ON THE RELATIONSHIP BETWEEN QUANTIFIED REFLECTIVE LOGIC AND QUANTIFIED DEFAULT LOGIC

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Abstract: Reflective Logic and Default Logic are both generalized so as to allow universally quantified variables to cross modal scopes whereby the Barcan formula and its converse hold. This is done by representing both the fixed-point equation for Reflective Logic and the fixed-point equation for Default both as necessary equivalences in the Modal Quantificational Logic Z. and then inserting universal quantifiers before the defaults. The two resulting systems, called Quantified Reflective Logic and Quantified Default Logic, are then compared by deriving metatheorems of Z that express their relationships. The main result is to show that every solution to the equivalence for Quantified Default Logic is a strongly grounded solution to the equivalence for Quantified Reflective Logic. It is further shown that Quantified Reflective Logic and Quantified Default Logic have exactly the same solutions when no default has an entailment condition.

Keywords: Quantified Reflective Logic, Quantified Default Logic, Modal Logic, Nonmonotonic Logic.

1. Introduction

Two nonmonotonic logics which inherently deal with entailment conditions in addition to possibility conditions in their defaults; are Reflective Logic and Default Logic [Reiter 1980] [Antoniou 1997]. The fixed-point solutions to Default Logic are defined by the set theoretic equation $\kappa=(\text{dl}\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)$ where:

$$(\text{dl}\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i) =_{df} \cap\{p{:}(p\supseteq(\text{fol}\ p))\wedge(p\supseteq\Gamma)\wedge \wedge_i(((\alpha_i\varepsilon p)\wedge \wedge_{j=1,mi}((\neg\beta_{ij})\notin\kappa))\rightarrow(\chi_i\varepsilon p))\}$$

where α_i, β_{ij}, and χ_i are closed sentences of First Order Logic (i.e. FOL) and Γ is a set of closed sentences of FOL $\wedge_{j=1,mi}$ stands for the conjunction of the formula that follows it as j ranges from 1 to m_i. If m_i=0 then it specifies #t. $\wedge_i$ is also a conjunction. By closed it is meant that no sentence may contain a free variable. (fol p) is the set of theorems deducible in FOL from the set p. The fixed-point solutions for Reflective Logic, can be defined by the simpler set theoretic equation $\kappa=(\text{rl}\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)$ given in [Brown 1989] where:

$$(\text{rl}\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i) =_{df} \text{fol}(\Gamma\cup\{\chi_i{:}(\alpha_i\varepsilon\kappa)\wedge \wedge_{j=1,mi}((\neg\beta_{ij})\notin\kappa)\})$$

where α_i, β_{ij}, and χ_i are again closed sentences of FOL and Γ is a set of closed sentences of FOL.

These two nonmonotonic systems have the basic problem that they do not explicate the case where free variables occur in the α_i, β_{ij}, and χ_i sentences and which are universally quantified just over the scope of those sentences. To carry out such an explication we want to transform $(\text{dl}\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)$ into something like:

$$\cap\{p{:}(p\supseteq(\text{fol}\ p))\wedge(p\supseteq\Gamma)\wedge \wedge_i\forall\xi_i(((\alpha_i\varepsilon p)\wedge \wedge_{j=1,mi}((\neg\beta_{ij})\notin\kappa))\rightarrow(\chi_i\varepsilon p))\}$$

and $(\text{rl}\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)$ into something like:[5] $\text{fol}(\Gamma\cup\{\Psi{:}\ \vee_i\exists\xi_i(\Psi=\chi_i\wedge(\alpha_i\varepsilon\kappa)\wedge \wedge_{j=1,mi}((\neg\beta_{ij})\notin\kappa))\})$

where ξ_i is a sequence of variables and the universal quantifier really means universal quantification. That is, the Barcan formula and its converse hold [Carnap 1946] so that a property universally holds (in κ) if and only if it holds (in κ) for everything: $((\forall\xi\alpha)\varepsilon\kappa)\leftrightarrow(\forall\xi(\alpha\varepsilon\kappa))$. The problem lies in the fact that α_i, β_{ij}, and χ_i are necessarily closed sentences of FOL.[6]

However, [Brown 2003a] showed how Reflective Logic can be represented in Modal Logic by the necessary equivalence: $\kappa\equiv(\text{RL}\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)$ where:

$$(\text{RL}\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i) =_{df} \Gamma\wedge\wedge_i((([\kappa]\alpha_i)\wedge(\wedge_{j=1,mi}(\langle\kappa\rangle\beta_{ij})))\rightarrow\chi_i)$$

