ON THE RELATIONSHIPS AMONG QUANTIFIED AUTOEPISTEMIC LOGIC, ITS KERNEL, AND QUANTIFIED REFLECTIVE LOGIC

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Abstract: A Quantified Autoepistemic Logic is axiomatized in a monotonic Modal Quantificational Logic whose modal laws are slightly stronger than S5. This Quantified Autoepistemic Logic obeys all the laws of First Order Logic and its L predicate obeys the laws of S5 Modal Logic in every fixed-point. It is proven that this Logic has a kernel not containing L such that L holds for a sentence if and only if that sentence is in the kernel. This result is important because it shows that L is superfluous thereby allowing the ori ginal equivalence to be simplified by eliminating L from it. It is also shown that the Kernel of Quantified Autoepistemic Logic is a generalization of Quantified Reflective Logic, which coincides with it in the propositional case.

Keywords: Quantified Autoepistemic Logic, Quantified Reflective Logic, Modal Logic, Nonmonotonic Logic.

1. Introduction

Quantified Autoepistemic Logic (i.e. QAEL) is a generalization of Autoepistemic Logic [Moore, Konolige87, Konolige87b], where both universally and existentially quantified variables are allowed to cross the scope of the L predicate. In a recent paper [Brown 2003b, 2003d] showed that Autoepistemic Logic could be represented in an extension of S5 Modal Logic. This modal representation may be generalized to provide a Quantified Autoepistemic Logic with the following necessary equivalence:

$$\kappa\equiv(\text{QAEL } \kappa\ \Gamma)$$

where QAEL is defined as follows:

$(\text{QAEL } \kappa\ \Gamma) =\text{df } \Gamma\wedge\forall i\forall\xi_i((L\ '\chi_i)\leftrightarrow([\kappa]\chi_i))$

$(L\ '\chi_i) =\text{df } (L'\chi_i\ a_i)$,

where χ_i is the ith sentence with or without free variables of a First Order Logic (i.e. FOL) and a_i is an association list associating the free variables in $'\chi_i$ to values specified by the sequence of variables ξ_i. The $\forall i$ quantifier ranges across the natural numbers. This Quantified Autoepistemic Logic is important because unlike some other attempts [Konolidge1989] to generalize Autoepistemic Logic, its quantifiers obey both the Barcan Formulae, the converse of the Barcan formula, and also all the laws of S5 Modal Logic and First Order Logic (i.e. FOL). Interpreted doxastically this necessary equivalence states that:

that which is believed is equivalent to: Γ and for all i and for all ξ_i $(L\ '\chi_i)$ if and only if χ_i is believed.

The purpose of this paper is to show that the L predicate is not essential to solving for κ and can be eliminated thereby allowing the above necessary equivalence to be replaced by a simpler necessary equivalence which when interpreted as a doxastic logic states:

that which is believed is equivalent to: Γ (with each *L'* replaced by $[\kappa]$) .

thereby eliminating every occurrence of the L predicate, all the (quoted) names of sentences, and the bi-implication containing *L*.

The remainder of this paper proves that the L predicate can be eliminated. Section 2 describes the First Order Logic (i.e.FOL) used herein. Section 3 describes the Modal Logic used herein. QAEL is defined in more detail in section 4. The *L* eliminated form of Quantified Autoepistemic Logic herein called the Quantified Autoepistemic Kernel (i.e. QAEK) is defined in section 5 and is explicated with theorems LEXT1 and LEXT2. In section 6, QAEK is shown to be related to QAEL by theorems QAEK1, QAEK2, QAEK3. The relationship between QAEK and Quantified Reflective Logic (i.e. QRL) [Brown 2003a] is given in section 7. Finally, in section 8, some consequences of all these results are discussed. Figure 1 outlines the relationship of all these theorems in producing the final theorems LEXT2, QAEK3, and AR2.

