[16] *Watson G.N.* A Treatise on the Theory of Bessel Functions. Cambridge: Cambridge University Press. 1952.

[17] *Titchmarch E.C.* The theory of functions. Oxford Univ. Press, 1939.

[18] *Ahiezer N.I.* The lectures of the approximating theory. Moscow: Nauka, 1965. 408p. (in Russian).

[19] *Chumachenko S.V., Govhar Malik, Imran Saif Chattha.* Series Summation Method in HSRK // Proc. of the international Conference TCSET'2004 "Modern Problems of Radio Engineering Telecommunications and Computer Science". February 24-28. 2004. Lviv-Slavsko, Ukraine. P.248-250.

[20] *Chumachenko S.V.* Solving Electrodynamics Problems by Reproducing Transformations Method // Proc. BEC 2004. Tallinn. October 3-6, 2004. PP. 319-322.

[21] *Chumachenko S.V., Gowher Malik, Khawar Parvez*. Reproducing Kernel Hilbert Space Methods FOR CAD Tools // Proc. EWDTW, 2004. Yalta-Alushta, September 23-26. PP. 247-250.

[22] *Svetlana Chumachenko, Vladimir Hahanov.* Reproducing Transformations method for IP-core of summatory and integral equations solving // Proc. of DSD 2004 Euromicro Symposium on Digital System Design: Architectures, Methods and Tools. August 31 - September 3, 2004, Rennes – France (Work in progress).

[23] *Titchmarch E.C.* The theory of functions. Oxford Univ. Press, 1939.

[24] *Prudnikov A.P., Bryichkov Yu.A., Marichev O.I.* Integrals and series. Moscow: Nauka, 1981. 800p. (in Russian).

Authors' Information

Chumachenko Svetlana – Ph. D. Senior Scientist and a professor assistant. Address: Kharkov National University of Radio Electronics, Ukraine, 61166, Kharkov, Lenin Avenue, 14, Phone: (+380)-57-7021-326, e-mail: ri@kture.kharkov.ua.

Kirichenko Ludmila – Ph. D. Senior Scientist and a professor assistant. Address: Kharkov National University of Radio Electronics, Ukraine, 61166, Kharkov, Lenin Avenue, 14, Phone: (+380)-57-7021-335, e-mail: ludmila@kture.kharkov.ua.

FUZZY SETS: ABSTRACTION AXIOM, STATISTICAL INTERPRETATION, OBSERVATIONS OF FUZZY SETS

Volodymyr Donchenko

Abstract: The issues relating fuzzy sets definition are under consideration including the analogue for separation axiom, statistical interpretation and membership function representation by the conditional Probabilities.

Keywords: fuzzy sets, membership function, conditional distribution

ACM Classification Keywords: I.5.1. Pattern Recognition: Models - Fuzzy set

Introduction

Fussy sets introduced by Lotfi Zadeh [Zadeh, 1965] (see also [Kaufmann, 1982]) were considered on the one hand as the modeling method alternative to statistical ones which ought to realized the idea of uncertainty, vagueness in the situation under consideration, on the other – as the theory which should to generalize the classical Set theory.

Such generalized pretensions syringed up on the fuzzy logic, which considered as the algebra with two binary operations namely max and min on the real numbers from the interval [0,1] for which all the properties of classical Boolean operations are valid excepting the low of excluded middle. Nevertheless nothing out of this frame in the fuzzy logic was considered. Particularly, such axiom of paramount importance known as abstraction [Stoll, 1960] or separation [Kuratovski K, Mostowski A.6 1967] principle was out of consideration. As is well known this principle establishes the correspondence between classical subsets and the properties of the elements of the universal set – namely, predicates on the universal set. In the Theory of Fuzzy sets this axiom is out of consideration by the reference to the notice that only subsets is under consideration. And nevertheless having the notation $\mu_{\underline{A}}(e)$ with reference to $\underline{A}$ as to set (here fuzzy) it means nothing because denote fuzzy subset as the graphic of membership function [Kaufmann, 1982]:

$$\underline{A} = \{(e, \mu_{\underline{A}}(e)) : e \in E)\},\ 0 \le \mu_{\underline{A}} \le 1.$$

This gap with the absence of an analogue of separation cause the question about what is just the object or property characterized uncertainly by the fuzzy subset (E, $\mu_{\underline{A}}(e)$).