Likewise [Brown 2003b] showed how Default Logic can be represented in Modal Logic by the necessary equivalence: $\kappa\equiv(\text{DL}\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)$ where:

$$(\text{DL}\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i) =_{df} \exists p(p\wedge([p]\Gamma)\wedge\wedge_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(\langle\kappa\rangle\beta_{ij})))\rightarrow([p]\chi_i)))$$

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[5] When the set theoretic notation is unravelled the existential quantifiers specified herein are essentially universally quantified over the defaults as can be seen in the equivalent equation: $k=\cap\{p{:}(p\supseteq(\text{fol}\ p))\wedge(p\supseteq\Gamma)\wedge \wedge_i\forall\xi_i(((\alpha_i\varepsilon k)\wedge\wedge_{j=1,mi}((\neg\beta_{ij})\notin k))\rightarrow(\chi_i\varepsilon p))\}$

[6] Of course one generally gives a meaning to such a sentence by saying that all the free variables are implicitly universally quantified or that all such variables are implicitly existentially quantified. However, neither approach allows a quantifier to refer to the same free variable in α_i, β_{ij}, and χ_i. This issue is discussed in more detail in section 3.2 in [Antoniou 1997].

The advantage of the modal representations is that quantifiers can be embedded in them wherever we wish thus allowing inserted universal quantifiers to capture the free variables in α_i, β_{ij}, and χ_i , giving the generalizations:

$(QRL\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i) =df\ \Gamma\wedge\wedge_i \underline{\forall\xi_i}((([\kappa]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow\chi_i)$

$(QDL\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i) =df\ \exists p(p\wedge([p]\Gamma)\wedge\wedge_i \underline{\forall\xi_i}((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))$

Having created two new nonmonotonic systems (i.e. QRL and QDL) the question arises as to how their fixed-point solutions are related. Herein we address this question. Section 2 axiomatizes the Z Modal Quantificational Logic. Quantified Reflective Logic (i.e., QRL) is defined in section 3 and some basic theorem schemata about it are proven. Quantified Default Logic (i.e., QDL) is defined in section 4 and some basic theorem schemata about it are proven. The main result is proven in section 5. Finally, some conclusions are drawn in section 6.

2. Axiomatization of Z Modal Logic

The Modal Quantificational Logic Z [Brown 1987] is a seven tuple: ($\rightarrow$, #f, $\forall$, [], *vars*, *predicates*, *functions*) where $\rightarrow$, #f, $\forall$, and [] are logical symbols, *vars* is a set of variable symbols, *predicates* is a set of predicate symbols each of which has an implicit arity specifying the number of terms associated with that predicate, and *functions* is a set of function symbols each of which has an implicit arity specifying the number of terms associated with that function. The sets of logical symbols, variables, predicate symbols, and function symbols are pairwise disjoint. The set of terms is the smallest set which includes the variables and is closed under the process of forming new terms from other terms using the function symbols of the language. The set of sentences is the smallest set which includes #f, the *variables*, and each of the *predicates* followed by an appropriate number of terms, and is closed under the process of forming new sentences from other sentences using the logical symbols of the language, provided that no variable in any subexpression has free occurrences both as a sentence and as a term. Variables that occur only in term positions are called concept variables. Variables which occur only in sentence positions are called propositional variables. Lower case Roman letters possibly indexed with digits are used as variables of Z. Greek letters are used as syntactic metavariables. γ, γ_1,...γ_n, range over the variables, ξ, ξ_1,...ξ_n range over a sequence of variables of an appropriate arity, $\pi,\pi_1...\pi_n,\rho,\rho_1...\rho_n$ range over the predicate symbols, ϕ, $\phi_1...\phi_n$ range over function symbols, δ, $\delta_1...\delta_n$ range over terms, Δ, $\Delta_1...\Delta_n$ range over a sequence of terms of an appropriate arity, and α, $\alpha_1...\alpha_n$, β, $\beta_1...\beta_n$,χ, $\chi_1...\chi_n$, Γ, and Ψ range over sentences. Thus, the terms are of the forms γ and (ϕ $\delta1...\delta n$), and the sentences are of the forms ($\alpha \rightarrow\beta$), #f, ($\forall\gamma\ \alpha$), ([]α), (π $\delta_1...\delta_n$), and γ. A nullary predicate π or function ϕ is written as a sentence or term without parentheses. The primitive symbols of Z are shown in Figure 1.

Symbol	Meaning		Symbol	Meaning
$\alpha\rightarrow\beta$	if α then β.		$\forall\gamma\ \alpha$	for all γ, α.
#f	falsity		[] α	α is logically necessary

Figure 1: Primitive Symbols of Z

The defined symbols of Z are listed in Figure 2 below with their intuitive interpretations.

