REPRESENTING REFLECTIVE LOGIC IN MODAL LOGIC

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Abstract: The nonmonotonic logic called Reflective Logic is shown to be representable in a monotonic Modal Quantificational Logic whose modal laws are stronger than S5. Specifically, it is proven that a set of sentences of First Order Logic is a fixed-point of the fixed-point equation of Reflective Logic with an initial set of axioms and defaults if and only if the meaning of that set of sentences is logically equivalent to a particular modal functor of the meanings of that initial set of sentences and of the sentences in those defaults. This result is important because the modal representation allows the use of powerful automatic deduction systems for Modal Logic and because unlike the original Reflective Logic, it is easily generalized to the case where quantified variables may be shared across the scope of the components of the defaults thus allowing such defaults to produce quantified consequences. Furthermore, this generalization properly treats such quantifiers since all the laws of First Order Logic hold and since both the Barcan Formula and its converse hold.

Keywords: Reflective Logic, Modal Logic, Nonmonotonic Logic.

1. Introduction

One of the simplest nonmonotonic logics which inherently deals with entailment conditions in addition to possibility conditions in its defaults is the so-called Reflective Logic [Brown 1989]. The basic idea of Reflective Logic is that there are some assumptions Γ and some non-logical "inference rules" of the form:

$$\frac{\alpha : \beta_1,...,\beta_m}{\chi}$$

which suggest that χ may be inferred whenever α is inferable and each $\beta_1,...,\beta_m$ is consistent with everything that is inferable. Such "inference rules" are not recursive and are circular in that the determination as to whether χ_i is derivable depends on whether β_j is consistent which in turn depends on what was derivable from this and other defaults. Thus, tentatively applying such inference rules by checking the consistency of $\beta_1,...,\beta_m$ with only the current set of inferences produces a χ_i result which may later have to be retracted. For this reason valid inferences in a nonmonotonic logic such as Reflective Logic are essentially carried out not in the original nonmonotonic logic, but rather in some (monotonic) metatheory in which that nonmonotonic logic is defined. [Brown 1989] explicated this intuition[2] by defining Reflective Logic in terms of the set theoretic proof theory metalanguage of First Order Logic (i.e. FOL) with the following fixed-point expression:

$$'\kappa=(\text{rl } '\kappa \ \{'\Gamma_i\} \ '\alpha_i{:}'\beta_{ij}/'\chi_i)$$

where rl is defined as: $(\text{rl } '\kappa \ \{'\Gamma_i\} \ '\alpha_i{:}'\beta_{ij}/'\chi_i) =_{df} (\text{fol}(\{'\Gamma_i\}\cup\{'\chi_i{:}('\alpha_i\varepsilon'\kappa)\wedge\wedge_{j=1,m_i}('(\neg\beta_{ij})\notin'\kappa)\}))$

where $'\alpha_i$, β_{ij}, and $'\chi_i$ are the closed sentences of FOL occurring in the ith "inference rule" and $\{'\Gamma_i\}$ is a set of closed sentences of FOL and $'\Gamma_i$ is the ith sentence in that set. A closed sentence is a sentence without any free variables. fol is a function which produces the set of theorems derivable in FOL from the set of sentences to which it is applied. The quotations appended to the front of these Greek letters indicate references in the metalanguage to sentences of the FOL object language. Interpreted doxastically this fixed-point equation states:

the set of closed sentences which are believed is equal to: the set of closed sentences derived in FOL from the union of the set of closed sentences: $\{'\Gamma_i\}$, and the set of closed sentences of the form $'\chi_i$ such that for each i, the closed sentence $'\alpha_i$ is believed and for each j, the closed sentence $'\beta_{ij}$ is believable.

[2] This explication is simpler but less sophisticated in its properties than that of Default Logic [Reiter 1980]. The fixed-points of both logics obey the laws: $'\kappa=(\text{fol } '\kappa)$, $'\kappa\supseteq\{'\Gamma_i\}$, and $((\alpha_i\varepsilon'\kappa)\wedge\wedge_{j=1,m_i}('(\neg\beta_{ij})\notin'\kappa))\rightarrow('\chi_i\varepsilon'\kappa)$. However, the fixed points of Default Logic are a subset of the fixed-points of Reflective Logic, but the converse is in general not true. Moreover, the fixed-points of Reflective Logic are the kernels of the fixed points of Autoepistemic Logic [Moore 1985].

The purpose of this paper is to show that all this metatheoretic machinery including the formalized syntax of FOL, the proof theory of FOL, the axioms of a strong set theory, and the set theoretic fixed-point equation is not needed and that the essence of Reflective Logic is representable as a necessary equivalence in a simple (monotonic) Modal Quantificational Logic. Interpreted as a doxastic logic this necessary equivalence states:

that which is believed is logically equivalent to for each i , Γ_i and for each i, if α_i is believed and for each j, β_{ij} is believable then χ_i

thereby eliminating all mention of any metatheoretic machinery.