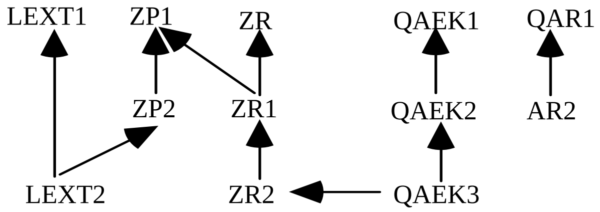

Figure 1: Dependencies among the Theorems

2. First Order Logic

We use a First Order Logic (i.e. FOL) defined as the six tuple: ($\rightarrow$, #f, $\forall$, *vars*, *predicates*, *functions*) where $\rightarrow$, #f, and $\forall$ are logical symbols, *vars* is a set of variable symbols, *predicates* is a set of predicate symbols each of which has an implicit arity specifying the number of associated terms, and *functions* is a set of function symbols each of which has an implicit arity specifying the number of associated terms. Lower case Roman letters possibly indexed with digits are used as variables. Greek letters possibly indexed with digits or lower case roman letters are used as syntactic metavariables. γ, $\gamma_1,...\gamma_n$, range over the variables, ξ, $\xi_1...\xi_n$ range over sequences of variables of an appropriate arity, $\pi,\pi_1...\pi_n$ range over the predicate symbols, ϕ, $\phi_1...\phi_n$ range over function symbols, δ, $\delta_1...\delta_n$, σ range over terms, and α, $\alpha_1...\alpha_n$, β, $\beta_1...\beta_n$,χ, $\chi_1...\chi_n$,Γ,Γ_1,...Γ_n,κ,κ_1,...κ_n, φ, range over sentences (including sentences with free variables). The terms are of the forms γ and (ϕ $\delta1...\delta n$), and the sentences are of the forms ($\alpha\rightarrow\beta$), #f, ($\forall\gamma$ α), and (π $\delta_1...\delta_n$). A nullary predicate π or function ϕ is written without parentheses. $\varphi\{\pi/\lambda\xi\alpha\}$ represents the replacement of all occurrences of π in φ by $\lambda\xi\alpha$ followed by lambda conversion. The primitive symbols are shown in Figure 2 with their interpretations. The particular FOL used herein includes the binary predicate symbol L and a denumerably infinite number of 0-ary function symbols representing the names (i.e. 'α) of the sentences (i.e. α) of this FOL.

Symbol	Meaning
$\alpha\rightarrow\beta$	if α then β.
#f	falsity
$\forall\gamma$ α	for all γ, α.

Figure 2: Primitive Symbols of First Order Logic

The defined symbols are listed in Figure 3 with their definitions and intuitive interpretations.

Symbol	Definition	Meaning		Symbol	Definition	Meaning
$\neg\alpha$	$\alpha\rightarrow$ #f	not α		$\alpha\wedge\beta$	$\neg(\alpha\rightarrow\neg\beta)$	α and β
#t	$\neg$ #f	truth		$\alpha\leftrightarrow\beta$	$(\alpha\rightarrow\beta)\wedge(\beta\rightarrow\alpha)$	α if and only if β
$\alpha\vee\beta$	$(\neg\alpha)\rightarrow\beta$	α or β		$\exists\gamma$ α	$\neg\forall\gamma\neg\alpha$	for some γ, α

Figure 3: Defined Symbols of First Order Logic

3. Modal Logic

We extend First Order Logic with a necessity symbol as given in Figure 4 below:

Symbol	Meaning
[]α	α is logically necessary

Figure 4: Primitive Symbols of Modal Logic

and with the laws of an S5 Modal Logic [Hughes & Cresswell 1968] as given in Figure 5 below:

R0: from α infer ([] α)	A2: ([]($\alpha\rightarrow\beta$)) $\rightarrow$ (([]α)$\rightarrow$ ([]β))
A1: ([]α) $\rightarrow\alpha$	A3: ([]α) $\vee$ ([]$\neg$[]α)

Figure 5: The Laws of S5 Modal Logic

These S5 modal laws and the laws of FOL constitute an S5 Modal Quantificational Logic similar to [Carnap 1946; Carnap 1956], and a FOL version [Parks 1976] of [Bressan 1972] in which the Barcan formula: $(\forall\gamma([]\alpha))\rightarrow([]\forall\gamma\alpha)$ and its converse hold. The defined Modal symbols used herein are listed in Figure 6 with their definitions and intuitive interpretations.

Symbol	Definition	Meaning		Symbol	Definition	Meaning
$<>\alpha$	$\neg [] \neg\alpha$	α is logically possible		$\alpha\equiv \beta$	$[] (\alpha\leftrightarrow\beta)$	α is synonymous to β
$<\beta>\alpha$	$<> (\beta\wedge\alpha)$	α and β is logically possible		$\delta=\sigma$	$(\pi\delta)\equiv(\pi\ \sigma)$	δ is logically equal to σ
$[\beta]\alpha$	$[] (\beta\rightarrow\alpha)$	β entails α		$\delta\neq\sigma$	$\neg(\delta=\sigma)$	δ is not logically equal to σ.