Symbol	Definition	Meaning		Symbol	Definition	Meaning
$\neg\alpha$	$\alpha\rightarrow$ #f	not α		$\alpha\wedge\beta$	$\neg(\alpha\rightarrow\neg\beta)$	α and β
#t	$\neg$ #f	truth		$\alpha\leftrightarrow\beta$	$(\alpha\rightarrow\beta)\wedge(\beta\rightarrow\alpha)$	α if and only if β
$\alpha\vee\beta$	$(\neg\alpha)\rightarrow\beta$	α or β		$\exists\gamma\ \alpha$	$\neg\forall\gamma\neg\alpha$	for some γ , α
<> α	$\neg$[]$\neg\alpha$	α is logically possible		[β] α	([]($\beta\rightarrow\alpha$))	β entails α
$\alpha\equiv\beta$	[]($\alpha\leftrightarrow\beta$)	α is logically equivalent toβ		<β> α	(<>($\beta\wedge\alpha$))	α is possible with β
$\delta_1=\delta_2$	$(\pi\delta_1)\equiv(\pi\ \delta_1)$	$\delta_1$is logically equal to δ_2				

Figure 2: Defined Symbols of Z

Z is effectively axiomatized with a recursively enumerable set of theorems as the set of axioms is itself recursively enumerable and its inference rules are recursive. The classical (i.e., non-modal) axioms and inference rules of Z include those of Quantificational Logic [Mendelson 1964] given in Figure 3. The laws MR1, MR2, MA1-MA7 are a standard set of axioms and inference rules for First Order Quantificational Logic except for the following: point: Because γ in MR2, MA4, and MA5 may be a propositional variable these laws constitute a fragment of Second Order Logic. Propositional quantifiers in modal logics have been investigated in [Fine 1970].

MA1: $\alpha \rightarrow (\beta \rightarrow \alpha)$ MR1: from α and $(\alpha \rightarrow \beta)$ infer β

MA2: $(\alpha \rightarrow (\beta \rightarrow \rho)) \rightarrow ((\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \rho))$ MR2: from α infer $(\forall\gamma\, \alpha)$

MA3: $((\neg\alpha) \rightarrow (\neg\beta)) \rightarrow (((\neg\alpha) \rightarrow \beta) \rightarrow \alpha)$

MA4: $(\forall\gamma\, \alpha) \rightarrow \beta$ where β is the result of substituting an expression (which is free for the free positions of γ in α) for all the free occurrences of γ in α.

MA5: $((\forall\gamma\, (\alpha \rightarrow \beta)) \rightarrow (\alpha \rightarrow (\forall\gamma\, \beta)))$ where γ does not occur in α.

Figure 3: The Classical Rules and Axioms of Z

The modal inference rule and axioms of Z about logical necessity (i.e., []) are given in Figure 4. R0, A1, A2, and A3 constitute an S5 Modal Logic [Hughes and Cresswell 1968] which, with the nonmodal laws, is an S5 modal quantificational logic similar to [Carnap 1946], [Carnap 1956], and a First Order Logic version [Parks 1976] of [Bressan 1972] in which the Barcan formula: $(\forall\gamma([]\alpha)) \rightarrow ([]\forall\gamma\alpha)$ and its converse hold. R0 implies that all assertions are logically necessary. Thus, in any logic with R0, contingent facts Γ holding in a knowledgebase κ are specified by asserting $([\kappa]\Gamma)$. If Γ is all that is in κ then $\kappa \equiv \Gamma$ is asserted. The variable κ may occur in Γ.

R0: from α infer $([]\, \alpha)$ A4: $([]\alpha) \rightarrow ([](\alpha\{\pi / \lambda\xi\beta\}$

A1: $([]p) \rightarrow p$ A5: $([]\alpha) \rightarrow ([](\alpha\{\phi / \lambda\xi\delta\}$

A2: $([p]q) \rightarrow (([]p) \rightarrow ([]q))$ A6: $\neg(\forall x \forall y(x=y))$

A3: $([]p) \vee ([]\neg[]p))$

Figure 4: The Modal Inference Rule and Axioms of Z

A4 is the key axiom schema of Z. It is far stronger than the trivial possibility axioms such as $\exists pq((\neg[p]q) \wedge (\neg[p]\neg q))$ assumed in [Lewis 1936] and $\exists p((<>p) \wedge (<>\neg p))$ assumed in [Bressan 1972]. It also extends certain axiom schemata used in propositional logic, including the PropPosAx schema in [Brown 1979], S13 [Cocchiarella 1984], and S5c [Hendry and Pokriefka 1985].