The remainder of this paper proves that this modal representation is equivalent to Reflective Logic. Section 2 describes a formalized syntax for a FOL object language. Section 3 describes the part of the proof theory of FOL needed herein (i.e. theorems FOL1-FOL4). Section 4 describes the Intensional Semantics of FOL which includes laws giving the meaning of FOL sentences: M0-M7, theorems giving the meaning of sets of FOL sentences: MS1, MS2, MS3, and laws specifying the relationship of meaning and modality to the proof theory of FOL (i.e. the laws R0, A1, A2, and A3 and the theorems: C1, C2, C3, and C4). The modal version of Reflective Logic, called RL, is defined in section 5 and explicated with theorems MR1-MR6 and SS1-SS2. In section 6, this modal version is shown by theorems RL1 and RL2 to be equivalent to the set theoretic fixed-point equation for Reflective Logic. Figure 1 outlines the relationship of all these theorems in producing the final theorems RL2, FOL4, and MR6.

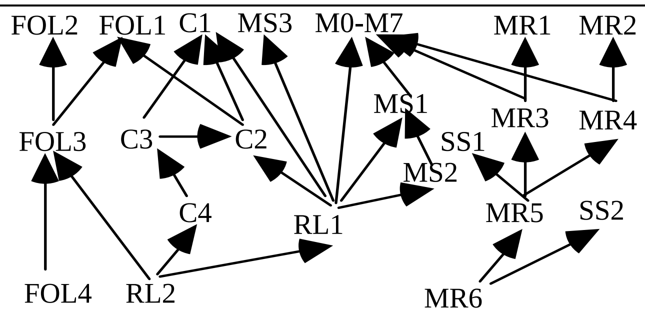

Figure 1: Dependencies among the Theorems

2. Formal Syntax of First Order Logic

We use a First Order Logic (i.e. FOL) defined as the six tuple: ($\rightarrow$, #f, $\forall$, *vars*, *predicates*, *functions*) where $\rightarrow$, #f, and $\forall$ are logical symbols, *vars* is a set of variable symbols, *predicates* is a set of predicate symbols each of which has an implicit arity specifying the number of associated terms, and *functions* is a set of function symbols each of which has an implicit arity specifying the number of associated terms. The sets of logical symbols, variables, predicate symbols, and function symbols are pairwise disjoint. Lower case Roman letters possibly indexed with digits are used as variables. Greek letters possibly indexed with digits are used as syntactic metavariables. γ, γ_1,...γ_n, range over the variables, ξ, ξ_1...ξ_n range over sequences of variables of an appropriate arity, π,π_1...π_n range over the predicate symbols, ϕ, ϕ_1...ϕ_n range over function symbols, δ, δ_1...δ_n, σ range over terms, and α, α_1...α_n, β, β_1...β_n,χ, χ_1...χ_n, Γ_1,...Γ_n,φ range over sentences. The terms are of the forms γ and (ϕ δ1...δn), and the sentences are of the forms ($\alpha\rightarrow\beta$), #f, ($\forall\gamma$ α), and (π δ_1...δ_n). A nullary predicate π or function ϕ is written as a sentence or a term without parentheses. $\varphi\{\pi/\lambda\xi\alpha\}$ represents the replacement of all occurrences of π in φ by $\lambda\xi\alpha$ followed by lambda conversion. The primitive symbols are shown in Figure 2 with their intuitive interpretations.

Symbol	Meaning
$\alpha\rightarrow\beta$	if α then β.
#f	falsity
$\forall\gamma\ \alpha$	for all γ, α.

Figure 2: Primitive Symbols of First Order Logic

The defined symbols are listed in Figure 3 with their definitions and intuitive interpretations.

Symbol	Definition	Meaning		Symbol	Definition	Meaning
$\neg\alpha$	$\alpha \rightarrow$ #f	not α		$\alpha\wedge\beta$	$\neg(\alpha \rightarrow \neg\beta)$	α and β
#t	$\neg$ #f	truth		$\alpha\leftrightarrow \beta$	$(\alpha\rightarrow \beta) \wedge (\beta\rightarrow \alpha)$	α if and only if β
$\alpha\vee\beta$	$(\neg \alpha)\rightarrow \beta$	α or β		$\exists\gamma\, \alpha$	$\neg\forall\gamma\, \neg\alpha$	for some γ , α

Figure 3: Defined Symbols of First Order Logic

The FOL object language expressions are referred in the metalanguage (which also includes a FOL syntax) by inserting a quote sign in front of the object language entity thereby making a structural descriptive name of that entity. In addition to referring to object language sentences, the formalized metalanguage also needs to refer to sets of sentences of FOL. Generally, a set of sentences is represented as: {'Γ_i} which is defined as: {'Γ_i: #t} which in turn is defined as: {s: $\exists$i(s='Γ_i)} where i ranges over some range of numbers (which may be finite or non-infinite). With a slight abuse of notation we also write 'κ, 'Γ to refer to such sets.

3. Proof Theory of First Order Logic

First Order Logic (i.e. FOL) is axiomatized with a recursively enumerable set of theorems as the set of axioms is itself recursively enumerable and its inference rules are recursive. The axioms and inference rules of FOL [Mendelson 1964] are those given in Figure 4. They form a standard set of axioms and inference rules for FOL.

