,u)):u\in U_m(\omega,x)\right].$$

From the induction hypothesis and Theorem 2.2, we know that

$$(\omega,x,u)\to E^{\Sigma_m}v_{m+1}(\omega,f_m(\omega,x,u))\text{ is }\Sigma_m\times B(X_m)\times B(Y_m)\text{-measurable}$$

and $(x,u)\to E^{\Sigma_m}v_{m+1}(\omega,f_m(\omega,x,u))$ is *u.s.c.* So as above, by approximating $(\omega,x,u)\to L_m(\omega,x,u)+E^{\Sigma_m}v_{m+1}(\omega,f_m(\omega,x,u))$ with Caratheodory integrands, we can get that $(\omega,x)\to v_m(\omega,x)$ is $\Sigma_m\times B(X_m)$-measurable and $x\to v_m(\omega,x)$ is *u.s.c.* So by induction we have proved the proposition.

Q.E.D.

Now we can state and prove a necessary and sufficient condition for optimality in (2)

Theorem 4.1. *If hypotheses $H(f)$, $H(U)$, $H(L)'$ hold, $\gamma_0\in L^0(\Sigma_0,X_0)$ and $\left(\{x_k^0\}_{k=0}^N,\{u_k^0\}_{k=0}^N\right)$ is a feasible state-control policy, then $\left(\{x_k^0\}_{k=0}^N,\{u_k^0\}_{k=0}^N\right)$ is* *an optimal policy if and only if*

$$v_k(\omega, x_k^0(\omega)) = L_k(\omega, x_k^0(\omega), u_k^0(\omega)) + E^{\Sigma_k} v_{k+1}(\omega, x_{k+1}^0(\omega)) \ \mu\text{-}a.e.$$

P r o o f. $\Downarrow$: Again our proof proceeds by induction.

For $k = N$, by definition we have

$$v_N(\omega, x_N^0(\omega)) = \sup\left[L_N(\omega, x_N^0(\omega), u) : u \in U_N(\omega, x_N^0(\omega))\right].$$

We claim that for μ-almost all $\omega \in \Omega$, $u_N^0(\omega)$ realizes this supremum. Suppose not. Then there exists $A \in \Sigma_N = \Sigma$, $\mu(A) > 0$ such that

$$L_N(\omega, x_N^0(\omega), u_N^0(\omega)) < v_N(\omega, x_N^0(\omega)) \quad \text{for} \quad \omega \in A.$$

Let $H : A \to 2^{Y_N} \setminus \{\emptyset\}$ be defined by

$$H(\omega) = \{u \in Y_N : L_N(\omega, x_N^0(\omega), u) = v_N(\omega, x_N^0(\omega)), u \in U_N(\omega, x_N^0(\omega))\}.$$

From hypothesis $H(U)$ (1), we know that $Gr\, U_N(\cdot, x_N^0(\cdot)) \in \Sigma_N \times B(Y_N)$, while from hypothesis $H(L)'$ and Proposition 4.1 above, we know that

$$(\omega, u) \to L_N(\omega, x_N^0(\omega), u) - v_N(\omega, x_N^0(\omega)) = \theta_N(\omega, u)$$

is $\Sigma_N \times B(Y_N)$-measurable. Therefore

$$Gr\, H = \{(\omega, u) \in A \times Y_N : \theta_N(\omega, u) = 0\} \cap Gr\, U_N(\cdot, x_N^0(\cdot)) \in (\Sigma_N \cap A) \times B(Y_N).$$

Apply Aumann's selection theorem (see Theorem 2.1 in this paper), to get $w : A \to Y$ a $(\Sigma_N \cap A)$-measurable function *s.t.* $w(\omega) \in H(\omega)$ for all $\omega \in A$. Set $\overline{u}_N^0 = \chi_{A^c} u_N^0 + \chi_A w$ and $\overline{u}_k^0 = u_k^0$, $k = 0, 1, \ldots, N-1$. Clearly then $\left(\{(x_k^0\}_{k=0}^N, \{\overline{u}_k^0\}_{k=0}^N\right)$ is feasible and

$$\sum_{k=0}^N J_k(x_k^0, u_k^0) < \sum_{k=0}^N J_k(x_k^0, \overline{u}_k^0)$$

where as in Section 3, $J_k(x, y) = \int_\Omega L_k(\omega, x(\omega), y(\omega)) d\mu(\omega)$. But then this last inequality contradicts the optimality of $\left(\{(x_k^0\}_{k=0}^N, \{u_k^0\}_{k=0}^N\right)$. So our claim follows.