3. Quantified Reflective Logic

The formula for Quantified Reflective Logic[7] (i.e., QRL) [Brown 1989] is defined in Z as follows:

RL0: $(\mathrm{QRL}\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i) =\mathrm{df}\ \Gamma \wedge \wedge_i \forall\xi_i((([\kappa]\alpha_i) \wedge (\wedge_{j=1,mi}(<\kappa>\beta_{ij}))) \rightarrow \chi_i)$

where Γ, α_i, β_{ij}, and χ_i are sentences of Z and κ does not occur in ξ. These sentences may contain free variables some of which may be captured by the $\forall\xi_i$ quantifiers. When the context is obvious $\Gamma\ \alpha_i{:}\beta_{ij}/\chi_i$ is omitted and instead just $(\mathrm{QRL}\ \kappa)$ is written. Interpreted as a doxastic logic, the equivalence:

$$\kappa \equiv (\mathrm{QRL}\ \kappa)$$

states:

[7] In the QRL generalization of Reflective Logic the Barcan formula and its converse hold for [k]: $([k]\forall\xi\alpha) \leftrightarrow (\forall\xi[k]\alpha)$ since they are inherited from the S5 modal properties of []. In terms of set theoretic fixed-points this amounts to saying that $('(\underline{\forall\xi}\alpha)\varepsilon k) \rightarrow (\underline{\forall\xi}('\alpha\varepsilon k))$ holds except for the problem that that in the set theory representation $'\alpha$ is a closed sentence.

that which is believed is logically equivalent to Γ and for each i, for all ξ_i if α_i is believed and for each j, β_{ij} is believable then χ_i

Here are some simple properties of QRL, namely that (QRL κ) entails Γ and any conclusion χ_i of a default whose conditions hold:

R1: $[(QRL\ \kappa)]\Gamma$

proof: Unfolding QRL gives: $[\Gamma\wedge\wedge_i\forall\xi_i((([\kappa]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow\chi_i)]\Gamma$ which is a tautology. QED.

R2: $(([\kappa]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([(QRL\ \kappa)]\chi_i)$

proof: Unfolding QRL gives: $(([\kappa]\alpha_i)\wedge(\wedge j=1,m_i(<\kappa>\beta_{ij})))\rightarrow([\Gamma\wedge\wedge_i\forall\xi_i((([\kappa]\alpha_i)\wedge(\wedge j=1,m_i(<\kappa>\beta_{ij})))\rightarrow\chi_i)]\chi_i)$

Using the hypotheses on the ith instance and where the quantified ξ_i is instantiated to ξ_i gives:

$(([\kappa]\alpha_i)\wedge(\wedge j=1,m_i(<\kappa>\beta_{ij})))\rightarrow([\Gamma\wedge\wedge_i\forall\xi_i((([\kappa]\alpha_i)\wedge(\wedge j=1,m_i(<\kappa>\beta_{ij})))\rightarrow\chi_i)\wedge\chi_i]\chi_i)$ which is a tautology. QED.

4. Quantified Default Logic

The formula for Quantified Default Logic (i.e., QDL) [Brown 1989] is defined in Z as follows:

D0: $(QDL\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i) =df\ \exists p(p\wedge([p]\Gamma)\wedge\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))$

where Γ, α_i, β_{ij}, and χ_i are sentences of Z without any free occurences of p and neither p nor κ occur in ξ_i. These sentences may contain free variables some of which may be captured by the $\forall\xi_i$ quantifiers. When the context is obvious $\Gamma\ \alpha_i{:}\beta_{ij}/\chi_i$ is omitted and just (QDL κ) is written. Interpreted as a doxastic logic the equivalence:

$$\kappa\equiv(QDL\ \kappa)$$

states:

that which is believed is logically equivalent to the disjunction of all potential belief states such that: Γ is potentially believed and for each i, for all ξ if α_i is potentially believed and for each j, β_{ij} is believable then χ_i is potentially believed.

Given below are some simple properties of QDL. The first two state that QDL entails Γ and any conclusion χ_i of a default whose entailment condition holds in QDL and whose possible conditions are possible with κ.

D1: $[(QDL\ \kappa)]\Gamma$

proof: Unfolding QDL gives: $[\exists p(p\wedge([p]\Gamma)\wedge\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))]\Gamma$

Since p is not free in Γ, pulling $\exists$p out of the hypothesis of the entailment gives:

$\forall p((([p]\Gamma)\wedge\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))\rightarrow([p]\Gamma))$ which is a tautology. QED.