MA1: $\alpha \rightarrow (\beta\rightarrow \alpha)$

MA2: $(\alpha\rightarrow (\beta\rightarrow \rho)) \rightarrow ((\alpha\rightarrow \beta)\rightarrow (\alpha\rightarrow \rho))$

MA3: $((\neg \alpha)\rightarrow (\neg \beta))\rightarrow (((\neg \alpha)\rightarrow \beta)\rightarrow\alpha)$

MA4: $(\forall\gamma\, \alpha)\rightarrow \beta$ where β is the result of substituting an expression (which is free for the free positions of γ in α) for all the free occurrences of γ in α.

MA5: $((\forall\gamma(\alpha \rightarrow \beta)) \rightarrow (\alpha\rightarrow(\forall\gamma\, \beta)))$ where γ does not occur in α.

MR1: from α and $(\alpha\rightarrow \beta)$ infer β

MR2: from α infer $(\forall\gamma\, \alpha)$

Figure 4: Inferences Rules and Axioms of FOL

In order to talk about sets of sentences we include in the metatheory set theory symbolism as developed along the lines of [Quine 1969]. This set theory includes the symbols ε, $\not\in$, $\supseteq$, $=$, $\cup$ as is defined therein.
The derivation operation (i.e. fol) of any First Order Logic obeys the Inclusion (i.e. FOL1) and Idempotence (i.e. FOL2) properties:

FOL1: (fol 'κ)$\supseteq$'κ Inclusion

FOL2: (fol 'κ)$\supseteq$(fol(fol 'κ)) Idempotence

From these two properties we prove:

FOL3: (rl 'κ 'Γ 'α_i:'β_{ij}/'χ_i))=(fol(rl 'κ 'Γ 'α_i:'β_{ij}/'χ_i))

proof: FOL1 and FOL2 imply that (fol(fol 'κ))=(fol 'κ). Since rl begins with fol this implies: 'κ=(fol(rl 'κ)) QED.

FOL4: ('κ=(rl 'κ 'Γ 'α_i:'β_{ij}/'χ_i))$\rightarrow$('κ=(fol 'κ))

proof: From the hypothesis and FOL3: 'κ=(fol(rl 'κ 'Γ 'α_i:'β_{ij}/'χ_i)) is derived. Using the hypothesis to replace (rl 'κ 'Γ 'α_i:'β_{ij}/'χ_i) by 'κ in this result gives: 'κ=(fol 'κ). QED.

4. Intensional Semantics of FOL

The meaning (i.e. mg) [Brown 1978, Boyer&Moore 1981] or rather disquotation of a sentence of First Order Logic (i.e. FOL) is defined to satisfy the laws given in Figure 5 below[3]. mg is defined in terms of mgs which maps each FOL object language sentence and an association list into a meaning. Likewise, mgn maps a FOL object language term and an association list into a meaning. An association list is simply a list of pairs consisting of an object language variable and the meaning to which it is bound.

[3] The laws M0-M7 are analogous to Tarski's definition of truth except that finite association lists are used to bind variables to values rather than infinite sequences. M4 is different since mg is interpreted as being meaning rather than truth.

M0: (mg 'α) =df (mgs '($\forall\gamma$1...γn α)'()) where 'γ1...'γn are all the free variables in 'α
M1: (mgs '($\alpha \rightarrow \beta$)a) $\leftrightarrow$ ((mgs 'α a)$\rightarrow$(mgs 'β a))
M2: (mgs '#f a) $\leftrightarrow$ #f
M3: (mgs '($\forall$ γ α)a) $\leftrightarrow$ $\forall$x(mgs 'α(cons(cons 'γ x)a))
M4: (mgs '(π δ_1...δ_n)a) $\leftrightarrow$ (π(mgn 'δ_1 a)...(mgn 'δ_n a)) for each predicate symbol 'π.
M5: (mgn '(ϕ δ_1...δ_n)a) = (ϕ(mgn 'δ_1 a)...(mgn 'δ_n a))   for each function symbol 'ϕ.
M6: (mgn 'γ a) = (cdr(assoc 'γ a))
M7: (assoc v L) = (if(eq? v(car(car L))) (car L) (assoc v(cdr L)))
where: cons, car, cdr, eq?, if are axiomatized as they are axiomatized in Scheme.

Figure 5: The Meaning of FOL Sentences

For example, the meaning of the sentence "Everything is less than something" is the proposition that everything is less than something. Thus the meaning operator disquotes its argument. Here is an example derivation:

(mg '($\forall$x$\exists$y(< x y)))

Replacing the defined symbols of the object language by primitive symbols of the object language gives:

(mg '($\forall$x(($\forall$y((< x y)$\rightarrow$#f))$\rightarrow$#f))). By M0 this is equivalent to: (mgs '($\forall$x(($\forall$y((< x y)$\rightarrow$#f))$\rightarrow$#f)) '())

By M3 this is equivalent to: $\forall$x(mgs '(($\forall$y((< x y)$\rightarrow$#f))$\rightarrow$#f) (cons(cons 'x x)'()))