Figure 3: Defined Symbols of Modal Logic

Next, we extend the FOL + S5 Modal Quantificational Logic with the A4 axiom scheme given in Figure 7.

A4: $<>\Gamma\{\pi/\lambda\xi\alpha\} \rightarrow <>\Gamma$

where $\Gamma\{\pi/\lambda\xi\alpha\}$ is the simultaneous replacement in Γ of all unmodalized occurrences of π by α.

Figure 7: The Possibility Axiom Scheme

Intuitively, A4 specifies that a sentence Γ is logically possible whenever the result obtained by "interpreting" all the unmodalized occurrences of a predicate within it, is logically possible. If A4 is successively applied to all the unmodalized predicates then it follows that a sentence Γ is logically possible if the result of interpreting all the unmodalized predicates is logically possible. The possibility axiom A4 extends the trivial possibility axiom (i.e. some proposition is neither #t nor #f) given in [Lewis 1936] and [Bressan 1972], the S5c possibility axiom schema (i.e. every conjunction of distinct negated or unnegated propositional constants is logically possible) given in [Hendry & Pokriefka 1985], and is implied by the possibility axiom schema used in the Z Modal Quantificational Logic described in [Brown 1987; Brown 1989]. The following metatheorems are derivable:

ZP1: The Possibility of a Separable Predicates: If (1) Γ, α, and β are sentences of FOL extended whereby any modalized sentence may occur in the place of predicates and (2) π does not occur unmodalized in any of Γ, α, and β then: $(<>(\Gamma\wedge(\forall\xi(\alpha\rightarrow(\pi\ \xi)))\wedge(\forall\xi(\beta\rightarrow\neg(\pi\ \xi)))))\leftrightarrow(<>(\Gamma\wedge(\neg\exists\xi(\alpha\wedge\beta))))$

ZP2: The Possibility of a Defined Predicate: If (1) Γ and α are sentences of FOL extended whereby any modalized sentence may occur in the place of predicates and (2) π does not occur unmodalized in any of Γ and α then: $(<>(\Gamma\wedge(\forall\xi((\pi\ \xi)\leftrightarrow\alpha))))\leftrightarrow(<>\Gamma)$ proof: Let β be $\neg\alpha$ in ZP1 and simplify. QED.

ZR: The Reduction Lemma: If (1) κ occurs in Γ and Ψ only in the context: $<\kappa>\varphi$ for some φ (or in the context $[\kappa]\mu$ which is essentially of the same modal form: $\neg<\kappa>\neg\mu$ and (2) for all such φ: $\forall p(((<(\Gamma\wedge\Psi)>\varphi)\leftrightarrow(<(\Gamma>\varphi))\{\kappa/p\})$ then : $(\kappa\equiv(\Gamma\wedge\Psi)) \leftrightarrow \exists p((\kappa\equiv(p\wedge(\Psi\{\kappa/p\})))\wedge(p\equiv(\Gamma\{\kappa/p\})))$

ZR1: Reducing a Reflection with a Separable Predicate: If (1) κ occurs in Γ, α, and β only in the context: $<\kappa>\varphi$ for some φ (or in the context $[\kappa]\mu$ which is essentially of the same modal form: $(\neg<\kappa>\neg\mu)$, (2) Γ, α, β, and φ are sentences of FOL extended whereby any modalized sentence may occur in the place of predicates, (3) π does not occur unmodalized in any of Γ, α, β, and φ then: $(\kappa\equiv(\Gamma\wedge(\forall\xi(\alpha\rightarrow(\pi\ \xi)))\wedge(\forall\xi(\beta\rightarrow\neg(\pi\ \xi)))))$

$\leftrightarrow \exists p((\kappa\equiv(p\wedge(((\forall\xi(\alpha\rightarrow(\pi\ \xi)))\wedge(\forall\xi(\beta\rightarrow\neg(\pi\ \xi))))\{\kappa/p\})))\wedge(p\equiv((\Gamma\wedge(\neg\exists\xi(\alpha\wedge\beta)))\{\kappa/p\})))$