D2: $(([(QDL\ \kappa)]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([(QDL\ \kappa)]\chi_i)$

proof: Unfolding both occurrences of QDL gives:

$(([\exists p(p\wedge([p]\Gamma)\wedge\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))]\alpha_i)\wedge(\wedge j=1,m_i(<\kappa>\beta_{ij})))$

$\rightarrow([\exists p(p\wedge([p]\Gamma)\wedge\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))]\chi_i)$

Since p is not free in α_i and χ_i, pulling $\exists$p out of the hypotheses of the entailments gives:

$((\forall p((([p]\Gamma)\wedge\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))\rightarrow([p]\alpha_i))\wedge(\wedge j=1,m_i(<\kappa>\beta_{ij}))))$

$\rightarrow\forall p((([p]\Gamma)\ \wedge\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))\rightarrow([p]\chi_i))$

Instantiating the p in the hypothesis to the p in the conclusion gives:

$(((([p]\Gamma)\wedge\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))\rightarrow([p]\alpha_i))$

$\wedge(\wedge j=1,m_i(<\kappa>\beta_{ij}))\wedge([p]\Gamma)\wedge\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))\rightarrow([p]\chi_i)$

which simplifies to just: $(([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij}))\wedge([p]\Gamma)\wedge\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))\rightarrow([p]\chi_i)$

Since p is not in ξ, forward chaining using the first and second hypotheses on the fourth proves the theorem. QED.

A slightly stronger version of QDL is defined below:

D3: $(QDL^*\ \kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i)$ =df $\exists p(p\wedge([\kappa]p)\wedge([p]\Gamma)\wedge\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))$

D4: $[(QDL^*\ \kappa)](QDL\ \kappa)$

proof: Unfolding QDL* and QDL gives: $[\exists p(p\wedge([\kappa]p)\wedge([p]\Gamma)\ \wedge\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))]$
$\exists p(p\wedge([p]\Gamma)\wedge\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))$

Letting p in the conclusion be the p in the hypothesis results in a tautology. QED.

Theorem D5 shows that QDL and QDL* are logically equivalent whenever κ entails the QDL formula:

D5: $([\kappa](QDL\ \kappa))\rightarrow((QDL\ \kappa)\equiv(QDL^*\ \kappa))$

proof: From Theorem D4, it suffices to prove: $([\kappa](QDL\ \kappa))\rightarrow([(QDL\ \kappa)](QDL^*\ \kappa))$

Unfolding QDL* gives: $([\kappa](QDL\ \kappa))\rightarrow([(QDL\ \kappa)]\exists p(p\wedge([\kappa]p)\wedge([p]\Gamma)\wedge\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i))))$

Since p and κ are not in ξ and p is not free in Γ, α_i, β_{ij}, and χ_i, letting p be (QDL κ) gives:

$([\kappa](QDL\ \kappa))\rightarrow$
$([(QDL\ \kappa)]((QDL\ \kappa)\wedge([\kappa](QDL\ \kappa))\wedge([(QDL\ \kappa)]\Gamma)\wedge\wedge_i\forall\xi_i((([(QDL\ \kappa)]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([(QDL\ \kappa)]\chi_i))))$

which holds by D1, D2, and the hypothesis. QED

5. Relationship between QRL and QDL

The following theorems characterize the relationship between QDL and QRL:

RD1: $(\kappa\equiv(QDL\ \kappa))\rightarrow[\kappa](QRL\ \kappa)$

proof: Unfolding the definition of QRL gives: $(\kappa\equiv(QDL\ \kappa))\rightarrow[\kappa](\Gamma\wedge\wedge_i\forall\xi_i((([\kappa]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow\chi_i))$

Since κ is not in ξ, pushing $[\kappa]$ to lowest scope using the laws of KU45 modal logic on $[\kappa]$ gives:
$(\kappa\equiv(QDL\ \kappa))\rightarrow(([\kappa]\Gamma)\wedge\wedge_i\forall\xi_i((([\kappa]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([\kappa]\chi_i)))$

Since κ is not in ξ, using the hypothesis to replace the first κ in the conclusion by (QDL κ) gives $[(QDL\ \kappa)]\Gamma$ which by theorem D1 is true. It remains only to prove: $(\kappa\equiv(QDL\ \kappa))\rightarrow((([\kappa]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([\kappa]\chi_i))$

Since κ is not in ξ_i, replacing two occurrences of κ by using the hypothesis and then dropping the hypothesis gives: $(([(QDL\ \kappa)]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([(QDL\ \kappa)]\chi_i)$ which by theorem D2 is true. QED.