By M1 this is equivalent to: $\forall$x((mgs '($\forall$y((< x y)$\rightarrow$#f)) (cons(cons 'x x)'())) $\rightarrow$(mgs '#f (cons(cons 'x x)'())))

By M2 this is equivalent to: $\forall$x((mgs '($\forall$y((< x y)$\rightarrow$#f)) (cons(cons 'x x)'()))$\rightarrow$#f)

We would now like to apply M3 to: (mgs '($\forall$y((< x y)$\rightarrow$#f)) (cons(cons 'x x)'()))

but we cannot since the bound variable x in M3 would capture the variable x which is free in this expression. In order to apply M3 we must first rename the bound variable x in M3 to be some other variable which will not capture any free variables in this expression. In this case we rename the bound x in M3 to be y, and then use that version of M3 to produce the equivalent expression:

$\forall$x(($\forall$y(mgs '((< x y)$\rightarrow$#f) (cons(cons 'y y)(cons(cons 'x x)'())))) $\rightarrow$#f)

By M1 this is equivalent to:

$\forall$x(($\forall$y((mgs '(< x y) (cons(cons 'y y)(cons(cons 'x x)'()))) $\rightarrow$(mgs '#f (cons(cons 'y y)(cons(cons 'x x)'()))))) $\rightarrow$#f)

By M2 this is equivalent to: $\forall$x$\exists$y((mgs '(< x y) (cons(cons 'y y)(cons(cons 'x x)'()))) By M4 this is equivalent to: $\forall$x$\exists$y(<(mgn 'x(cons(cons 'y y)(cons(cons 'x x)'()))) (mgn 'y (cons(cons 'y y)(cons(cons 'x x)'()))))

By M6 twice this is equivalent to: $\forall$x$\exists$y(< x y)

The meaning of a set of sentences is defined in terms of the meanings of the sentences in the set as:

(ms 'κ) =df $\forall$s((sε'κ)$\rightarrow$(mg s))

MS1: (ms{'α: Γ}) $\leftrightarrow$ $\forall\xi$($\Gamma\rightarrow\alpha$) where ξ is the sequence of all the free variables in 'α and where Γ is any sentence of the intensional semantics.

proof: (ms{'α:Γ}) Unfolding ms and the set pattern abstraction symbol gives: $\forall$s((sε\{s: $\exists\xi$((s='α)$\wedge\Gamma$)\})$\rightarrow$(mg s))

where ξ is a sequence of the free variables in 'a. This is equivalent to: $\forall$s(($\exists\xi$((s='α)$\wedge\Gamma$)))$\rightarrow$(mg s))

which is logically equivalent to: $\forall$s$\forall\xi$ (((s='κ)$\wedge\Gamma$)$\rightarrow$(mg s)) which is equivalent to: $\forall\xi$($\Gamma\rightarrow$(mg 'α))

Unfolding mg using M0-M7 then gives: $\forall\xi$($\Gamma\rightarrow\alpha$) QED

The meaning of the union of two sets of FOL sentences is the conjunction of their meanings (i.e. MS1) and the meaning of a set is the meaning of all the sentences in the set (i.e. MS2):

MS2: (ms{'Γ_i}) $\leftrightarrow$ $\forall$i$\forall\xi_i\Gamma_i$

proof: (ms{'Γ_i}) Unfolding the set notation gives: (ms{'Γ_i: #t})

By MS1 this is equivalent to: $\forall_i\forall\xi_i$(#t$\rightarrow\Gamma_i$) which is equivalent to: $\forall$i$\forall\xi_i\Gamma_i$ QED.

MS3: $(ms('\kappa \cup '\Gamma)) \leftrightarrow ((ms\ '\kappa) \wedge (ms\ '\Gamma))$

proof: Unfolding ms and union in: $(ms('\kappa \cup '\Gamma))$ gives: $\forall s((s \varepsilon \{s: (s \varepsilon '\kappa) \vee (s \varepsilon '\Gamma)\}) \rightarrow (mg\ s))$ or rather: $\forall s(((s \varepsilon '\kappa) \vee (s \varepsilon '\Gamma)) \rightarrow (mg\ s))$ which is logically equivalent to: $(\forall \alpha((s \varepsilon '\kappa) \rightarrow (mg\ s))) \wedge (\forall s((s \varepsilon '\Gamma) \rightarrow (mg\ s)))$

Folding ms twice then gives: $((ms\ '\kappa) \wedge (ms\ '\Gamma))$ QED.

The meaning operation may be used to develop an Intensional Semantics for a FOL object language by axiomatizing the modal concept of necessity so that it satisfies the theorem:

C1: $('\alpha \varepsilon (fol\ '\kappa)) \leftrightarrow ([]\ ((ms\ '\kappa) \rightarrow (mg\ '\alpha)))$

for every sentence $'\alpha$ and every set of sentences $'\kappa$ of that FOL object language. The necessity symbol is represented by a box: []. C1 states that a sentence of FOL is a FOL-theorem (i.e. fol) of a set of sentences of FOL if and only if the meaning of that set of sentences necessarily implies the meaning of that sentence. One modal logic which satisfies C1 for FOL[4] is the Z Modal Quantificational Logic described in [Brown 1987; Brown 1989] whose theorems are recursively enumerable. Z has the metatheorem: $(<>\Gamma)\{\pi/\lambda\xi\alpha\} \rightarrow (<>\Gamma)$ where Γ is a sentence of FOL and includes all the laws of S5 Modal Logic [Hughes & Cresswell 1968] whose modal axioms and inference rules are given in Figure 6. Therein, κ and Γ represent arbitrary sentences of the intentional semantics.