ZR2: Reducing a Reflection with a Defined Predicate: If (1) κ occurs in Γ, and α only in the context: $<\kappa>\varphi$ for some φ (or in the context $[\kappa]\mu$ which is essentially of the same modal form: $(\neg<\kappa>\neg\mu)$, (2) Γ, α, , and φ are sentences of FOL extended whereby any modalized sentence may occur in the place of predicates, (3) π does not occur unmodalized in any of Γ, α, and φ then: $\kappa\equiv(\Gamma\wedge\forall\xi((\pi\ \xi)\leftrightarrow\alpha))\leftrightarrow \exists p\ \kappa\equiv(p\wedge\forall\xi((\pi\ \xi)\leftrightarrow\alpha\{\kappa/p\})) \wedge p\equiv\Gamma\{\kappa/p\}$. proof: Let β be $\neg\alpha$ in ZR1 and simplify. QED.

4. Quantified Autoepistemic Logic

Quantified Autoepistemic Logic (i.e. QAEL) is defined in Modal Logic by a necessary equivalence of the form:

$$\kappa\equiv(\text{QAEL}\ \kappa\ \Gamma)$$

where QAEL is defined as follows: (QAEL κ Γ) =df $\Gamma\wedge\forall i\forall\xi_i((L\ '\chi_i)\leftrightarrow([\kappa]\chi_i))$ where $(L\ '\chi_i)$ =df (L $'\chi_i$ a_i), χ_i is the ith sentence with or without free variables of FOL and a_i is an association list binding the free variables in $'\chi_i$ to values specified by the sequence of metalanguage variables ξ_i. The $\forall i$ quantifier ranges across the natural numbers. Any FOL proposition κ which makes this necessary equivalence true is a solution. QAEL addresses the problem of how quantified variables whose scopes cross the L predicate may be represented. Furthermore these quantifiers obey not only the Barcan formula but unlike the generalization of Autoepistemic Logic given in [Konolige 1989] its converse and therefore does not suffer the anomalies therein discussed. For example we could then state in QAEL that everything is a bird and that all things believed to be birds for which flying is believable do in fact fly as follows: $\kappa\equiv$(QAEL $\kappa((\forall x(\text{Bird } x))\wedge\forall x(((L\ '(\text{Bird } x))\wedge(\neg L\ '(\neg(\text{Fly } x))))\rightarrow(\text{Fly } x))))$

5. Quantified Autoepistemic Kernel

The Quantified Autoepistemic Kernel [Brown 1989] is defined in Modal Quantificational Logic by the necessary equivalence:

$$\varphi\equiv(\text{QAEK } \varphi\ \Gamma)$$

where QAEK is defined as: (QAEK φ Γ) =df $\Gamma\{L\ '/[\varphi]\}$

The *L* predicate does not occur unmodalized in QAEK. However, the kernel may be used to define an extension containing facts involving *L* as follows:

(L-EXT φ) =df $(\varphi\wedge\forall i\forall\xi_i((L\ '\chi_i)\leftrightarrow(([\varphi]\chi_i)\{L\ '/[\varphi]\})))$

The kernel φ possesses two important properties with respect to L-extensions, namely that the L-extension of φ entails φ, and φ entails every FOL sentence not containing an occurrence of *L* which the L-extension entails

LEXT1: $[(\text{L-ext } \varphi)]\varphi$ proof: Unfolding L-ext gives a tautology. QED.

LEXT2: If *L* is not in s and if φ contains no unmodalized occurrence of L, then: $\forall s([(\text{L-ext } \varphi)]s)\leftrightarrow([\varphi]s)$

proof: Pushing negation through gives the equivalent sentence: $\forall r((<(\text{L-ext } \varphi)>r)\leftrightarrow(<\varphi>r))$

Unfolding L-ext gives: $\forall r((<(\varphi\wedge\forall i\forall\xi_i((L\ \chi_i)\leftrightarrow(([\varphi]\chi_i)\{L\ '/[\varphi]\}))>r)\leftrightarrow(<\varphi>r))$ or rather:

$\forall r((<\varphi>(\forall i\forall\xi_i((L\ '\chi_i)\leftrightarrow(([\varphi]\chi_i)\{L\ '/[\varphi]\}))\wedge r)\leftrightarrow(<\varphi>r))$ which is an instance of theorem ZP2. QED

LEXT1 and LEXT2 show that the kernel determines all the non-kernel sentences in the L-extension. Representing problems in the Quantified Autoepistemic Kernel simplifies their solution since the pre-processing step of eliminating the *L* predicate from Γ is eliminated.