RD2: $(\kappa\equiv(QDL\ \kappa))\rightarrow[(QRL\ \kappa)]\kappa$

proof: Using the hypothesis to replace the entailed κ in the conclusion gives: $(\kappa\equiv(QDL\ \kappa))\rightarrow([(QRL\ \kappa)](QDL\ \kappa))$

Unfolding QDL in the conclusion gives:
$(\kappa\equiv(QDL\ \kappa))\rightarrow[(QRL\ \kappa)]\exists p(p\wedge([p]\Gamma)\ \wedge\wedge_i\forall\xi_i\ ((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))$

Since p and κ are not in ξ and p is not free in Γ, α_i, β_{ij}, and χ_i, letting p be (QRL κ) gives:
$(\kappa\equiv(QDL\ \kappa))\rightarrow[(QRL\ \kappa)]((QRL\ \kappa)\wedge([(QRL\ \kappa)]\Gamma)\wedge\wedge_i\forall\xi_i((([(QRL\ \kappa)]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([(QRL\ \kappa)]\chi_i)))$

The hypothesis $\kappa\equiv(QDL\ \kappa)$ and RD1 imply $([\kappa](QRL\ \kappa))$ which, since κ is not in ξ, allows the above sentence to be generalized to:
$(\kappa\equiv(QDL\ \kappa))\rightarrow[(QRL\ \kappa)]((QRL\ \kappa)\wedge([(QRL\ \kappa)]\Gamma)\wedge\wedge_i\forall\xi_i((([\kappa]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([(QRL\ \kappa)]\chi_i)))$

which by RL1 and RL2 is true. QED.

From RD1 and RD2 we may infer that every solution of the reflective equivalence of Quantified Default Logic is a solution of the equivalence for Quantified Reflective Logic:

RD3: $(\kappa\equiv(\mathrm{QDL}\ \kappa))\rightarrow(\kappa\equiv(\mathrm{QRL}\ \kappa))$

It also follows that every solution to Quantified Reflective Logic entails (QDL κ).

RD4: $([\kappa](\mathrm{QRL}\ \kappa))\rightarrow[\kappa](\mathrm{QDL}\ \kappa)$

proof: Unfolding the definition of QDL gives:

$([\kappa](\mathrm{QRL}\ \kappa))\rightarrow[\kappa]\exists p(p\wedge([p]\Gamma)\wedge\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))$

Since p and κ are not in ξ_i and p is not free in Γ, α_i, β_{ij}, and χ_i, letting p be κ gives:

$([\kappa](\mathrm{QRL}\ \kappa))\rightarrow[\kappa](\kappa\wedge([\kappa]\Gamma)\wedge\wedge_i\forall\xi_i((([\kappa]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([\kappa]\chi_i)))$

Since κ is not in ξ_i, using the hypothesis to replace two occurrences of κ by (QRL κ) gives the generalization:

$([\kappa](\mathrm{QRL}\ \kappa))\rightarrow[\kappa](\kappa\wedge([(\mathrm{QRL}\ \kappa)]\Gamma)\wedge\wedge_i\forall\xi_i((([\kappa]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([(\mathrm{QRL}\ \kappa)]\chi_i)))$

which is true by RL1 and RL2. QED

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From RD3 and RD4 we may infer that the solutions to QDL are precisely those solutions to QRL which are entailed by (QDL κ):

RD5: $(\kappa\equiv(\mathrm{QDL}\ \kappa))\leftrightarrow((\kappa\equiv(\mathrm{QRL}\ \kappa))\wedge([(\mathrm{QDL}\ \kappa)]\kappa))$

Likewise since $\kappa\equiv(\mathrm{QRL}\ \kappa)$ in RD5 implies $([\kappa](\mathrm{QDL}\ \kappa))$ by RD3 and since $([\kappa](\mathrm{QDL}\ \kappa))$ implies that (QDL κ) is logically equivalent to (QDL* κ) by D5, it follows that:

RD6: $(\kappa\equiv(\mathrm{QDL}\ \kappa))\leftrightarrow((\kappa\equiv(\mathrm{QRL}\ \kappa))\wedge([(\mathrm{QDL}^*\ \kappa)]\kappa))$

RD6 characterizes the relationship between QRL and QDL in terms of $[(\mathrm{QDL}^*\ \kappa)]\kappa$. We now show that $([(\mathrm{QDL}^*\ \kappa)]\kappa)$ is equivalent to the notion of being constructive, defined as follows: a Reflectivec solution κ is constructive iff it is not the case that there exists a proposition which satisfies the following four conditions: (1) κ entails that proposition, (2) the proposition does not entail κ, (3) the proposition entails Γ, and (4) for each i and for all ξ the proposition entails the conclusion χ_i of each default whose presupposition α_i is entailed by that proposition and whose β_{ij} formulas are possible with κ.