Implicitly and only partially the problem of the gap between subsets and properties are bringing about in the conception of the linguistic variables and its values.

It is just within the framework of linguistic variables the separation principle is realizing implicitly, i.e. the equivalency of the properties and subsets is adopted. Such latent realization discover itself by consideration for this or that value: for example for the “distance” – defined properties, for example: “long”, “middle”, ”short”. This properties correspond with certain interval of the “distance, and these intervals are just the supporters for defining the appropriate membership functions. Each of these membership functions namely realizes the uncertain description of the correspondent property. Thus the linguistic variable is not simply a fuzzy set or collection of such sets. It is fuzzy set plus property described implicitly by this variable. In such a way linguistic variable embodies tacitly the idea of the object for uncertainty. In the example with a “distance” these objects of uncertainty are the properties “long”, “middle”, ”short”. Besides, linguistic variable establishes the relations between fuzzy sets and correspondent properties.

Such approach: approach by the mean of linguistic variable is realizing the relation between property and fuzzy sets only partially, because all the supporters of these sets are different from each others. In the example with a “distance” the supporters of the correspondent fuzzy sets are the subintervals of the general interval for the “distance”.

In the author’s publications [Donchenko, 1998, №3], [Donchenko, 1998, №3] probabilistic interpretation for fuzzy sets is proposed: the probabilistic interpretation for membership function to be more explicit.

In the publication [Donchenko, 2004] the more precise definition for fuzzy set is given. In that one appropriate predicate P on the supporter is introduced in the definition by itself under retaining the general approach to definition of fuzzy sets. A membership function is dependent of the supporter’s element and the predicate P within the notation $\mu^{\{P\}}(e)$.

Generalized Logit- and Probit – regressions exemplify such approach. In the ones membership function is associated with certain property – event: purchasing of that or this goods for example. An element from the supporter (appropriate R^p) describes additional conditions: income and so on. The property – event is inhered originally. It is fixed by event $\{Y = 1\}$, correspondent to purchasing or something of that type, with some Bernoulli’s random variable Y.

1. Modified Determination of Fuzzy Sets

The way for the solving the problem of constructing the analogue of the separation principle may be on the author opinion the straight reference on the object or property described uncertainly. This reference ought to be reflected evidently in the definition of the membership function: $\mu^{\{P\}}(e)$, where P – correspondent property (predicate) on E. Taking into account that for classical sets separation principle take place the property may be replaced by appropriate subset(classical surely). Two membership functions $\mu^{\{P_1\}}(e)$ and $\mu^{\{P_2\}}(e)$ c $P_1 \neq P_2$, specify two different Fuzzy sets, even if they are equal as the function of e, $e \in E$.

Definition. By the Fuzzy subset (modified) of the E, which uncertainly describe P on E (or subset $P_E \subseteq E$), will be called the pair (E, $\mu^{(P)}(e)$) or (E, $\mu^{(P_E)}(e)$), where:

- E – is the abstract set, which will be referenced to as universal or supporter;
- P – predicate on E, P_E – subset of E, which corresponds to P;
- $\mu^{(P)}(e)$ – function of two arguments: e, $e \in E$ and P from the set of all predicates on E. This function just as in classical theory of Fuzzy sets will be referenced to as membership function, adding that it characterize uncertainly property P(or P_E) .

2. Probabilistic Interpretation for Fuzzy Sets

Generally speaking, the authors – founders of the Fuzzy set Theory time and again underlined distinction and fundamental alternatively distinction relatively Statistics.

The thesis that the Fuzzy set theory is alternative to statistical methods underlined emphatically. The opinion that the membership function reflects only the degree of the expert subjective confidence became commonplace. But the object of the characterization stayed out of Fuzzy sets definition.