6. The Relationship between Quantified Autoepistemic Logic and its Kernel

We now show how all occurrences of L including those within quotes as parts of structural descriptive names of sentences of Autoepistemic Logic may be eliminated from Γ: For example, if Γ consisted of the single default: $(\neg(L\ '(L\ '(\neg\pi))))\rightarrow\pi$ then the necessary equivalence is: $\kappa\equiv(\text{QAEL } \kappa((\neg(L\ '(L\ '(\neg\pi))))\rightarrow\pi))$

Unfolding AEL gives: $\kappa\equiv(((\neg(L'(L\ '(\neg\pi))))\rightarrow\pi)\wedge\forall i\forall\xi_i((L'\chi_i)\leftrightarrow([\kappa]\chi_i)))$. Since the quantified statement is connected to: $(\neg(L\ '(L\ '(\neg\pi))))\rightarrow\pi$ by a conjunction it may be assumed when simplifying that expression. Instantiating i so that χ_i is $(L\ '(\neg\pi))$ and using that instance gives the equivalent expression: $\kappa\equiv(((\neg([\kappa](L\ '(\neg\pi))))\rightarrow\pi)\wedge\forall i\forall\xi_i((L\ '\chi_i)\leftrightarrow([\kappa]\chi_i)))$. We would like to eliminate the remaining *L* in the first formulae but it is inside the scope of an entailment and therefore the (non-necessary) equivalence: $\forall i\forall\xi_i((L\ '\chi_i)\leftrightarrow([\kappa]\chi_i)))$ does not justify such a reduction merely by virtue of the two formulas being connected by conjunction. However, the entire formula allows the derivation of: $[\kappa](\forall i\forall\xi_i((L\ '\chi_i))\leftrightarrow([\kappa]\chi_i))$ which shows that $\forall i\forall\xi_i((L\ '\chi_i)\leftrightarrow([\kappa]\chi_i))$ may be assumed in any scope entailed by κ. Thus we can still reduce occurrences of *L* even embedded within an entailment. Thus, the above equation is equivalent to: $\kappa\equiv(((\neg([\kappa]([\kappa](\neg\pi))))\rightarrow\pi)\wedge\forall i\forall\xi_i((L\ '\chi_i)\leftrightarrow([\kappa]\chi_i))$ in which no occurrence of *L* nor quotation appears in the first formulae in the conjunction. Notating the above described process (i.e. sequence of deductions) as $(\Gamma\{L\ '/[\kappa]\})$ or rather the substitution of *L* ' by $[\kappa]$ gives the theorem:

QAEK1: (κ≡(QAEL κ Γ)) ↔ (κ≡(QAEL κ (Γ{*L* '/[κ]})))

The process which eliminated L from Γ can also be used to eliminate *L* from χ_i in the formulae: κ≡((Γ{*L* '/[κ]})∧∀i∀ξ_i((*L* 'χ_i)↔([κ]χ_i))). Since χ_i occurs within the modal scope of a κ entailment we are justified in replacing an instance of it by another formulae by assuming ∀i∀ξ_i((*L* 'χ_i)↔([κ]χ_i))) for any other instance of χ_i since [κ]∀i∀ξ_i((*L* 'χ_i)↔([κ]χ_i))) follows from the overall equation. To replace each χ_i by a sentence which no longer contains *L*, we specify an ordering of all the sentences based on the maximum depth of *L*s as they occur through the structural descriptive names of the sentences. A sentence χ_i with no *L* would be of depth 0, a sentence with L would be at depth 1, a sentence with 'L would be of depth 2, a sentence with ''*L* would be of depth 3 and so forth. The proof is by induction. The base case is always true since *L* is not in those sentences. The induction step proceeds by using ∀i∀ξ_i((*L* 'χ_i)↔([κ]χ_i)))) on sentences whose *L* depth is less than n to prove that relation for sentences whose depth is n. Notating the result of the above described process gives:

QAEK2: (κ≡(QAEL κ Γ))↔ (κ≡((Γ{*L* '/[κ]})∧∀i∀ξ_i((*L* 'χ_i)↔(([κ]χ_i){*L* '/[κ]})))