RD7: (Constructive $\kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i$)

$=_{df}\neg\exists p(([\kappa]p)\wedge(\neg([p]\kappa))\wedge([p]\Gamma)\wedge\wedge_i\forall\xi((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))$

RD8: $([(\mathrm{QDL}^*\ \kappa)]\kappa)\leftrightarrow(\mathrm{Constructive}\ \kappa)$

proof: Unfolding the (QDL* κ) in $([(\mathrm{QDL}^*\ \kappa)]\kappa)$ gives:

$[\exists p(p\wedge([p]\Gamma)\wedge([\kappa]p)\wedge\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))]\kappa$

Pulling $\exists$p out of the hypothesis of the entailment gives:

$\forall p((\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i))\wedge([p]\Gamma)\wedge([\kappa]p))\rightarrow([p]\kappa))$

Pushing a negation through the formula gives:

$\neg\exists p(([\kappa]p)\wedge(\neg([p]\kappa))\wedge([p]\Gamma)\ \wedge\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(<\kappa>\beta_{ij})))\rightarrow([p]\chi_i)))$

which is the definition of being constructive. QED.

Being constructive isequivalent to the notion of being strongly grounded. A Quantified Reflective solution κ is strongly grounded iff it is not the case that there exists a proposition which satisfies the following four conditions: (1) κ entails that proposition, (2) the proposition does not entail κ, (3) the proposition entails Γ, and (4) for each i and all ξ the proposition entails the conclusion χ_i of each default whose β_{ij} formulas are also possible with κ in addition to being such that the default's presupposition α_i is entailed by that proposition and the default's β_{ij} formulas are possible with that proposition:[8]

[8] When no variables cross modal scopesthis concept may be defined in set theory as:

RD9: (Strongly-grounded $\kappa\ \Gamma\ \alpha_i{:}\beta_{ij}/\chi_i$) =df

$\neg\exists p(([\kappa]p)\wedge(\neg([p]\kappa))\wedge([p]\Gamma)\ \wedge\wedge_i\forall\xi((\wedge_{j=1,mi}(\langle\kappa\rangle\beta_{ij}))\rightarrow((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(\langle p\rangle\beta_{ij})))\rightarrow([p]\chi_i))))$

RD10: (Constructive κ)$\leftrightarrow$(Strongly-grounded κ)
proof: Unfolding Strongly-grounded gives:

$\neg\exists p(([\kappa]p)\wedge(\neg([p]\kappa))\wedge([p]\Gamma)\wedge\wedge_i\forall\xi_i((\wedge_{j=1,mi}(\langle\kappa\rangle\beta_{ij}))\rightarrow((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(\langle p\rangle\beta_{ij})))\rightarrow([p]\chi_i))))$

Since ($[\kappa]p$), ($\langle\kappa\rangle\beta_{ij}$) implies ($\langle p\rangle\beta_{ij}$). Since p and κ do not occur in ξ, the above sentence is equivalent to:

$\neg\exists p(([\kappa]p)\wedge(\neg([p]\kappa))\wedge([p]\Gamma)\wedge\wedge_i\forall\xi_i((\wedge_{j=1,mi}(\langle\kappa\rangle\beta_{ij}))\rightarrow((([p]\alpha_i)\wedge\#t)\rightarrow([p]\chi_i))))$

or rather: $\neg\exists p(([\kappa]p)\wedge(\neg([p]\kappa))\wedge([p]\Gamma)\wedge\wedge_i\forall\xi_i((([p]\alpha_i)\wedge(\wedge_{j=1,mi}(\langle\kappa\rangle\beta_{ij})))\rightarrow([p]\chi_i)))$

which is the definition of being constructive. QED.

The above theorems give five characterizations of QDL in terms of QRL:[9]
RD11: All the following are equivalent:
(1) $\kappa\equiv$(QDL κ), (2) ($\kappa\equiv$(QRL κ))$\wedge$($\kappa\equiv$(QDL κ)), (3) ($\kappa\equiv$(QRL κ))$\wedge$([(QDL κ)]κ), (4) ($\kappa\equiv$(QRL κ))$\wedge$([(QDL* κ)]κ),
(5) ($\kappa\equiv$(QRL κ))$\wedge$(Constructive κ), (6) ($\kappa\equiv$(QRL κ))$\wedge$(Strongly-grounded κ)
proof: The second formula follows from RD3, the third from RD5, the fourth from RD6, the fifth from RD8 and the sixth from RD10. QED.