Now suppose we have proved the validity of the optimality equation for $k = N, N-1, \ldots, m+1$. We have:

$$v_m(\omega, x_m^0(\omega)) =$$

$$\sup\left[L_m(\omega, x_m^0(\omega), u) + E^{\Sigma_m} v_{m+1}(\omega, f_m(\omega, x_m^0(\omega), u)) : u \in U_m(\omega, x_m^0(\omega))\right].$$

Let $w \in S_{U_m(\cdot, x_m^0(\cdot))}$ and set $y_{m+1}(\omega) = f_m(\omega, x_m^0(\omega), w(\omega))$. Then via straightforward applications of Aumann's selection theorem as above, we can produce $w_{m+k} \in S_{U_{m+k}(\cdot, y_{m+k}(\cdot))}$ *s.t.* $y_{m+k}(\omega) = f_{m+k-1}(\omega, y_{m+k-1}(\omega), w_{m+k-1}(\omega))$ and

$$L_{m+k}(\omega, y_{m+k}(\omega), w_{m+k}(\omega)) + E^{\Sigma_{m+k}} v_{m+k+1}(\omega, f_{m+k}(\omega, x_{m+k}^0(\omega), w_{m+k}(\omega)))$$

$$= \sup[L_{m+k}(\omega, y_{m+k}(\omega, z) + E^{\Sigma_{m+k}} v_{m+k+1}(\omega, f_{m+k}(\omega, x_{m+k}^0(\omega), z)) :$$

$$z \in U_{m+k}(\omega, y_{m+k}(\omega))]$$

for $k = 1, 2, \ldots, N - m - 1$, while

$$y_N(\omega) = f_{N-1}(\omega, y_{N-1}(\omega), w_{N-1}(\omega)) \text{ and } w_N \in S_{U_N(\cdot, y_N(\cdot))} \text{ is such that}$$

$$L_N(\omega, y_N(\omega), w_N(\omega)) = \sup\left[L_N(\omega, y_N(\omega), u) : u \in U_N(\omega, y_N(\omega))\right].$$

Then define the following feasible policy:

$$z_k = \begin{cases} x_k^0 & k = 0, 1, \ldots, m \\ y_k & k = m+1, \ldots, N \end{cases} \quad \text{and} \quad h_k = \begin{cases} u_k & k = 0, 1, \ldots, m-1 \\ w & k = m \\ w_k & k = m+1, \ldots, N. \end{cases}$$

From the optimality of the policy $\left(\{(x_k^0\}_{k=0}^N, \{u_k^0\}_{k=0}^N\right)$, we have

$$\sum_{k=m}^N J_k(z_k, h_k) \le \sum_{k=m}^N J_k(x_k^0, u_k^0).$$

Using the induction hypothesis, we get

$$\int_\Omega \left[L_m(\omega, x_m^0(\omega), w(\omega)) + v_{m+1}(\omega, z_{m+1}(\omega))\right] d\mu(\omega)$$

$$\le \int_\Omega \left[L_m(\omega, x_m^0(\omega), u_m^0(\omega)) + v_{m+1}(\omega, x_{m+1}^0(\omega))\right] d\mu(\omega).$$

Since $w \in S_{U_m(\cdot, x_m^0(\cdot))}$ was arbitrarily, using Theorem 2.2 of Hiai-Umegaki [6], we get

$$\int_\Omega \sup\left[L_m(\omega, x_m^0(\omega), w) + E^{\Sigma_m} v_{m+1}(\omega, x_{m+1}^0(\omega)) : u \in U_m(\omega, x_m^0(\omega))\right] d\mu(\omega)$$

$$= \int_\Omega \left[L_m(\omega, x_m^0(\omega), u_m^0(\omega)) + E^{\Sigma_m} v_{m+1}(\omega, x_{m+1}^0(\omega))\right] d\mu(\omega)$$

$$\Rightarrow \sup \left[L_m(\omega, x_m^0(\omega), w) + E^{\Sigma_m} v_{m+1}(\omega, x_{m+1}^0(\omega)) : w \in U_m(\omega, x_m^0(\omega)) \right]$$
$$= L_m(\omega, x_m^0(\omega), u_m^0(\omega)) + E^{\Sigma_m} v_{m+1}(\omega, x_{m+1}^0(\omega)) \ \mu\text{-}a.e.$$

So by induction, we have proved the necessity of the optimality equation.