QAEK2 shows how all but one occurrence of *L* may be eliminated from the equivalence. Essentially κ is logically equivalent to a modal formula Γ{*L* '/[κ]}) not containing L conjoined to what is essentially a "definition" of *L* in terms of another modal formulae not containing *L*. This suggests that *L* is superfluous notation and that the essence of κ lies only in the first formulae. This intuition is easily proven:

QAEK3: (κ≡(QAEL κ Γ))↔ ∃p((κ≡(L-EXT p))∧(p≡(QAEK p Γ)))

proof: By QAEK2 (κ≡(QAEL κ Γ)) is equivalent to: (κ≡((Γ{*L* '/[κ]})∧∀i∀ξ_i((*L* 'χ_i)↔(([κ]χ_i){*L* '/[κ]}))

Instantiating ZR2 with: Γ:= Γ{*L* '/[κ]}, ξ:=iξ_i, π:=*L*, α:=([κ]χ_i{*L* '/[κ]}))) shows that the above expression is equivalent to: ∃p((κ≡(p∧∀i∀ξ_i((*L* ' χ_i)↔(([p]χ_i){*L* '/[p]}))) ∧(p≡(Γ{*L* '/[p]}))). Folding L-EXT and QAEK gives ∃p((κ≡(L-EXT p))∧(p≡(QAEK p Γ))) QED.

QAEK3 divides the Autoepistemic equation into two distinct equivalences, one axiomatizing the kernel p and the other defining the stronger proposition κ which is the L-extension of p containing additional facts about the *L* predicate. LEXT1 and LEXT2 show that the L-extension κ is a conservative extension of the kernel and therefore it is not essential. For this reason it suffices to deal with just the necessary equivalence for the Quantified Autoepistemic Kernel in studying Quantified Autoepistemic Logic: φ≡(QAEK φ Γ).

7. The Relationship between Quantified Autoepistemic Kernel and Quantified Reflective Logic

The modal representation of Reflective Logic [Brown 1989, 2003a, 2003c] may be generalized to a Quantified Reflective Logic as:

κ≡(QRL κ Γ α_i:β_{ij}/χ_i)

where QRL is defined in Modal Logic as follows:

(QRL κ Γ α_i:β_{ij}/χ_i) =df Γ∧∀i∀ξ_i((([κ]α_i)∧$\wedge_{j=1,m_i}$<κ>β_{ij})→χ_i)

where Γ, α_i, β_{ij}, and χ_i are sentences of FOL which may contain free variables. The variables in ξ_i may occur in any of α_i, β_{ij}, and χ_i. When the context is obvious Γ α_i:β_{ij}/χ_i is omitted and instead just (QRL κ) is written. $\wedge_{j=1,m_i}$ stands for the conjunction of the formula which follows it as j ranges from 1 to m_i. If m_i=0 then it specifies #t. If i ranges over a finite number of defaults then $\forall_i$ may be replaced in this definition by a conjunction: $\wedge_i$. Interpreted as a doxastic logic, the necessary equivalence states:

that which is believed is logically equivalent to:

Γ and for each i, if α_i is believed and for each j, β_{ij} is believable then χ_i

Quantified Reflective Logic is an instance of the Quantified Autoepistemic Kernel. Specifically:

QAR1: (QRL κ Γ α_i:β_{ij}/χ_i)≡(QAEK κ Γ∧∀i∀ξ_i(((*L* 'α_i)∧$\wedge_{j=1,m_i}$(¬(*L* '(¬β_{ij}))))→χ_i))

proof: Unfolding QRL and QAEK gives identical formulas. QED.

We call the instance of the Quantified Autoepistemic Kernel in which no quantified variables in Γ cross a modal scope simply the Autoepistemic Kernel. (i.e. AEK). Likewise, we call the instance of Quantified Reflective Logic with no variables in the any sequence ξ_i simply Reflective Logic (i.e. RL).

(AEK φ Γ) =df $\Gamma\{L/[\varphi]\}$

(RL κ Γ $\alpha_i{:}\beta_{ij}/\chi_i$) =df $\Gamma\wedge\forall_i((([\kappa]\alpha_i)\wedge\wedge_{j=1,mi}<\kappa>\beta_{ij})\rightarrow\chi_i)$

where Γ, α_i, β_{ij}, and χ_i are closed sentences of FOL. By closed it is meant that no sentence may contain a free variable.