This example of logit- and probit– likely regressions mentioned above is not universally kind example because special character of the supporter.

Nevertheless the practices of mathematical modeling using Fuzzy sets demand emphatically such interpretation. The one ought to bring out Fuzzy sets out subjective confidence that or those experts and put at the researchers disposal possibilities to establish the correspondence between fuzzy set and the correspondent object under the modeling. It ought to answer the question also what is the observation of Fuzzy set.

2.1. Probabilistic Interpretation for Fuzzy Sets: Discrete Supporter

This subsection deals with the probabilistic interpretation for the classical variant of the Fuzzy Set: Fuzzy Set with discrete supporter – discrete E. This interpretation keeps true for modified variant also.

The interpretation namely is the consequence of the next theorem.

Theorem 1. For any classical Fuzzy Set (E, $\mu_{\underline{A}}(e)$) with discrete support E exist such discrete probability space (Ω, B_Ω, P), event $A \in B_\Omega$ and complete collection of events H_e: $H_e \in B_\Omega$, $e \in E$, – within this probability space such that membership function $\mu_{\underline{A}}(e)$ is represented by the system of conditional probabilities in the next form:

$$\mu_{\underline{A}}(e) = P\{A|H_e\},\ e \in E .$$

Proof. Let's choose and fix any two-element set with the elements say α and $\bar{\alpha}$. Let's also construct the Ω in the next way:

$$\Omega = \{\alpha, \bar{\alpha}\} \times E .$$

Any element $\omega \in \Omega$ is the pair (α ,e) or ω=($\bar{\alpha}$,e) for appropriate $e \in E$. As the E is discrete Ω is also discrete. Let p_e, $p_e > 0$, $e \in E$ – be the probabilities of any distribution row with the only constraint: all of them are positive. One may determine on the Boolean of the Ω probability P by the distribution row $\bar{p}_\omega, \omega \in \Omega$ in the next way:

$$\bar{p}_\omega = \begin{cases} \mu_{\underline{A}}(e)p_e, \text{ для } \omega = (e,\alpha) \\ (1-\mu_{\underline{A}}(e))p_e, \text{ для } \omega = (e,\bar{\alpha}) \end{cases}$$

Indeed:

- For any $\omega \in \Omega$ $\bar{p}_\omega \geq 0$;
- $\sum_{\omega \in \Omega} \bar{p}_\omega = \sum_{e \in E} p_{(\alpha,e)} + \sum_{e \in E} p_{(\bar{\alpha},e)} = \sum_{e \in E} \mu_{\underline{A}}(e)p_e + \sum_{e \in E}(1-\mu_{\underline{A}}(e))p_e = 1$

We also determine the event $A \in B_\Omega$ and complete collection of the events $H_e \in B_\Omega$, e∈E by the next relations correspondingly:

$$A = \{\alpha\} \times E$$

$$H_e=\{(\alpha, e), (\bar{\alpha}, e\}, e \in E.$$

It is obviously, that $A \cap H_e=\{(\alpha, e)\}$.
Besides,

$$P\{H_e\}= P\{(\alpha, e)+ P\{(\bar{\alpha}, e)= p_e > 0,$$

$$P(A\cap H_e)=P\{(\alpha, e)\}= \mu_{\underline{A}}(e) \cdot p_e.$$

Also:

$$P(A \mid H_e) = \frac{P(A \cap H_e)}{P(H_e)} = \frac{P\{(\alpha, e)\}}{p_e} = \frac{\mu_{\underline{A}}(e)p_e}{p_e} = \mu_{\underline{A}}(e).$$

Then we get $\mu_{\underline{A}}(e) = P(A \mid H_e)$, and the proof finishes.

The result of the Th1 may be extended on the special collection of the Fuzzy Sets which we will designate as complete in the sense of the next definition.