R0: from α infer $([]\ \kappa)$	A2: $([](\kappa \rightarrow \Gamma)) \rightarrow (([]\kappa) \rightarrow ([]\Gamma))$
A1: $([]\kappa) \rightarrow \kappa$	A3: $([]\kappa) \vee ([]\neg[]\kappa)$

Figure 6: The Laws of S5 Modal Logic

These S5 modal laws and the laws of FOL given in Figure 4 constitute an S5 Modal Quantificational Logic similar to [Carnap 1946; Carnap 1956], and a FOL version [Parks 1976] of [Bressan 1972] in which the Barcan formula: $(\forall\gamma([]\kappa)) \rightarrow ([]\forall\gamma\kappa)$ and its converse hold. The R0 inference rule implies that anything derivable in the metatheory is necessary. Thus, in any logic with R0, contingent facts would never be asserted as additional axioms of the metatheory. For example, we would not assert $([](\kappa \leftrightarrow \Gamma))$ as an axiom and then try to prove $([](\kappa \rightarrow \alpha))$. Instead we would try to prove that $([](\kappa \leftrightarrow \Gamma)) \rightarrow ([](\kappa \rightarrow \alpha))$.

The defined Modal symbols used herein are listed in Figure 7 with their definitions and interpretations.

Symbol	Definition	Meaning		Symbol	Definition	Meaning
$<>\kappa$	$\neg\ []\ \neg\kappa$	α is logically possible		$[\kappa]\ \Gamma$	$[]\ (\kappa \rightarrow \Gamma)$	β entails α
$\kappa \equiv \Gamma$	$[]\ (\kappa \leftrightarrow \Gamma)$	α is logically equivalent to β		$<\kappa>\ \Gamma$	$<>\ (\kappa \wedge \Gamma)$	α and β is logically possible

Figure 7: Defined Symbols of Modal Logic

For example, folding the definition of entailment, C1 may be rewritten more compactly as:

C1': $('\alpha \varepsilon (fol\ '\kappa)) \leftrightarrow ([(ms\ '\kappa)](mg\ '\alpha))$

This compact notation for entailment is used hereafter.

From the laws of the Intensional Semantics we prove that the meaning of the set of FOL consequences of a set of sentences is the meaning of that set of sentences (C2), the FOL consequences of a set of sentences contain the FOL consequences of another set if and only if the meaning of the first set entails the meaning of the second set (C3), and the sets of FOL consequences of two sets of sentences are equal if and only if the meanings of the two sets are logically equivalent (C4):

C2: $(ms(fol\ '\kappa)) \equiv (ms\ '\kappa)$

proof: The proof divides into two cases:

(1) $[(ms\ '\kappa)](ms(fol\ '\kappa)))$ Unfolding the second ms gives: $[(ms\ '\kappa)]\forall s((s \varepsilon (fol\ '\kappa)) \rightarrow (mg\ s))$

By the soundness part of C1 this is equivalent to: $[(ms\ '\kappa)]\forall s(([(ms\ '\kappa)](mg\ s)) \rightarrow (mg\ s))$

By the S5 laws this is equivalent to: $\forall s(([(ms\ '\kappa)](mg\ s)) \rightarrow [(ms\ '\kappa)](mg\ s))$ which is a tautology.

(2) $[(ms(fol\ '\kappa))](ms\ '\kappa)$ Unfolding ms twice gives: $[\forall s((s \varepsilon (fol\ '\kappa)) \rightarrow (mg\ s))]\forall s((s \varepsilon '\kappa) \rightarrow (mg\ s))$ which is: $[\forall s((s \varepsilon (fol\ '\kappa)) \rightarrow (mg\ s))]((s \varepsilon '\kappa) \rightarrow (mg\ s))$ Backchaining on the hypothesis and then dropping it gives: $(s \varepsilon '\kappa) \rightarrow (s \varepsilon (fol\ '\kappa))$. Folding $\supseteq$ gives an instance of FOL1. QED.

[4] An S5 modal logic which satisfies a metatheorem analogous to C1 for Propositional Logic is the system S5c given in [Hendry and Pokriefka 1985] which has axiom schemes stating that every conjunction of distinct propositional constants is logically possible. This extends the trivial possibility axiom that some proposition is neither #t nor #f used in [Lewis 1936; Bressan 1972].