$\Uparrow$: Let $\left(\{y_k\}_{k=0}^N, \{w_k\}_{k=0}^N\right)$ be a feasible policy. Then by definition

$$L_k(\omega, y_k(\omega), w_k(\omega)) + E^{\Sigma_k} v_{k+1}(\omega, y_{k+1}(\omega)) \le v_k(\omega, y_k(\omega)), \quad k = 0, 1, \dots, N-1.$$

Successive applications of this inequality give us

$$\sum_{k=0}^{N} J_k(y_k, w_k) \le \int_\Omega v_0(\omega, \gamma_0(\omega)) d\mu(\omega).$$

On the other hand by hypothesis, we have

$$L_k(\omega, x_k^0(\omega), u_k^0(\omega)) + E^{\Sigma_k} v_{k+1}(\omega, x_{k+1}^0(\omega)) = v_k(\omega, x_k^0(\omega)), \quad k = 0, 1, \dots, N-1.$$

Successive applications of this equality give us

$$\sum_{k=0}^{N} J_k(x_k^0, u_k^0) = \int_\Omega v_0(\omega, \gamma_0(\omega)) d\mu(\omega)$$
$$\Rightarrow \sum_{k=0}^{N} J_k(y_k, w_k) \le \sum_{k=0}^{N} J_k(x_k^0, u_k^0)$$

$\Rightarrow \left(\{x_k^0\}_{k=0}^N, \{u_k^0\}_{k=0}^N\right)$ is indeed an optimal feasible policy.

Q.E.D.

By solving the optimization problems in the necessary and sufficient condition of Theorem 4.1, we can prodice step-by-step an optimal policy for problem (2).

Theorem 4.2. *If hypotheses $H(f)$, $H(U)$, $H(L)'$ hold and $v_0 \in L^0(\Sigma_0, X_0)$, then there exists an optimal policy $\left(\{x_k^0\}_{k=0}^N, \{u_k^0\}_{k=0}^N\right)$ for problem* (2).

Proof. Consider the following optimization problem

$$\eta_0(\omega) = \sup \left[L_0(\omega, \gamma_0(\omega), u) + E^{\Sigma_0} v_1(\omega, f_0(\omega, \gamma_0(\omega), u)) : u \in U_0(\omega, \gamma_0(\omega)) \right].$$

Using Proposition 4.1 and Theorem 2.2, we see that $(\omega, u) \to L_0(\omega, \gamma_0(\omega), u) + E^{\Sigma_0} v_1(\omega, f_0(\omega, \gamma_0(\omega), u))$ is jointly measurable and $u.s.c.$ in u. So by Weierstrass theorem and as before via Aumann's selection theorem, we can find $u_0^0 \in S_{U_0(\cdot, \gamma_0(\cdot))}$ such that

$$L_0(\omega, \gamma_0(\omega), u_0^0(\omega)) + E^{\Sigma_0} v_1(\omega, f_0(\omega, \gamma_0(\omega), u_0^0(\omega))) = \eta_0(\omega).$$

Set $x_1^0(\omega) = f_0(\omega, \gamma_0(\omega), u_0^0(\omega))$ and consider the next step optimization problem

$$\eta_1(\omega) = \sup \left[L_1(\omega, x_1^0(\omega), u) + E^{\Sigma_1} v_2(\omega, f_1(\omega, x_1^0(\omega), u)) : u \in U_1(\omega, x_1^0(\omega)) \right].$$

With the same technique as above, we can produce $u_1^0 \in S_{U_1(\cdot, x_1^0(\cdot))}$ such that

$$\eta_1(\omega) = L_1(\omega, x_0^1(\omega), u_1^0(\omega)) + E^{\Sigma_1} v_2(\omega, f_1(\omega, x_1^0(\omega), u_1^0(\omega))) \ \mu\text{-}a.e.$$

Set $x_2^0(\omega) = f_1(\omega, x_1^0(\omega), u_1^0(\omega))$ and continue constructing this way two sepuences $\{x_k^0\}_{k=0}^N$, $\{u_k^0\}_{k=0}^N$. Then from Theorem 4.1., we conclude that $\left(\{x_k^0\}_{k=0}^N, \{u_k^0\}_{k=0}^N\right)$ is an optimal feasible policy for problem (2).

Q.E.D.

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