Having shown that the Quantified Default solutions are the strongly grounded Quantified Reflective solutions, it is now shown that being strongly grounded essentially applies only to the defaults with entailment conditions since if there are essentially no entailment conditions in the defaults (i.e., α_i is #t for every ith default since #t is entailed by anything) then the Quantified Default solutions are precisely the Quantified Reflective solutions:
RD12: (QDL $\kappa\ \Gamma$ #t:β_{ij}/χ_i) $\equiv$ (QRL $\kappa\ \Gamma$ #t:β_{ij}/χ_i)

proof: Unfolding (QDL $\kappa\ \Gamma$ #t:β_{ij}/χ_i) gives: $\exists p(p\wedge([p]\Gamma)\wedge\wedge_i\forall\xi_i((([p]\#t)\wedge(\wedge_{j=1,mi}(\langle\kappa\rangle\beta_{ij})))\rightarrow([p]\chi_i)))$

which simplifies to: $\exists p(p\wedge([p]\Gamma)\wedge\wedge_i\forall\xi_i((\wedge_{j=1,mi}(\langle\kappa\rangle\beta_{ij}))\rightarrow([p]\chi_i)))$

Since p does not occur in ξ_i, the KU45 modal laws of [p] allow it to be pulled out giving:

$\exists p(p\wedge([p]\ (\Gamma\wedge\wedge_i\forall\xi((\wedge_{j=1,mi}(\langle\kappa\rangle\beta_{ij}))\rightarrow\chi_i))))$ which is: $\Gamma\wedge\wedge_i\forall\xi_i((\wedge_{j=1,mi}(\langle\kappa\rangle\beta_{ij}))\rightarrow\chi_i)$

which may be rewritten as: $\Gamma\wedge\wedge_i\forall\xi_i((([\kappa]\#t)\wedge(\wedge_{j=1,mi}(\langle\kappa\rangle\beta_{ij})))\rightarrow\chi_i)$ which is: (QRL $\kappa\ \Gamma$ #t:β_{ij}/χ_i). QED.

6. Conclusion

Theorem RD11 shows that the solutions to Quantified Default Logic (i.e., QDL) are precisely the strongly grounded solutions to Quantified Reflective Logic (i.e., QRL). These results apply where variables cross modal scopes in any combination of the following two cases:

(1) where variables are universally quantified precisely over the scope of a default (or equivalently across the scope of all defaults and the initial theory Γ since they are connected by conjunction and since the universal quantifier commutes with conjunction),

(2) where variables are not quantified within the scope of the reflective equivalence in which case they are free within the scope of the theorem schemata proven herein and those schemata lie within the scope of any universal or existential quantification of such variables.

This paper does not address the important case where existential quantification occurs precisely over the scope of one or more defaults nor more complicated systems whereby quantifiers and modal symbols are

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(strongly-grounded k) =d $\neg\exists p((k\supseteq p)\wedge(\neg(p\supseteq k))\wedge(p\supseteq(\text{folth}\ p))\wedge(p\supseteq\Gamma)\wedge\wedge_i((\wedge_{j=1,mi}((\neg\beta_{ij})\notin k))\rightarrow(((\alpha_i\varepsilon p)\wedge\wedge_{j=1,mi}((\neg\beta_{ij})\notin p))\rightarrow(\chi_i\varepsilon p))))$. .

[9] [Konolige 1987a 1987b] previously attempted to prove a theorem relating the kernels of the "strongly grounded" fixed-points of Autoepistemic logic to the fixed-points of Default Logic (i.e. dl). A correct version of that attempt is described in [Antoniou 1997]. Since the fixed-points of Reflective Logic (i.e. rl) are the kernels of the fixed-points of Autoepistemic Logic that result is related to the result given herein. However, that result is not as general as the result given herein because it does not explain the relationship between <u>Quantified</u> Default Logic (i.e. QDL) and <u>Quantified</u> Reflective Logic (i.e. QRL) where variables may occur free in the 'α, 'β_{ij}, and 'χ_i sentences or in the sentences in Γ thereby being quantified across the modal scopes of the defaults (which is the subject of this paper).

nested in complex ways. (It is noted, however, that [Brown 1978] showed how an additional modal axiom allows modal scopes to be reduced to a depth of one even in the presence of quantifiers.)

This paper has not addressed automatic deduction systems for QDL and QRL, but there is the obvious point that theorems RD11 and RD12 suggest that a good deduction system for one logic may form the basis for a deduction system for the other logic. In particular, a deduction system that produced the QRL solutions could be used to produce the QDL solutions by checking which of those solutions satisfied a supporting condition (e.g. being strongly grounded) in RD11. The cost of checking a solution once it is produced would seem to be less than the cost of mechanically computing it.

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