C3: $(\text{fol}\ '\kappa)\supseteq(\text{fol}\ '\Gamma) \leftrightarrow ([(\text{ms}\ '\kappa)](\text{ms}\ '\Gamma))$

proof: Unfolding $\supseteq$ gives: $\forall s((s\varepsilon(\text{fol}\ '\Gamma))\rightarrow(s\varepsilon(\text{fol}\ '\kappa)))$

By C1 twice this is equivalent to: $\forall s(([(\text{ms}\ '\Gamma)](\text{mg}\ s))\rightarrow([(\text{ms}\ '\kappa)](\text{mg}\ s)))$

By the laws of S5 modal logic this is equivalent to: $([(\text{ms}\ '\kappa)]\forall s(([(\text{ms}\ '\Gamma)](\text{mg}\ s)))\rightarrow(\text{mg}\ s)))$

By C1 this is equivalent to: $[(\text{ms}\ '\kappa)]\forall s((s\varepsilon(\text{fol}\ '\Gamma))\rightarrow(\text{mg}\ s))$. Folding ms then gives: $[(\text{ms}\ '\kappa)](\text{ms}(\text{fol}\ '\Gamma))$

By C2 this is equivalent to: $[(\text{ms}\ '\kappa)](\text{ms}\ '\Gamma)$. QED.

C4: $((\text{fol}\ '\kappa)=(\text{fol}\ '\Gamma)) \leftrightarrow ((\text{ms}\ '\kappa)\equiv(\text{ms}\ '\Gamma))$

proof: This is equivalent to $(((\text{fol}\ '\kappa)\supseteq(\text{fol}\ '\Gamma))\wedge((\text{fol}\ '\Gamma)\supseteq(\text{fol}\ '\kappa))) \leftrightarrow ([(\text{ms}\ '\kappa)](\text{ms}\ '\Gamma))\wedge([(\text{ms}\ '\Gamma)](\text{ms}\ '\kappa))$ which follows by using C3 twice.

5. Reflective Logic Represented in Modal Logic

The fixed-point equation for Reflective Logic may be expressed as a necessary equivalence in an S5 Modal Quantificational Logic as follows: $\kappa\equiv(\text{RL}\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)$ where RL is defined as: $(\text{RL}\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i) =\text{df}\ \Gamma\wedge\forall i((([\kappa]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow\chi_i)$ where Γ, α_i, β_{ij}, and χ_i are propositions of FOL. When the context is obvious $\Gamma\ \alpha_i{:}\beta_{ij}/\chi_i$ is omitted and just $(\text{RL}\ \kappa)$ is written. Given below are some simple properties of RL. The first two theorems state that RL entails Γ and any conclusion χ_i of a default whose entailment condition holds in κ and whose possible conditions are possible with κ.

MR1: $[(\text{RL}\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\Gamma$

proof: By R0 it suffices to prove: $(\text{RL}\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)\rightarrow\Gamma$. Unfolding RL gives:
$\Gamma\wedge\forall_i((([\kappa]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow\chi_i)\rightarrow\Gamma$ which is a tautology. QED.

MR2: $(([\kappa]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([(\text{RL}\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\chi_i)$

proof: Unfolding RL gives: $(([\kappa]\alpha_i)\wedge(\wedge j=1,m_i(<\kappa>\beta_{ij})))\rightarrow([\Gamma\wedge\forall_i((([\kappa]\alpha_i)\wedge(\wedge j=1,m_i(<\kappa>\beta_{ij})))\rightarrow\chi_i)]\chi_i)$

Using the hypotheses on the ith instance gives:
$(([\kappa]\alpha_i)\wedge(\wedge j=1,m_i(<\kappa>\beta_{ij})))\rightarrow ([\Gamma\wedge\forall_i((([\kappa]\alpha_i)\wedge(\wedge j=1,m_i(<\kappa>\beta_{ij})))\rightarrow\chi_i)\wedge\chi_i]\chi_i)$ which is a tautology. QED.

The concept (i.e. ss) of the combined meaning of all the sentences of the FOL object language whose meanings are entailed by a proposition is defined as follows:

$(\text{ss}\ \kappa) =\text{df}\ \forall s(([\kappa](\text{mg}\ s))\rightarrow(\text{mg}\ s))$

SS1 shows that a proposition entails the combined meaning of the FOL object language sentences that it entails. SS2 shows that if a proposition is necessarily equivalent to the combined meaning of the FOL object language sentences that it entails, then there exists a set of FOL object language sentences whose meaning is necessarily equivalent to that proposition:

SS1: $[\kappa](\text{ss}\ \kappa)$

proof: By R0 it suffices to prove: $\kappa\rightarrow(\text{ss}\ \kappa)$. Unfolding ss gives: $\kappa\rightarrow\forall s(([\kappa](\text{mg}\ s))\rightarrow(\text{mg}\ s))$
which is equivalent to: $\forall s(([\kappa](\text{mg}\ s))\rightarrow(\kappa\rightarrow(\text{mg}\ s)))$ which is an instance of A1. QED.

SS2: $(\kappa\equiv(\text{ss}\ \kappa))\rightarrow \exists s(\kappa\equiv(\text{ms}\ s))$

proof: Letting s be $\{s{:}\ ([\kappa](\text{mg}\ s))$ gives $(\kappa\equiv(\text{ss}\ \kappa))\rightarrow (\kappa\equiv(\text{ms}\{s{:}\ ([\kappa](\text{mg}\ s))))$. Unfolding ms and lambda conversion gives: $(\kappa\equiv(\text{ss}\ \kappa))\leftrightarrow (\kappa\equiv\forall s(([\kappa](\text{mg}\ s))\rightarrow(\text{mg}\ s)))$. Folding ss gives a tautology. QED.