By QAR1 Reflective Logic is clearly an instance of the Autoepistemic Kernel. However, in addition, it turns out that the Autoepistemic Kernel is also an instance of Reflective Logic:

AR2. The Autoepistemic Kernel is an instance of Reflective Logic. Specifically, for every FOL formulae Γ_i there exist FOL formulas: α_i, β_{ij}, and χ_i. such that: (AEK κ $(\forall_i\Gamma_i)$)≡(RL κ #t $\alpha_i{:}\beta_{ij}/\chi_i$)

proof: By QAR1 it suffices to prove that each $\Gamma_i\{L'/[\kappa]\}$, which we herebelow call Ψ is representable as

$\wedge_i((([\kappa]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow\chi_i)$

We choose a k-entailment: $([\kappa]\varphi)$ in Ψ of lowest scope that has not already been chosen. We use the laws of classical logic to place φ into conjunctive normal form (treating any embedded κ-entailment as another predicate). The following five theorem schemata of Z are then used to reduce the scope of $[\kappa]$[1] .

KU1: $([\kappa](\alpha_1\wedge...\wedge\alpha_n))\equiv(([\kappa]\alpha_1)\wedge...\wedge([\kappa]\alpha_n))$

KU2: $([\kappa](\alpha_1\vee...\vee\alpha_m\vee([\kappa]\varphi)\vee\beta_1\vee...\vee\beta_n))\equiv(([\kappa]\varphi)\vee([\kappa](\alpha_1\vee...\vee\alpha_m\vee\beta_1\vee...\vee\beta_n)))$

KU3: $([\kappa](\alpha_1\vee...\vee\alpha_m\vee(\neg([\kappa]\varphi))\vee\beta_1\vee...\vee\beta_n))\equiv((\neg[\kappa]\varphi)\vee([\kappa](\alpha_1\vee...\vee\alpha_m\vee\beta_1\vee...\vee\beta_n)))$

KU4: $([\kappa]([\kappa]\varphi))\equiv([\kappa]\varphi)$

KU5: $([\kappa](\neg[\kappa]\varphi))\equiv((\neg[\kappa]\varphi)\vee([\kappa]\#f))$

If the result begins with a conjunction, KU1 is applied. If the result begins with a disjunction with an embedded κ entailment or negation of a κ entailment then respectively KU2 or KU3 is applied. If the result is itself a k-entailment of the negation of a k-entailment then respectively KU4 or KU5 is applied. The over all process is repeated until no further KU rule is applicable. When the process finishes since none of the above rules is applicable if the overall formula is put into conjunctive normal form then every resulting disjunction must be of the following form when negations of entailments are ordered before entailments which are ordered before other expressions: $((\vee_{j=1,a}\neg([\kappa]\alpha_j))\vee(\vee_{j=1,b}([\kappa]\beta_j))\vee(\vee_{j=1,c}\chi_j))$ Pulling the first negation out and noting that $(\wedge_{j=1,a}([\kappa]\alpha_j))$ is equivalent to $([\kappa](\wedge_{j=1,a}\alpha_j))$ gives:

$((\neg([\kappa](\wedge_{j=1,a}\alpha_j)))\vee(\vee_{j=1,b}([\kappa]\beta_j))\vee(\vee_{j=1,c}\chi_j))$ or rather: $((\neg([\kappa](\wedge_{j=1,a}\alpha_j)))\vee(\vee_{j=1,b}([\kappa]\beta_j))\vee(\vee_{j=1,c}\chi_j))$

Letting α be $(\wedge_{j=1,a}\alpha_j)$ and χ be $(\vee_{j=1,c}\chi_j)$ gives: $((\neg([\kappa]\alpha))\vee(\vee_{j=1,b}([\kappa]\beta_j))\vee\chi)$

where α is #t if their are no α_j formulas (since that is the identity of conjunction) and where χ is #f if there are no χ_j formulas (since that is the identity of disjunction). Rewriting the above as an implication gives:

$(([\kappa]\alpha)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_j)))\rightarrow\chi_i$ where the resulting β_j are the negations of the previous ones. This formula is called a default. The conjunction of all the defaults is then written as:

$\wedge_i((([\kappa]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow\chi_i)$ where the defaults are not required to have any β subformulas. QED.