Definittion. A collection $(E, \mu_{\underline{A_i}}(e))$, $i = \overline{1, n}$ of the classical subset will be called complete if the next relation take place for any $e \in E$:

$$\sum_{i=1}^{n} \mu_{\underline{A_i}}(e) = 1.$$

Theorem 2. For any complete collection of Fuzzy subsets $(E, \mu_{\underline{A_i}}(e))$, $i = \overline{1, n}$ with the equal supporter E one may find discrete probability space (Ω, B_Ω, P), collection of the evens $A_i \in B_\Omega$, $i = \overline{1, n}$ and complete collection of the events H_e : $H_e \in B_\Omega$, e∈E, – within this probability space, that all of the membership functions $\mu_{\underline{A_i}}$, $i = \overline{1, n}$ may be represented as the systems of conditional probabilities in the next way:

$$\mu_{\underline{A_i}}(e) = P\{A_i|H_e\}, \text{ for any } e \in E,\ i = \overline{1, n}.$$

Proof. Just as in previous case let's choose and fix set $\Re$ but with n elements now: $\Re = \{\alpha_1, \ldots, \alpha_n\}$ and construct the Ω as the next Cartesian product $\Omega = \Re \times E$. An element $\omega \in \Omega$ is the pair of the type $\omega=(\alpha, e)$ for appropriate $e \in E$, $\alpha \in \Re$. The set Ω is discrete again. Let's p_e, $p_e > 0$, e∈E are chosen in just the same way as in the proof of the Th1, i.e. are the probabilities of the discrete distribution row on E. The probability P on the Boolean of Ω we construct by the membership function $\mu_{\underline{A_i}}(e)$, $i = \overline{1, n}$ and probabilities p_e, e∈E describing the distribution row $\bar{p}_\omega, \omega \in \Omega$ in the next way now:

$$\bar{p}_\omega = \begin{cases} \mu_{\underline{A_1}}(e) p_e, \text{ for } \omega = (e, \alpha_1) \\ \cdots \\ \mu_{\underline{A_n}}(e)) p_e, \text{ for } \omega = (e, \alpha_n) \end{cases}$$

Indeed:

- For any $\omega \in \Omega$ $\bar{p}_\omega \geq 0$;
- $\sum_{\omega \in \Omega} \bar{p}_\omega = \sum_{i=1}^{n} \sum_{e \in E} \bar{p}_{(\alpha_i, e)} = \sum_{i=1}^{n} \sum_{e \in E} \mu_{\underline{A_i}}(e) p_e = \sum_{i=1}^{n} \mu_{\underline{A_i}}(e) \sum_{e \in E} p_e = \sum_{i=1}^{n} \mu_{\underline{A_i}}(e) = 1.$

We determine the events $A_i \in B_\Omega$, $i=\overline{1,n}$ and complete collection of events $H_e \in B_\Omega$, e∈E then by the next relations correspondingly:

$$A_i = \{\alpha_i\} \times E,\ i=\overline{1,n},$$

$$H_e = \Re \times \{e\},\ e \in E.$$

Once again

$$A_i \cap H_e = \{(\alpha_i, e)\},\ i=\overline{1,n},\ e \in E.$$

and

$$P\{H_e\} = \sum_{\omega \in H_e} \overline{p}_\omega = \sum_{i=1}^{n} \overline{p}_{(\alpha_i, e)} = \sum_{i=1}^{n} \mu_{\underline{A_i}}(e) p_e = p_e > 0,$$

$$P(A_i \cap H_e) = P\{(\alpha_i, e)\} = \mu_{\underline{A_i}}(e) \cdot p_e.$$

Thus

$$P(A_i \mid H_e) = \frac{P(A_i \cap H_e)}{P(H_e)} = \frac{P\{(\alpha_i, e)\}}{p_e} = \frac{\mu_{\underline{A_i}}(e) p_e}{p_e} = \mu_{\underline{A_i}}(e).$$

And so

$$\mu_{\underline{A_i}}(e) = P(A_i \mid H_e),\ i=\overline{1,n},\ e \in E.$$

And this is the end of the proof.