The theorems MR3 and MR4 are analogous to MR1 and MR2 except that RL is replaced by the combined meanings of the sentences entailed by RL.

MR3: $[\text{ss}(\text{RL}\ \kappa\ \forall i\Gamma_i\ \alpha_i{:}\beta_{ij}/\chi_i)]\forall i\Gamma_i$

proof: By R0 it suffices to prove: $(\text{ss}(\text{RL}\ \kappa\ \forall i\Gamma_i\ \alpha_i{:}\beta_{ij}/\chi_i))\rightarrow\forall i\Gamma_I$ which is equivalent to:
$(\text{ss}(\text{RL}\ \kappa\ \forall i\Gamma_i\ \alpha_i{:}\beta_{ij}/\chi_i))\rightarrow\Gamma_i$. Unfolding ss gives: $(\forall s(([(\text{RL}\ \kappa\ \forall i\Gamma_i\ \alpha_i{:}\beta_{ij}/\chi_i)](\text{mg}\ s))\rightarrow(\text{mg}\ s)))\rightarrow\Gamma_i$

which by the meaning laws M0-M8 is equivalent to: $(\forall s(([(RL\ \kappa\ \forall i\Gamma_i\ \alpha_i{:}\beta_{ij}/\chi_i)](mg\ s))\rightarrow(mg\ s)))\rightarrow(mg\ '\Gamma_i)$
Backchaining on $(mg\ '\Gamma_i)$ with s in the hypothesis being $'\Gamma_i$ in the conclusion shows that it suffices to prove:
$([(RL\ \kappa\ \forall i\Gamma_i\ \alpha_i{:}\beta_{ij}/\chi_i)](mg\ '\Gamma_i))$ which by the meaning laws: M0-M7 is equivalent to: $([(RL\ \kappa\ \forall i\Gamma_i\ \alpha_i{:}\beta_{ij}/\chi_i)]\Gamma_i)$
which by the laws of S5 Modal Logic is equivalent to: $([(RL\ \kappa\ \forall i\Gamma_i\ \alpha_i{:}\beta_{ij}/\chi_i)]\forall i\Gamma_i)$
which is an instance of theorem MR1. QED.

MR4: $(([\kappa]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([ss(RL\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\chi_i)$
proof: Unfolding the last ss gives: $(([\kappa]\alpha_i)\wedge\wedge_{j=1,mi}(<\kappa>\beta_{ij}))\rightarrow([\forall s(([(RL\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)](mg\ s))\rightarrow(mg\ s))]\chi_i)$
Instantiating s in the hypothesis to $'\chi_i$ and then dropping the hypothesis gives:
$(([\kappa]\alpha_i)\wedge\wedge_{j=1,mi}(<\kappa>\beta_{ij}))\rightarrow([([(RL\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)](mg\ '\chi_i))\rightarrow(mg\ '\chi_i)]\chi_i)$
Using the meaning laws M0-M7 gives: $(([\kappa]\alpha_i)\wedge\wedge_{j=1,mi}(<\kappa>\beta_{ij}))\rightarrow([([(RL\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\chi_i)\rightarrow\chi_i]\chi_i)$
Backchaining on χ_i shows that it suffices to prove: $(([\kappa]\alpha_i)\wedge\wedge_{j=1,mi}(<\kappa>\beta_{ij}))\rightarrow([(RL\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)]\chi_i)$
which is an instance of theorem MR2. QED.

Finally MR5 and MR6 show that talking about the meanings of sets of FOL sentences in the modal representation of Reflective Logic is equivalent to talking about propositions in general.
MR5: $(ss(RL\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i))\equiv(RL\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)$ proof: In view of SS1, it suffices to prove
:($[(ss(RL\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i))](RL\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i))$. Unfolding the second occurrence of RL gives: $[(ss(RL\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i))](\forall i\Gamma_i\wedge\forall i((([\kappa]\alpha_i)\wedge\wedge_{j=1,mi}<\kappa>\beta_{ij})\rightarrow\chi_i)$ which holds by theorems MR3 and MR4. QED.

MR6: $(\kappa\equiv(RL\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i))\rightarrow\exists s(\kappa\equiv(ms\ s))$
proof: From the hypothesis and MR5 $\kappa\equiv(ss(RL\ \kappa\ \forall i\Gamma_i\ \alpha_i{:}\beta_{ij}/\chi_i))$ is derived. Using the hypothesis to replace $(RL\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)$ by κ in this result gives: $\kappa\equiv(ss(RL\ \kappa(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i))$, By SS2 this implies the conclusion. QED.

6. Conclusion: The Relationship between Reflective Logic and the Modal Logic

The relationship between the proof theoretic definition of Reflective Logic [Brown 1989] and the modal representation is developed and proven in two steps. First theorem RL1 shows that the meaning of the set rl is the proposition RL and then theorem RL2 shows that a set of FOL sentences which contains its FOL theorems is a fixed-point of the fixed-point equation of Reflective Logic with an initial set of axioms and defaults if and only if the meaning (or rather disquotation) of that set of sentences is logically equivalent to RL of the meanings of that initial set of sentences and those defaults.