(RL κ #t $\alpha_i{:}\beta_{ij}/\chi_i$) is often written as: (RL κ Γ $\alpha_i{:}\beta_{ij}/\chi_i$) where Γ is all those defaults having no α (or where α is #t) nor β subformulas (and hence no modals) and i ranges over just the "real" defaults containing modals.

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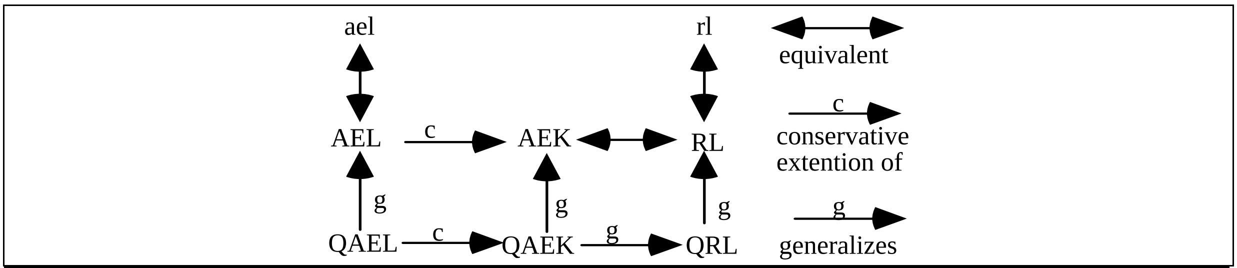
[1]When $([K]\psi)$ is viewed with K fixed as a unary symbol, it has the properties of a KU45 modal logic [Park] .

8. Conclusion

The nonmonotonic systems discussed herein are related as described in Figure 8.



ael = Autoepistemic Logic [Moore 1985]	rl = Reflective Logic [Brown 2003a, Brown 2003c]
AEL = Autoepistemic Logic in Modal Logic	QAEL = Quantified Autoepistemic Logic in Modal Logic
AEK = Autoepistemic Kernel in Modal Logic	QAEK = Quantified Autoepistemic Kernel in Modal Logic
RL = Reflective Logic in Modal Logic	QRL = Quantified Reflective Logic in Modal Logic

Figure 8: Relationships among NonMonotonic Systems

The original set theoretic description of Autoepistemic Logic (i.e. ael) is equivalent [Brown 2003b, 2003d] to the modal description AEL and the set theoretic description of Reflective Logic (i.e. rl) is equivalent [Brown 2003a, 2003c] to the modal description RL. Equivalence means that the meaning of the fixed-points of the set theoretic descriptions are identical to the solutions of the necessary equivalences of the modal systems whenever their inputs bear a similar relation. Since the modal systems (i.e. FOL+S5+A4) are much simpler than the set theoretic descriptions (i.e. FOL + Set Theory + FOL Syntax, + FOL Proof Theory) they provide a reduction in both conceptual and computational complexity. For this reason we focus on the modal systems: AEL and RL.

QAEL and QAEK, are respectively generalizations of AEL and AEK in which quantifiers are allowed to be inserted anywhere in the formulas and where such quantified variables may cross modal scopes. Since AEL and QAEL are proven by QAEK3 and LEXT2 to be conservative extensions (involving the superfluous *L* predicate) of AEK and QAEK respectively, these systems AEK and QAEK are said to be the kernels of AEL and QAEL respectively. Because the kernel systems eliminate all occurrences of *L*' and the biconditional relating *L*' to [k] they are more useful systems for both understanding and automatic theorem proving. For this reason we now focus on just the kernel systems: AEK and QAEK.

AEK is proven to be equivalent to RL by AR2. QRL is a generalization of RL where only universal quantifiers may be inserted and only inserted at the beginning of a default. By QAR1, QAEK is a generalization of QRL, But, in general, QRL is weaker than QAEK since for example it does not allow for existential quantifiers just before a default. Because QAEK and QRL differ while AEK and RL are equivalent, it follows that both QAEK and QRL can be said to be different quantificational generalizations of the Autoepistemic Kernel. Both are interesting systems with QAEK providing greater generality and QRL having deep relationships to nomonotonic logics with quantified default inference rules [Brown 2003e]. [Brown 2003f] describes an Automatic Deduction system for the propositional case of Autoepistemic Kernels (i.e. AEK) which reduces to the propositional case of Reflective Logic (i.e. RL)). Deduction Methods for the QAEL and QRL are discussed in [Brown 1987; Leasure 1993; Leasure & Brown 1995].

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