2.2. Probabilistic Interpretation for Fuzzy Sets : Non - Discrete Supporter

The result of the previous subsection may be improved noticeably and to extend to non-discrete case if the supporter E possesses certain structure, namely is the space with a measure.

Theorem 3. Given the:

- (E, $\Im$, m)- is the space with a measure;
- $(E, \mu_{\underline{A_i}}(e))$, $i=\overline{1,n}$ $\mu_i(e)$, i > 0 is the complete collection on Fuzzy subsets with the equal supporters E;
- all of the membership functions $\mu^{(A_i)}(e)$, $i=\overline{1,n}$ are $\Im$, £, measurable (£– Borel σ-algebra on R^1),

then:

- exist probability space (Ω, B_Ω, P),
- exist ξ discrete random S_p – valued random variable on (Ω, B_Ω, P) with S_p is n-element set with elemens say S_i, $i=\overline{1,n}$;
- exist η random E – valued random variable on (Ω, B_Ω, P)

such, that for any $i=\overline{1,n}$

$\mu^{(A_i)}(e)$ = P{ $\xi = S_i \mid \eta = e$ },

where P{$\xi = S_i \mid \eta$ } – conditional distribution of r.v. ξ respectively r.v.. η.

The conditional distribution is regular:for any $e \in E$ P(B | η= e } is a probability respectively B.

Proof. The proof the result extend the idea of previous states of the kind and being technically valuable is ommitted.

2.3. Observations of the Modified Fuzzy Sets

The modification of the definition of Fuzzy set introduced earlier in the paper imparts the objectivity to the Fuzzy set and it is possible now to say about observations of Fuzzy sets for modified ones. It's very important ontologic aspect for mathematical modeling using Fuzzy sets. The observation of modified Fuzzy sets is the pair (e, P(e)) – e, $e \in E$ – element from the supporter and P(e) is the predicate P( ) mean on this element. Namely e is the displayed element and P(e) is the fixed information about property P( ). It is just in such a way the observations are interpreting in the logit- and probit – regressions and in its generalizations.

The membership function may be estimated statistically by MLM for example as it is used to be in the regressions mentioned above. And experts may be used to estimate the function too but in new way additionally. Namely, they may be questioned about $\mu^{(A_i)}(e)$ as it is now, but about fulfilling P(e) also. So the combination of LSM and MLM may be used for the membership function to estimate.

Conclusion

The statistical interpretation and modification of the Fuzzy sets conception represented in the paper earlier make it possible to use the Fuzzy set characterization as the one existent objectively. This approach solves the question of the analogue of separation principle for the classical Fuzzy sets when these ones are considered in modified way.

Bibliography

[Zadeh, 1965] Zadeh, Lotfi. Fuzzy Sets/ Information and Control, 8(3). June 1965. pp. 338-53.

[Kaufmann, 1982], Kaufmann Arnold, Introduction in the Theory of Fuzzy Sets, - Moscow. 1982 - 322 p. (in Russian)

[Stoll, 1960] Stoll, Robert R, Sets,.Logic.and Axiomatic Theories. Freeman and Company, San Francisco,1960.

[Kuratovski K, Mostowski A.б 1967] Kuratovski K, Mostowski A. Set theory – North Holland Publishing Company, Amsterdam, 1967.

[Donchenko, 1998, №3] Donchenko V.S. Conditional distributions and Fuzzy sets. // Bulletin of Kiev University. Series: Physics and Mathematics, №3, 1998. (In Ukrainian)

[Donchenko, 1998, №4] Donchenko V.S. Probability and Fuzzy sets. // Bulletin of Kiev University. Series: Physics and Mathematics, №4,1998. (In Ukrainian)

[Donchenko, 2004] Donchenko V.S. Statistical models of observations and Fuzzy sets. // Вестник Киевского университета, №1, 2004.

Authors' Information

Volodymyr S. Donchenko – Kyiv National Taras Shevchenko Universiry, Professor.
2, Akademician Glushkov prospectus, building 6, Kyiv, 03680, Ukraine; e-mail: vsdon@unictb.kiev.ua