RL1: $(ms(rl(fol\ '\kappa)\{'\Gamma_i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))\equiv(RL(ms\ '\kappa)(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)$
proof: $(ms(rl(fol\ '\kappa)\{'\Gamma_i\}'\alpha_i{:}'\beta_{ij}/'\chi_i))$
Unfolding the definition of rl gives: $ms(fol(\{'\Gamma_i\}\cup\{'\chi_i{:}('\alpha_i\varepsilon(fol\ '\kappa))\wedge\wedge_{j=1,mi}('(\neg\beta_{ij})\notin(fol\ '\kappa))\}))$
By C2 this is equivalent to: $ms(\{'\Gamma_i\}\cup\{'\chi_i{:}('\alpha_i\varepsilon(fol\ '\kappa))\wedge\wedge_{j=1,mi}('(\neg\beta_{ij})\notin(fol\ '\kappa))\})$
Using C1 twice gives: $ms(\{'\Gamma_i\}\cup\{'\chi_i{:}\ (([(ms\ '\kappa)](mg\ '\alpha_i))\ \wedge\wedge_{j=1,mi}\neg([(ms\ '\kappa)](mg\ '(\neg\beta_{ij}))))\})$
Using MS3 gives: $(ms\ \{'\Gamma_i\})\wedge(ms\{'\chi_i{:}\ (([(ms\ '\kappa)](mg\ '\alpha_i))\ \wedge\wedge_{j=1,mi}\neg([(ms\ '\kappa)](mg\ '(\neg\beta_{ij}))))\})$
Using MS2 gives: $(\forall i\Gamma_i)\wedge(ms\{'\chi_i{:}\ (([(ms\ '\kappa)](mg\ '\alpha_i))\ \wedge\wedge_{j=1,mi}\neg([(ms\ '\kappa)](mg\ '(\neg\beta_{ij}))))\})$
Using MS1 gives: $(\forall i\Gamma_i)\wedge\forall_i((([(ms\ '\kappa)](mg\ '\alpha_i))\wedge\wedge_{j=1,mi}\neg([(ms\ '\kappa)](mg\ '(\neg\beta_{ij}))))\rightarrow(mg\ '\chi_i))$
Using M0-M7 gives: $(\forall i\Gamma_i)\wedge\forall_i((([(ms\ '\kappa)]\alpha_i)\wedge\wedge_{j=1,mi}\neg([(ms\ '\kappa)]\neg\beta_{ij}))\rightarrow\chi_i)$
Folding the definition of RL then gives: $(RL(ms\ '\kappa)(\forall i\Gamma_i)\alpha_i{:}\beta_{ij}/\chi_i)$ QED.

RL2: $((\text{fol } '\kappa)=(\text{rl}(\text{fol } '\kappa)\{'\Gamma_i\}\ '\alpha_i:'\beta_{ij}/'\chi_i))\leftrightarrow((\text{ms } '\kappa)\equiv(\text{RL}(\text{ms } '\kappa)(\forall i\Gamma_i)\alpha_i:\beta_{ij}/\chi_i))$

proof: $(\text{fol } '\kappa)=(\text{rl}(\text{fol } '\kappa)\{'\Gamma_i\}\ '\alpha_i:'\beta_{ij}/'\chi_i)$ By FOL3 this is equivalent to: $(\text{fol } '\kappa)=(\text{fol}(\text{rl}(\text{fol } '\kappa)\{'\Gamma_i\}\ '\alpha_i:'\beta_{ij}/'\chi_i))$

By C4 this is equivalent to: $((\text{ms } '\kappa)\equiv(\text{ms}(\text{rl}(\text{fol } '\kappa)\{'\Gamma_i\}\ '\alpha_i:'\beta_{ij}/'\chi_i)))$

By RL1 this is equivalent to: $(\text{ms } '\kappa)\equiv(\text{RL}(\text{ms } '\kappa)(\forall i\Gamma_i)\ \alpha_i:\beta_{ij}/\chi_i))$ QED.

Theorem RL2 shows that the set of theorems: (fol $'\kappa$) of a set $'\kappa$ is a fixed-point of a fixed-point equation of Reflective Logic if and only if the meaning (ms $'\kappa$) of $'\kappa$ is a solution to the necessary equivalence. Furthermore, by FOL4 there are no other fixed-points (such as a set not containing all its theorems) and by MR6 there are no other solutions (such as a proposition not representable as a sentence in the FOL object language). Therefore, the Modal representation of Reflective Logic (i.e. RL), faithfully represents the set theoretic description of Reflective Logic (i.e. rl).Finally, we note that $\forall i\Gamma_i$ and (ms $'\kappa$) may be generalized to be arbitrary propositions Γ and κ giving the more general modal representation: $\kappa\equiv(\text{RL}\ \kappa\ \Gamma\ \alpha_i:\beta_{ij}/\chi_i)$.

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